Last Update: October 20

# Study Guide - Exam # 2

## I Derivatives

**(1)** Given $y = f(x)$ and an interval $[a, b]$, then the *change* in $x$ is $\Delta x = b - a$ and the corresponding *change* in $f$ (or change in $y$) is $\Delta y = f(b) - f(a)$ and

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \begin{cases} \text{slope of secant line through } (a, f(a)),\ (b, f(b)) \\ \text{average velocity over the interval } [a, b] \\ \text{average rate of change of } f(x) \text{ over the interval } [a, b] \end{cases}$$

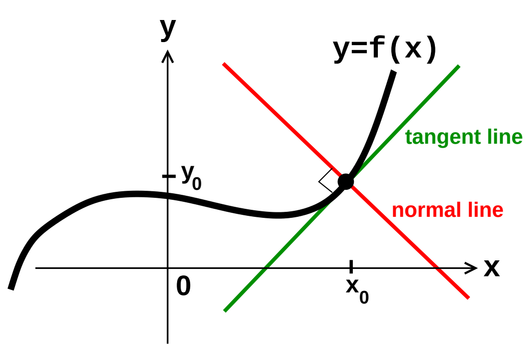
**(2)** Definition of **derivative** of $y = f(x)$ at the point $x = a$:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} \quad \text{or equivalently} \quad \boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

**(3)** Interpretation of derivative:

$$f'(a) = \begin{cases} \text{slope of tangent line to the graph of } y = f(x) \text{ at } a \\ \text{velocity at time } a \\ \text{(instantaneous) rate of change of } f(x) \text{ at } a \end{cases}$$

**(3)** Tangent line and Normal line to graph of $y = f(x)$:




<u>Point-Slope Formula</u> for the line through $(x_0, y_0)$ with slope $m$: $\boxed{y - y_0 = m(x - x_0)}$

Recall, if $m$ = slope of *tangent line*, then $-\dfrac{1}{m}$ = slope of *normal line*.

**(4)** Derivative as a function: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$; differentiable functions (i.e., $f'(x)$ exists); $f$ differentiable at $x \implies f$ continuous at $x$; know the conditions when $f$ is <u>not</u> differentiable at $x = a$

Higher order derivatives: $y''$ or equivalently $\dfrac{d^2y}{dx^2}$; $y'''$ or equivalently $\dfrac{d^3y}{dx^3}$; etc...

When $s(t)$ is the position of an object $\implies v = s'(t)$, $a = s''(t)$ and speed $= |s'(t)|$.

## II Basic Differentiation Rules: If $f$ and $g$ are differentiable functions, and $c$ is a constant:

**(1)** $\dfrac{d\{c\}}{dx} = 0\,;\quad \dfrac{d\{cf(x)\}}{dx} = cf'(x)\,;\quad \dfrac{d\{f(x)+g(x)\}}{dx} = f'(x)+g'(x)\,;$

$\dfrac{d\{f(x)-g(x)\}}{dx} = f'(x)-g'(x)$

**(2) Power Rule**: $\dfrac{d(x^n)}{dx} = nx^{n-1}$, where $n$ is any real number

**(3) Product Rule**: $\Big\{f(x)g(x)\Big\}' = f(x)g'(x) + g(x)f'(x)$

**(4) Quotient Rule**: $\left\{\dfrac{f(x)}{g(x)}\right\}' = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, provided $g(x) \neq 0$

## III Derivatives of Trig Functions

**(1)** If $x$ is in radians, then $\boxed{\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1}$ $\boxed{\lim\limits_{x\to 0}\dfrac{x}{\sin x} = 1}$ $\boxed{\lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0}$.

