Last Update: November 7

Study Guide - Exam # 3

I Related Rates Problem Method:

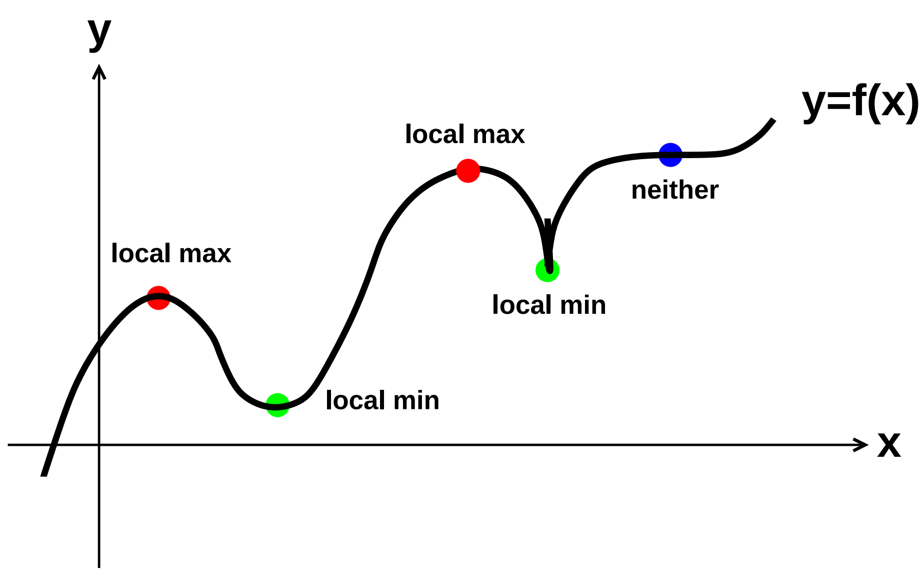
1. Read problem carefully several times.
2. Draw a picture and label variables.
3. Write down three important items: { Given rate; Desired rate; Equation (relating the variables) }
4. Use Chain Rule to differentiate Equation w.r.t to time t .
5. Solve for desired rate.

II Extrema (Maximum/Minimum):

(a) Definitions of absolute maximum value, absolute minimum value, local/relative maximum value, and local/relative minimum value; c is a **critical number** of f if c is in the domain of f and either $f'(c) = 0$ or $f'(c)$ DNE.

(b) **Absolute Extreme Value Theorem**: If $f(x)$ is continuous on a closed interval $[a, b]$, then f always has an absolute maximum value and an absolute minimum value on $[a, b]$.

(c) **Local Extreme Value Theorem**: If $f(x)$ has a local max/min value at c, then c must be a critical point of f, i.e., either $f'(c) = 0$ or $f'(c)$ DNE:



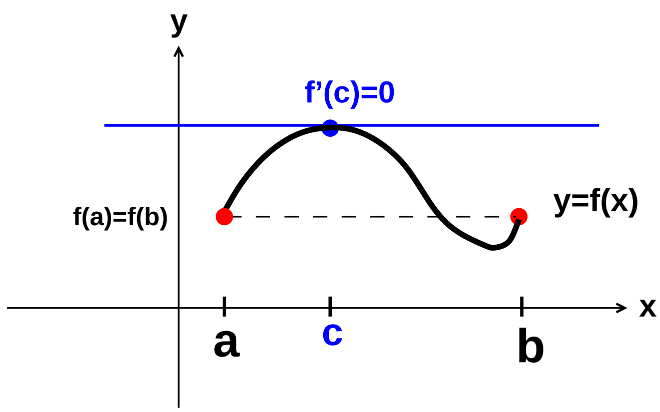
⚠ If $f'(c) = 0$, then this does **NOT** imply that $f(c)$ is a local max or local min value (see picture).

(d) **Finding Absolute Extrema Method** (f must be continuous on $[a, b]$)

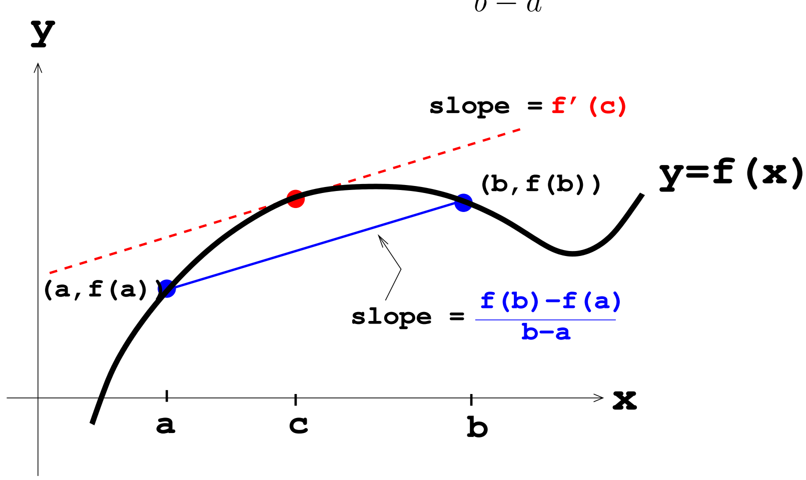
1. Find all *admissible* critical numbers c in (a, b)
2. Make table of values of $f(x)$ at the critical points and at the endpoints of the interval
3. Choose largest and smallest values of $f(x)$ in the table

