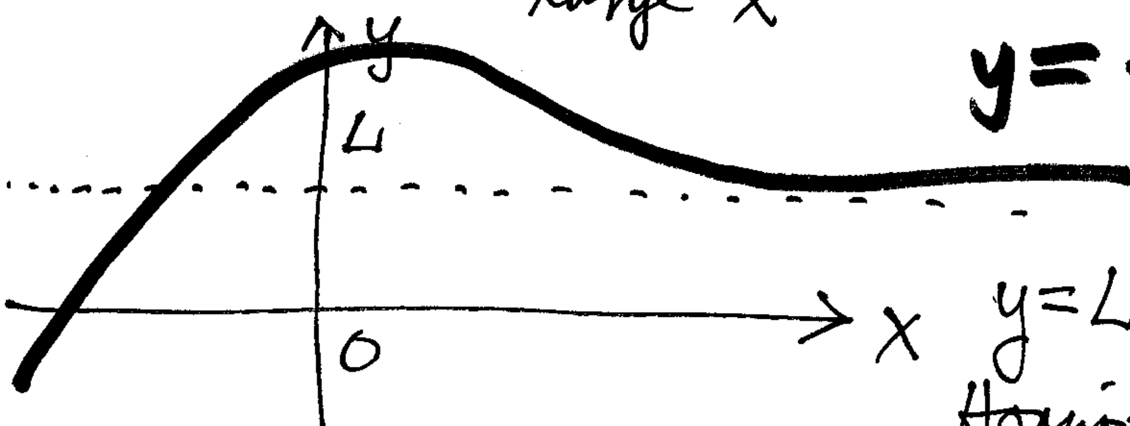
§2.5 – Limits at ∞

Last time: $\lim_{\substack{x \to 2^+ \\ (x>2)}} \dfrac{1-x^2}{x-2} = -\infty$ ($1-x^2 < 0$, $x-2 > 0$) $\Rightarrow$ $x=2$ is a vertical asymp.

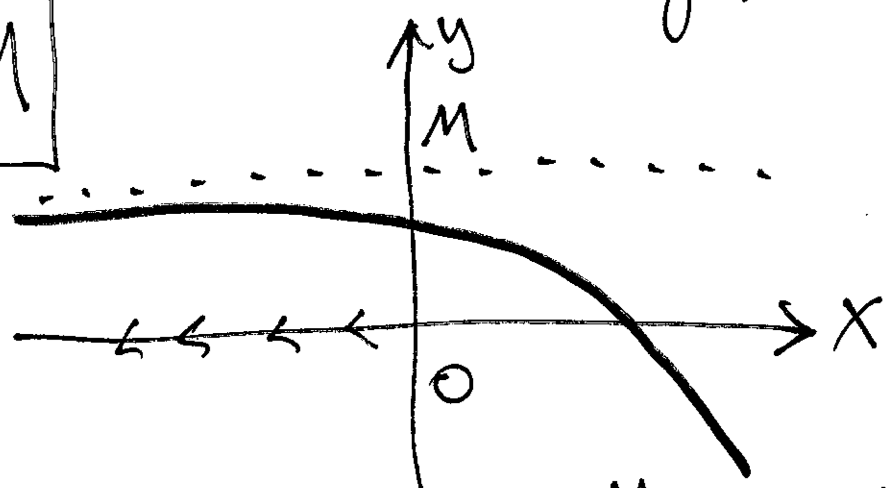
Def: $\boxed{\lim_{x\to\infty} f(x) = L}$ means $f(x)$ gets arbitrarily close to L for all suff. large x



$y = f(x)$

$y = L$ is Horizontal Asymp of f

$\boxed{\lim_{x\to -\infty} f(x) = M}$



$y = M$ Horizontal Asymp

$\boxed{\text{Ex1}}$ Compute (if they exist):

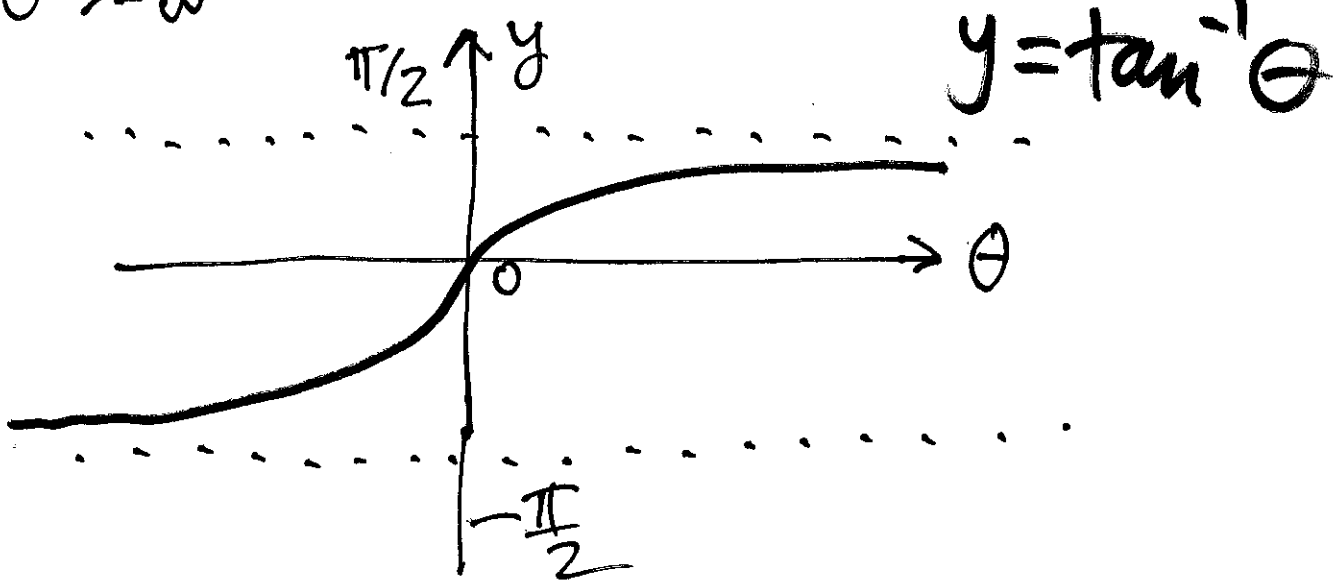
(a) $\lim_{x\to\infty}\left(3+\frac{\cos x}{x}\right) = 3+0 = 3$

