Last Update: December 6

Study Guide - Remaining Course Topics

I Integration Theory:

(a) Riemann Sums

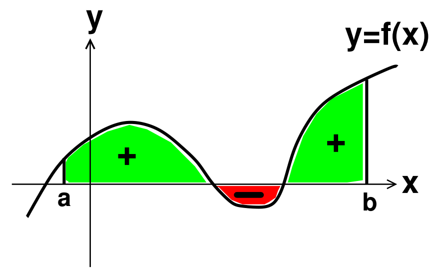
(i) Let $f(x)$ be continuous on $[a,b]$, divide this interval into n subintervals $[x_{k-1}, x_k]$ of equal length $\Delta x = \dfrac{b-a}{n}$, where $x_k = x_0 + k\Delta x$ for $k = 1, 2, 3, \cdots, n \quad (x_0 = a,\ x_n = b)$. For each k, choose a point x_k^* in $[x_{k-1}, x_k]$.

Then $\displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ is a Riemann Sum of f over $[a,b]$

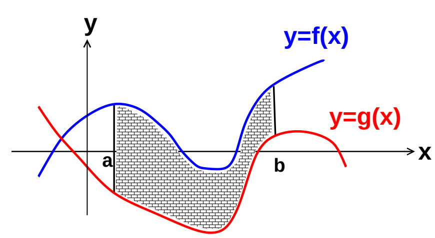
If
- $x_k^* = x_{k-1} \implies \displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ Left Riemann Sum of $f(x)$ over $[a,b]$
- $x_k^* = x_k \implies \displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ Right Riemann Sum of $f(x)$ over $[a,b]$
- $x_k^* = \dfrac{x_{k-1}+x_k}{2} \implies \displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x$ Midpoint Riemann Sum of $f(x)$ over $[a,b]$

(ii) Definite integral : $\boxed{\displaystyle\int_a^b f(x)\,dx = \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k}$, where limit is taken over all possible partitions of $[a,b]$, all possible sublengths $\Delta x_k = (x_k - x_{k-1})$ with all possible choices of x_k^* in $[x_{k-1}, x_k]$ and $\Delta = \max\left\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\right\}$

(b) $\displaystyle\int_a^b f(x)\,dx =$ Net area under the curve over $[a,b]$; properties of definite integrals.



Note that $\displaystyle\int_a^b \left\{f(x) - g(x)\right\} dx =$ Area between the curves $y = f(x)$ and $y = g(x)$ over $[a,b]$:



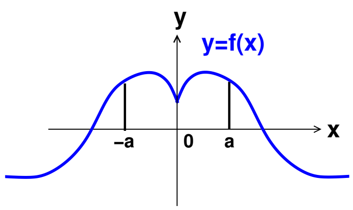
(c) <u>FUNDAMENTAL THEOREM OF CALCULUS (PART I):</u> $\dfrac{d}{dx}\left\{\int_a^x f(t)\,dt\right\} = f(x)$

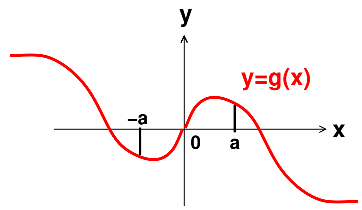
(This is also known as **Leibnitz' Rule**)

(d) <u>FUNDAMENTAL THEOREM OF CALCULUS (PART II):</u> $\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$

where $F(x)$ is any antiderivative of $f(x)$.

(e) A function $f(x)$ is **Even** if $f(-x) = f(x)$; a function $g(x)$ is **Odd** if $g(-x) = -g(x)$ and hence by symmetry: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \qquad \int_{-a}^{a} g(x)\,dx = 0:$

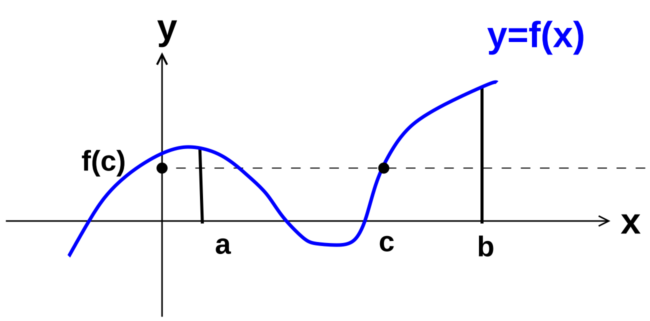




(f) *Average Value* of $f(x)$ over $[a, b]$ is $\overline{f} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$

(g) <u>Mean Value Theorem For Integrals</u>: If f is continuous over $[a, b]$, then there is a number c in (a, b) such that

$$\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\overline{f}} = f(c)$$



(h) <u>**Substitution Rule**</u>

(1) Indefinite Integrals $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, $u = g(x)$ and $du = g'(x)dx$.

(2) Definite Integrals $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$, $u = g(x)$ and $du = g'(x)dx$.

(i) <u>Generalized Leibnitz' Rule</u>: $\dfrac{d}{dx}\left\{\int_{\phi_1(x)}^{\phi_2(x)} f(t)\,dt\right\} = f(\phi_2(x))\phi_2'(x) - f(\phi_1(x))\phi_1'(x)$

II Exponential Models:

(a) The function $y(t)$ has exponential growth when $y'(t) = k\,y(t)$; hence $y(t) = y_0\,e^{kt}$ where k is the growth rate, $y(0) = y_0$, and if $\begin{cases} k > 0 \implies & y(t) \text{ has exponential growth} \\ k < 0 \implies & y(t) \text{ has exponential decay} \end{cases}$

(b) If $k > 0$ then the Doubling Time is time required to double size of y and is $T_2 = \dfrac{\ln 2}{k}$; the Tripling Time is $T_3 = \dfrac{\ln 3}{k}$

(c) If $k < 0$ then the Half-Life is time required to halve the size of y and is $T_{\frac{1}{2}} = \dfrac{\ln\left(\frac{1}{2}\right)}{k}$;

Basic TABLE OF DERIVATIVES and INTEGRALS ← Click here.