III Useful Theorems:

(a) **Rolle's Theorem**: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$:



(b) **Mean Value Theorem**: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c, where $a < c < b$, such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$:



i.e., $\boxed{f(b) - f(a) = f'(c)\,(b - a)}$

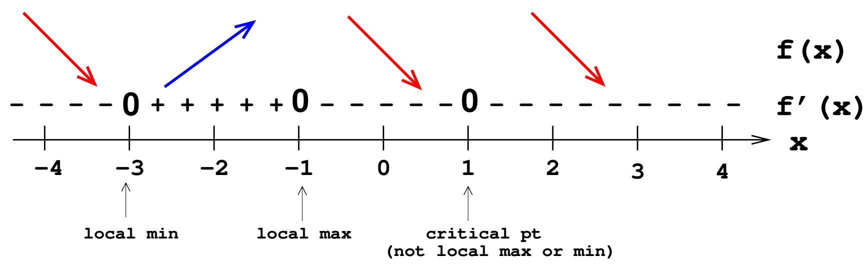
Remark: If something about f' is known, then something about the sizes of $f(a)$ and $f(b)$ can be found.

(c) **Theorem**: (Useful for integration theory later)

- If $f'(x) = 0$ for all x in I, then $f(x) = C$ for all x in I.
- If $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for all $x \in I$.

IV Increasing functions: $f'(x) > 0 \Longrightarrow f \nearrow$; decreasing functions: $f'(x) < 0 \Longrightarrow f \searrow$.

Increasing/Decreasing



V <u>**First Derivative Test**</u>: Suppose c is a critical number of a continuous function f.

(a) If f' changes from $+$ to $-$ at $c \Longrightarrow f(c)$ is local max value

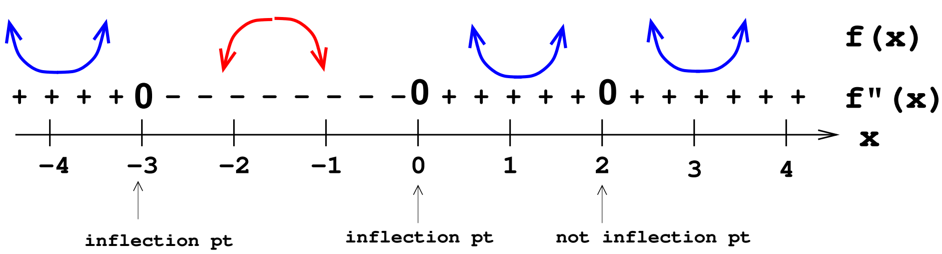
(b) If f' changes from $-$ to $+$ at $c \Longrightarrow f(c)$ is a local min value

(c) If f' <u>does not</u> change sign at $c \Longrightarrow f$ has <u>neither</u> local max nor local min value at c

(*Displaying this information on a number line is much more efficient, see above figure.*)

VI f concave up: $f''(x) > 0 \Longrightarrow f \bigcup$; and f concave down: $f''(x) < 0 \Longrightarrow f \bigcap$; inflection point (i.e. point where concavity changes).

Concave Up/Down



(*Displaying this information on a number line is much more efficient, see above figure.*)

VII <u>**Second Derivative Test**</u>: Suppose f'' is continuous near critical number c and $f'(c) = 0$.

(a) If $f''(c) > 0 \Longrightarrow f$ has a local min at c.

(b) If $f''(c) < 0 \Longrightarrow f$ has a local max at c.

⚠ If $f''(c) = 0$, then 2^{nd} Derivative Test cannot be used, so then just use 1^{st} Derivative Test.

VIII Graphing Functions $f(x)$ "Guidelines":

1. `Determine domain/interval of interest` of f
2. `Use symmetry, if available`:
   $f(-x) = f(x)$ for *even* function;
   $f(-x) = -f(x)$ for *odd* function;
   $f(x+p) = f(x)$ for *periodic* function
3. `Find intervals`: where f is increasing $\nearrow$ and decreasing $\searrow$; local max and local min
4. `Find intervals`: where f is concave up $\bigcup$ and concave down $\bigcap$; inflection points
5. `Locate asymptotes`:
   $x = a$ is a *Vertical Asymptote*: if either $\lim\limits_{x \to a^-} f(x)$ or $\lim\limits_{x \to a^+} f(x)$ is infinite

   $y = L$ is a *Horizontal Asymptote*: if either $\lim\limits_{x \to \infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$.