$$\frac{-1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$$

$\frac{-1}{x} \to 0$ as $x\to\infty$

$\frac{1}{x} \to 0$ as $x\to\infty$

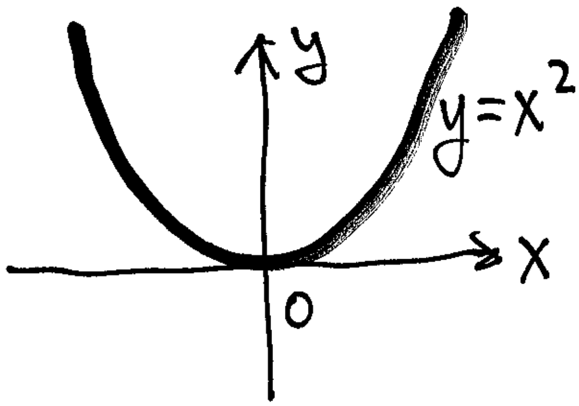
(b) $\lim_{\theta\to-\infty} \tan^{-1}\theta = -\frac{\pi}{2}$

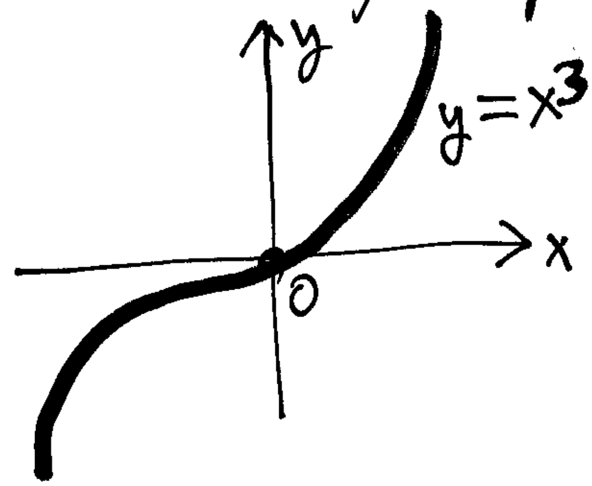


(c) $\lim_{x\to\infty} \sin x$ (DNE <u>and</u> is neither $\pm\infty$)

Def (Infinite limit at ∞)

$\boxed{\lim_{x\to\infty} f(x) = \infty}$ means f grows arb. large as x grows arbitrarily large





Thm (Limits of Powers) If $n = 1, 2, 3, \ldots$ then

① $\lim_{x\to\pm\infty} x^n = \infty$ (n even)

② $\left.\begin{array}{l} \lim_{x\to\infty} x^n = \infty \\ \lim_{x\to-\infty} x^n = -\infty \end{array}\right\}$ (n odd)

③ $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$

④ If $p(x) = \underbrace{a_n x^n}_{\text{dominant term}} + \cdots + a_1 x + a_0$ then

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$$

The behavior of $f(x)$ as $x \to \pm\infty$ is called <u>end behavior</u> of f.

<u>Def.</u>: If graph of f approaches a line that is neither horizontal nor vertical as $x \to \pm\infty$ the line is a <u>slant/oblique</u> asymptote

From page 97: $f(x) = \dfrac{2x^2+6x-2}{x+1}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\dfrac{2x^2+6x-2}{x^2}}{\dfrac{x+1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{6}{x}-\frac{2}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \infty$$

$\lim_{x\to-\infty} f(x) = -\infty$ ✓

$\therefore$ NO ~~Horizontal~~ asympt.

Slant asympt : Use "Long Division"

$$\frac{2x^2+6x-2}{x+1} = (2x+4) + \underbrace{\left(\frac{-6}{x+1}\right)}_{\to 0}$$

as $x \to \infty$

$$\begin{array}{r}
2x+4 \\
(x+1)\overline{)\,2x^2+6x-2} \\
\underline{2x^2+2x} \\
4x-2 \\
\underline{4x+4} \\
-6
\end{array}$$

$\therefore$ $y = 2x+4$

is a slant asymp.

<u>Thm</u> (End Behavior & Asymp. for Rational Functions)

Let $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x) = a_m x^m + \cdots + a_0$
$q(x) = b_n x^n + \cdots + b_0$

① if deg $p(x) <$ deg $q(x) \Rightarrow \lim\limits_{x\to\pm\infty} f(x) = 0$ $\therefore$ $y = 0$ is Horiz. Asym.

② if deg $p(x) =$ deg $q(x) \Rightarrow \lim\limits_{x\to\pm\infty} f(x) = \dfrac{a_m}{b_n}$ $\therefore$ $y = \dfrac{a_m}{b_n}$ Horiz. asym

③ if deg $p(x) >$ deg $q(x) \Rightarrow \lim\limits_{x\to\pm\infty} f(x) = \infty$ (or $-\infty$)

$\therefore$ <u>No</u> Horizon. Asymp

④ if deg $p(x) = 1 +$ deg $q(x) \Rightarrow$ get slant asymp.

⑤ If $f(x) = \dfrac{p(x)}{q(x)}$ is in reduced form

vertical asymp. are zeros of $q(x)$