Practice Problems - Remaining Topics

(1) $\displaystyle\int_1^4 \left|x-3\right|\,dx =$ **A.** $\frac{3}{2}$ **B.** $\frac{5}{2}$ **C.** $\frac{9}{2}$ **D.** 5 **E.** $-\frac{3}{2}$

(2) If $f(x)$ is continuous on $a \le x \le b$ and $a < c < b$, then $\displaystyle\int_c^b f(x)\,dx =$

A. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ **B.** $\displaystyle\int_a^c f(x)\,dx - \int_a^b f(x)\,dx$ **C.** $\displaystyle\int_c^a f(x)\,dx + \int_b^a f(x)\,dx$

D. $\displaystyle\int_a^b f(x)\,dx - \int_a^c f(x)\,dx$ **E.** $\displaystyle\int_a^c f(x)\,dx - \int_b^c f(x)\,dx$

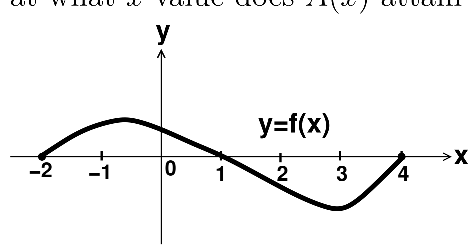
(3) $\displaystyle\int_0^2 x^2\,e^{x^3}\,dx =$ **A.** $\frac{1}{3}(e^8-1)$ **B.** (e^8-1) **C.** $\frac{1}{3}(e-1)$ **D.** $(e-1)$ **E.** $\frac{1}{3}\left(e^{\frac{1}{4}}-1\right)$

(4) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx =$ **A.** $\ln\left(\frac{1}{2}\right)$ **B.** $-\ln\left(\frac{1}{2}\right)$ **C.** $-\ln\left(2-\sqrt{3}\right)$ **D.** $\ln\left(\sqrt{3}-1\right)$
E. $\ln\left(\frac{1}{4}\right)$

(5) If $y(x) = \displaystyle\int_3^{\tan x} \sqrt{\sqrt{t}+6t}\;dt$, find $y'\left(\frac{\pi}{4}\right)$.

A. $\sqrt{7}$ **B.** $2\sqrt{7}$ **C.** $\sqrt{\frac{3\pi}{2}+\sqrt{\frac{\pi}{4}}}$ **D.** $2\sqrt{\frac{3\pi}{2}+\sqrt{\frac{\pi}{4}}}$ **E.** 2

(6) If $A(x) = \displaystyle\int_0^x f(t)\,dt$, where the graph of $y = f(t)$ is as shown, at what x value does $A(x)$ attain its maximum value on the closed interval $-2 \le x \le 4$?



A. $x=-2$ **B.** $x=0$ **C.** $x=1$ **D.** $x=3$ **E.** $x=4$

(7) If $a > 0$ and $\displaystyle\int_1^{\sqrt{a}} \frac{1}{x}\,dx = 3$, then $\displaystyle\int_1^{a} \frac{1}{x}\,dx =$

A. 9 **B.** 6 **C.** $\sqrt{3}$ **D.** 12 **E.** $\ln 9$

(8) If 40% of a radioactive substance decays in 50 days, what is the half-life of the substance?

A. $50\,\dfrac{\ln 0.5}{\ln 0.4}$ **B.** $50\,\dfrac{\ln 0.5}{\ln 0.6}$ **C.** $50\,\dfrac{\ln 0.6}{\ln 0.5}$ **D.** $50\,\dfrac{\ln 0.4}{\ln 0.5}$ **E.** $50\,\dfrac{\ln 0.4}{\ln 0.6}$

(9) Use a Riemann Sum to estimate the area under the graph of $y = \sqrt{x}$ from $x = 1$ to $x = 4$ using three approximating rectangles and left endpoints.

A. $\sqrt{1}+\sqrt{2}+\sqrt{3}$ **B.** $\sqrt{2}+\sqrt{3}+\sqrt{4}$ **C.** $\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}$ **D.** $\frac{1}{3}\left(\sqrt{1}+\sqrt{2}+\sqrt{3}\right)$
E. $\frac{1}{3}\left(\sqrt{2}+\sqrt{3}+\sqrt{4}\right)$

(10) Using 50 rectangles and the right endpoint of each subinterval as the sample point, the Riemann sum approximation of a certain definite integral is

$$\frac{1}{50}\left(\sqrt{\frac{1}{50}}+\sqrt{\frac{2}{50}}+\cdots+\sqrt{\frac{50}{50}}\right).$$

What is the definite integral that is being approximated?

A. $\displaystyle\int_0^1 \sqrt{\frac{x}{50}}\,dx$ **B.** $\displaystyle\int_0^1 \sqrt{x}\,dx$ **C.** $\displaystyle\frac{1}{50}\int_0^1 \sqrt{\frac{x}{50}}\,dx$ **D.** $\displaystyle\frac{1}{50}\int_0^1 \sqrt{x}\,dx$ **E.** $\displaystyle\frac{1}{50}\int_0^{50} \sqrt{x}\,dx$

Answers

1. B **2.** D **3.** A **4.** B **5.** B **6.** C **7.** B **8.** B
9. A **10.** B