(Note that $\sin kx \neq k\sin x$)

**(2)** If $x$ is in radians, then

| | |
|---|---|
| $\dfrac{d}{dx}\{\sin x\} = \cos x$ | $\dfrac{d}{dx}\{\cos x\} = -\sin x$ |
| $\dfrac{d}{dx}\{\tan x\} = \sec^2 x$ | $\dfrac{d}{dx}\{\cot x\} = -\csc^2 x$ |
| $\dfrac{d}{dx}\{\sec x\} = \sec x\tan x$ | $\dfrac{d}{dx}\{\csc x\} = -\csc x\cot x$ |

## IV CHAIN RULE: If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at $x$ and its derivative is

$$\boxed{(f\circ g)'(x) = \frac{d}{dx}\Big\{f(g(x))\Big\} = f'(g(x))\,g'(x)}$$

or, equivalently, if $y = f(u)$ and $u = g(x)$, then $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}}$

## V **Implicit Differentiation Method**: If $y = y(x)$ is a function of $x$ defined implicitly by the equation $F(x, y) = 0$ then

`Step 1:` Differentiate both sides of $F(x, y) = 0$ w.r.t. the independent variable $x$

`Step 2:` Solve for the desired derivative $\dfrac{dy}{dx}$.

**Remarks:**

- Since $y = y(x)$ is a function of $x$, then by the Chain Rule $\dfrac{d}{dx}\left\{y^n\right\} = ny^{n-1}\dfrac{dy}{dx}$.
- After using Implicit Differentiation, the derivative $\dfrac{dy}{dx}$ usually involves both $x$ and $y$.

## VI **More Differentiation Rules**:

**(1)** $\boxed{\dfrac{d\left(e^x\right)}{dx} = e^x}$ $\boxed{\dfrac{d\left(b^x\right)}{dx} = b^x \ln b}$

**(2)** Derivative of Natural Logarithm

| |
|---|
| $\dfrac{d}{dx}\Big(\ln x\Big) = \dfrac{1}{x}, \quad \text{if } x > 0$ |
| $\dfrac{d}{dx}\Big(\ln \lvert x\rvert\Big) = \dfrac{1}{x}, \quad \text{if } x \neq 0$ |
| $\dfrac{d}{dx}\Big(\ln \lvert u(x)\rvert\Big) = \dfrac{u'(x)}{u(x)}, \quad \text{if } \ u(x) \neq 0$ |

**(3)** Law of Logarithms: (Reminder)

- $\log_b\left(xy\right) = \log_b x + \log_b y$
- $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
- $\log_b\left(x^p\right) = p\,\log_b x$

**(4)** **Logarithmic Differentiation Method**

`Step 1`: Take natural log of both sides of the equation and simplify using Law of Logarithms
`Step 2`: Differentiate both sides implicitly w.r.t $x$
`Step 3`: Solve the resulting equation for $\dfrac{dy}{dx}$

## VII Derivatives of Inverse Trig Functions

Note, for example, $\sin^{-1} x$ is same as $\arcsin x$, but $\sin^{-1} x \neq (\sin x)^{-1}$

**(1)** Common Basic Derivatives:

| | | |
|---|---|---|
| $\dfrac{d(\sin^{-1} x)}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ | $\dfrac{d(\cos^{-1} x)}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$, | for $-1 < x < 1$ |
| $\dfrac{d(\tan^{-1} x)}{dx} = \dfrac{1}{1+x^2}$ | $\dfrac{d(\cot^{-1} x)}{dx} = -\dfrac{1}{1+x^2}$, | for $-\infty < x < \infty$ |
| $\dfrac{d(\sec^{-1} x)}{dx} = \dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\dfrac{d(\csc^{-1} x)}{dx} = -\dfrac{1}{\lvert x\rvert\sqrt{x^2-1}}$, | for $\lvert x\rvert > 1$ |

**(2)** If $u$ is a differentiable function of $x$, then by the Chain Rule, $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, etc.

## VIII Theorem (Derivative of the Inverse Function)

If $f$ is differentiable and has an inverse on an interval $I$, $x_0$ is a point in $I$ and $f'(x_0) \neq 0$, then the inverse function $f^{-1}$ is differentiable at the point $f(x_0)$ and

$$\boxed{(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}}$$

---

# Practice Problems - Exam #2

**(1)** If $f(x) = 2x^2 + 4$, which of the following is $f'(3)$?

**A.** $\displaystyle\lim_{h\to 0} \frac{2(3+h)^2 - 10}{h}$ **B.** $\displaystyle\lim_{x\to 3} \frac{2x^2 - 18}{x-3}$ **C.** $\displaystyle\lim_{h\to 0} \frac{2(3+h)^2 + 18}{h}$ **D.** $\displaystyle\lim_{x\to 3} \frac{2x^2 + 18x}{x-3}$

**E.** $\displaystyle\lim_{h\to 0} \frac{2h^2 - 18}{h-3}$

**(2)** Find all values of $x$ for which the graph of $f(x) = \dfrac{2x-1}{x+1}$ has a tangent line parallel to the line $y = 3x + 1$

**A.** $-2$ and $0$ **B.** $3$ and $0$ **C.** $0$ and $2$ **D.** $-3$ and $1$ **E.** $-3$ and $0$

**(3)** A particle is moving on a straight line so that its displacement from the origin is given by $s(t) = t^3 - 6t^2 + 12t - 8$. During which of the following time intervals is the particle speeding up?

**A.** $(2, \infty)$ only **B.** $(0, 3)$ only **C.** $(0, \infty)$ only **D.** $(0, 1) \cup (2, \infty)$ **E.** $(0, 2)$ only

**(4)** Which of these statements are **TRUE**?

(i) If $f(x) = |x|$, then $f'(0)$ **DNE**

(ii) If $g(x) = \dfrac{|x|}{x}$, then $g'(0)$ **DNE**

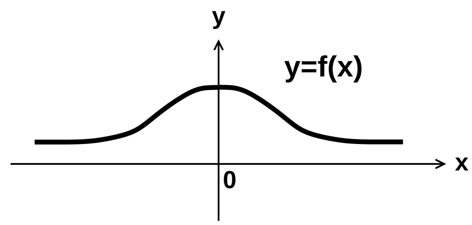
(iii) If $h(x) = x|x|$, then $h'(0) = 0$

**A.** Only (i) and (ii) **B.** Only (i) **C.** Only (i) and (iii) **D.** None are True **E.** All are True

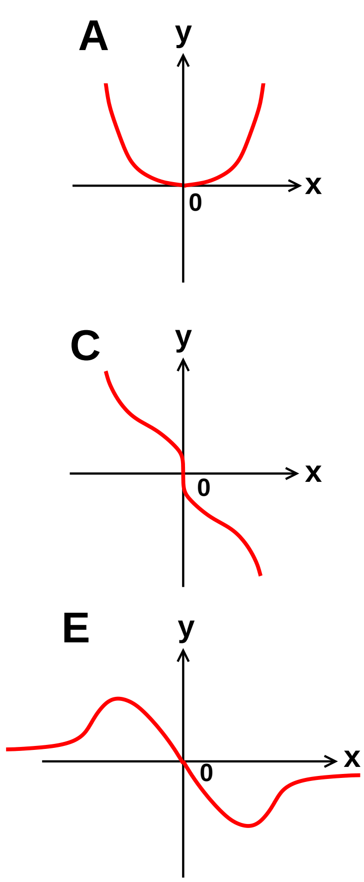
**(5)** If the normal line to the graph of $y = f(x)$ at $(2, 6)$ is $y = \frac{2}{3}x - 4$, what is $f'(2)$?

**A.** $\frac{3}{2}$ **B.** $\frac{2}{3}$ **C.** $-\frac{3}{2}$ **D.** 2 **E.** Cannot be determined

**(6)** Given the graph of $y = f(x)$ as shown here:




then which graph below looks most like the graph of the derivative $f'(x)$ ?




**(7)** The tangent line to $y = \left(2 + \sqrt{x}\right)^2$ at $x = 1$ intersects the $y$ axis where?

**A.** $(0, 6)$ **B.** $(0, 4)$ **C.** $(0, 2)$ **D.** $(0, 1)$ **E.** $(0, 3)$

**(8)** $\lim_{x\to 0} \dfrac{\tan \pi x}{x \sec x} =$

**A.** $\dfrac{1}{\pi}$ **B.** $0$ **C.** $1$ **D.** $\pi$ **E.** **DNE**

**(9)** Find the derivative of $f(x) = \dfrac{1}{\sqrt[3]{1-x^2}}$

**A.** $\dfrac{2x}{3(1-x^2)^{\frac{1}{3}}}$ **B.** $\dfrac{-2x}{3(1-x^2)^{\frac{4}{3}}}$ **C.** $\dfrac{-2x}{3(1-x^2)^{\frac{1}{3}}}$ **D.** $\dfrac{2x}{3(1-x^2)^{\frac{2}{3}}}$ **E.** $\dfrac{2x}{3(1-x^2)^{\frac{4}{3}}}$

**(10)** If $y = \sqrt{x + \sqrt{x}}$, then $y' =$

**A.** $\dfrac{1+\sqrt{x}}{2\sqrt{x+\sqrt{x}}}$ **B.** $\dfrac{1+\sqrt{x}}{4\sqrt{x+\sqrt{x}}}$ **C.** $\dfrac{1+2\sqrt{x}}{2\sqrt{x+\sqrt{x}}}$ **D.** $\dfrac{1+2\sqrt{x}}{2\sqrt{x}\sqrt{x+\sqrt{x}}}$ **E.** $\dfrac{1+2\sqrt{x}}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$

**(11)** Given the following data

| | |
|---|---|
| $f(0) = -3$ | $f(1) = 4$ |
| $f'(0) = 2$ | $f'(1) = 3$ |
| $g(0) = 1$ | $g(1) = 0$ |
| $g'(0) = -1$ | $g'(1) = -2$ |

if $h(x) = \dfrac{f(g(x))}{g(x)}$, find $h'(0)$.

**A.** 7 **B.** 2 **C.** 4 **D.** 3 **E.** 1

**(12)** $\dfrac{d^2}{dx^2}\big(x \sin x\big) =$

**A.** $2\cos x - x\sin x$ **B.** $x\cos x + \sin x$ **C.** $x\cos x - \sin x$ **D.** $x\sin x + \cos x$ **E.** $x\cos x - 2\sin x$

**(13)** Compute $\lim_{x\to 0} \dfrac{\sin x\, \cos x - \sin x + x\sin x}{x^2}$

**A.** $\frac{1}{2}$ **B.** **DNE** **C.** $0$ **D.** 2 **E.** 1

**(14)** $\dfrac{d}{dx}\Big(\sin^3(x) - \cos(x^3)\Big) =$

**A.** $3\sin^2(x)\cos(x) - 3x^2\sin(x^3)$ **B.** $3\sin^2(x)\cos(x) + 3x^2\sin(x^3)\cos(x^3)$

**C.** $3\sin^2(x)\cos(x) - 3x^2\sin(x^3)\cos(x^3)$ **D.** $3\sin^2(x)\cos(x) + 3x^2\cos(x^3)$

**E.** $3\sin^2(x)\cos(x) + 3x^2\sin(x^3)$

**(15)** Use Implicit Differentiation to find $y'$ if $x^2 + 5xy - 6y^4 = 18$

**A.** $\dfrac{2x+5y}{24y^3-5x}$ **B.** $\dfrac{2x+5y}{24y^3}$ **C.** $\dfrac{2x}{24y^3-5x}$ **D.** $\dfrac{2x+5y-18}{24y^3}$ **E.** $\dfrac{2x+5y-18}{24y^3-5x}$

**(16)** Use Logarithmic Differentiation to find the derivative of $f(x) = x^{\sin x}$

**A.** $(\sin x)\, x^{\sin x - 1}$ **B.** $x^{\sin x}\,(\cos x)\,(\ln x)$ **C.** $\dfrac{\sin x}{x} + (\cos x)\,(\ln x)$ **D.** $x^{\sin x}\left[\dfrac{\sin x}{x} + (\cos x)\,(\ln x)\right]$

**E.** $x\cos x + \sin x$

**(17)** If $f(x) = x^{\ln x}$, find $f'(e)$.

**A.** 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

**(18)** If $y = x^2 + 2^x$, then $\dfrac{dy}{dx} =$

**A.** $2x + x2^{x-1}$ **B.** $2x + 2^x$ **C.** $2x + 2^x \ln 2$ **D.** $(2x + 2^x)\ln 2$ **E.** $2x + \dfrac{2^x}{\ln 2}$

**(19)** Find $f'(1)$ if $f(x) = \ln\left[\dfrac{(3x-1)^2}{(x+1)^4}\right]$

**A.** 1 **B.** $-1$ **C.** 2 **D.** $-2$ **E.** 5

**(20)** If $f(x) = \tan^{-1}\left(\dfrac{2}{x^2}\right)$, $f'(-1) =$

**A.** 1 **B.** $\dfrac{4}{5}$ **C.** $-\dfrac{2}{5}$ **D.** $\dfrac{2}{5}$ **E.** $-\dfrac{4}{5}$

**(21)** If $y = \arcsin(x) - \sqrt{1-x^2}$, $\dfrac{dy}{dx} =$

**A.** $\dfrac{1}{2\sqrt{1-x^2}}$ **B.** $\dfrac{1+x}{\sqrt{1-x^2}}$ **C.** $\dfrac{2}{\sqrt{1-x^2}}$ **D.** $\dfrac{x^2}{\sqrt{1-x^2}}$ **E.** $\dfrac{1}{\sqrt{1+x}}$

**(22)** Compute $y''$ if $x^2 + y^2 = 2y + 5$.

**A.** $\dfrac{d^2y}{dx^2} = \dfrac{1}{1-y} + \dfrac{x^2}{(1-y)^2}$ **B.** $\dfrac{d^2y}{dx^2} = \dfrac{1}{1-y} + \dfrac{x^2}{(1-y)^3}$ **C.** $\dfrac{d^2y}{dx^2} = \dfrac{1+x^2}{(1-y)^2}$

**D.** $\dfrac{d^2y}{dx^2} = \dfrac{1+x^2}{(1-y)^3}$ **E.** $\dfrac{d^2y}{dx^2} = \dfrac{x^2}{1-y} - \dfrac{1}{(1-y)^3}$

**(23)** If $y = y(x)$ is defined implicitly by the equation $y\,e^{y^2} = 10x$, then $\dfrac{dy}{dx} =$

**A.** $y + 2ye^{y^2}$ **B.** $2y^2e^{y^2} + e^{y^2}$ **C.** $\dfrac{y}{x(2y^2+1)}$ **D.** $\dfrac{y}{(2y^2+1)}$ **E.** $\dfrac{y}{10(2y^2+1)}$

**(24)** If $f(x) = x^3 + x + 1$, compute $\left(f^{-1}\right)'(3)$.

**A.** 1 **B.** $\dfrac{1}{2}$ **C.** $\dfrac{1}{3}$ **D.** $\dfrac{1}{4}$ **E.** $\dfrac{1}{28}$

## Answers

**1.** B **2.** A **3.** A **4.** E **5.** C **6.** E **7.** A **8.** D **9.** E
**10.** E **11.** E **12.** A **13.** E **14.** E **15.** A **16.** D **17.** C
**18.** C **19.** A **20.** B **21.** B **22.** B **23.** C **24.** D