   $y = Ax + B$ is a *Slant Asymptote*: occurs when $f(x) = \dfrac{p(x)}{q(x)}$ and $\deg q(x) = 1 + \deg p(x)$
6. Determine x and y intercepts (if any)

IX Solving Optimization Problems "Guidelines":

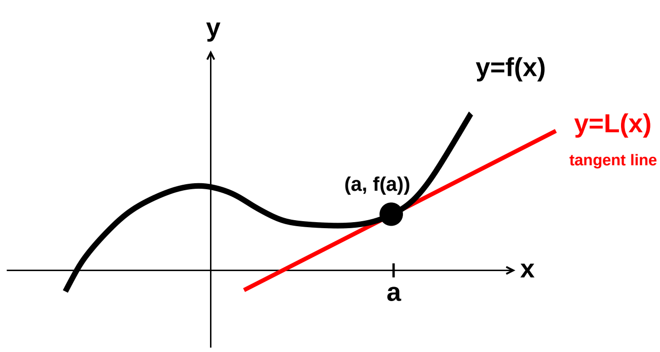
1. Read problem carefully.
2. Draw a picture and identify/label variables.
3. Write down the **objective function** (i.e., the function to be extremized). It is usually a function of more than one independent variable.
4. Write down the constraint(s) and use to express the objective function as a function of a one variable.
5. Use Calculus techniques to find the absolute extrema (absolute max and/or min) over the interval of interest. (Mostly use the First Derivative Test here)

X Differentials and Linear Approximations:

(a) **Theorem (Linear Approximation of $f(x)$ near $x = a$)**:

$$\boxed{f(x) \approx L(x) = f(a) + f'(a)(x-a)} \text{ for } x \text{ near } a$$

Remark: Note that $y = L(x) = f(a) + f'(a)(x-a)$ is just the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. The above linear approximation is sometimes also called the *tangent line approximation to $f(x)$ at $x = a$*:



(b) Given a function $y = f(x)$, if x changes from x to $x + \Delta x$, then the corresponding exact change in y is $\Delta y = f(x + \Delta x) - f(x)$. The *differential* of x is defined by $dx = \Delta x$, while the *differential* of y is defined by $\boxed{dy = f'(x)\,dx}$. In general, $\Delta y \neq dy$.

Remark: The theorem above states that $\Delta y \approx dy$, i.e., $\underbrace{f(x+dx) - f(x)}_{\Delta y} \approx \underbrace{f'(x)\,dx}_{dy}$

XI Indeterminate Forms:

(a) **Indeterminate Form (Types):** $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

(b) **L'Hôpital's Rule**: Let f and g be differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a).

If $\lim_{x\to a} f(x) = g(x) = 0$; or if $\lim_{x\to a} f(x) = g(x) = \pm\infty$ then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

provided the limit on the right side exists or is infinite.

⚠ *You may need to first do some algebra to convert one of the various Indeterminate Forms types indicated in Part* (a) *into expressions where the above formula can be used.*

Important Remark: L'Hôpital's Rule is also valid for one-sided limits, $x \to a^-$, $x \to a^+$, and also for limits when $x \to \infty$ or $x \to -\infty$.

XII Antiderivatives:

(a) $F(x)$ is called an *antiderivative* of $f(x)$ on I, if $F'(x) = f(x)$ for all x in I.

(b) **Theorem (All Antiderivatives)**: If $F(x)$ is one antiderivative of $f(x)$ on I, then $F(x) + C$ represents all antiderivatives of $f(x)$ on I.

(c) An antiderivative $F(x)$ is represented by the notation $\int f(x)\,dx$, this is called an indefinite integral of $f(x)$. Hence $\int f(x)\,dx = F(x) + C$.

(d) **Basic Properties of Indefinite Integrals**

(i) $\int kf(x)\,dx = k\int f(x)\,dx$, for every constant k

(ii) $\int \Big\{f(x) + g(x)\Big\}\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iii) $\int k\;dx = kx + C$

(e) **Power Rule for Integrals** $\boxed{\int x^p\,dx = \frac{x^{p+1}}{p+1}\,dx}$, for any number $p \neq -1$.

(f) Basic TABLE OF DERIVATIVES and INTEGRALS ← **Click here**.

Practice Problems - Exam #3

(0) **Exam #3, Fall 2017 ← Click here**. (Good Exam Questions)

ANSWERS ← Click here

Extra RR and MM Problems and Solutions (my web page) ← Click here.

(1) If each edge of a cube is increasing at a constant rate of 3 cm/s, how fast is the volume of the cube increasing when the length of each edge is 10 cm?

A. 270 cm^3/s **B.** 300 cm^3/s **C.** 900 cm^3/s **D.** 27 cm^3/s **E.** 3000 cm^3/s

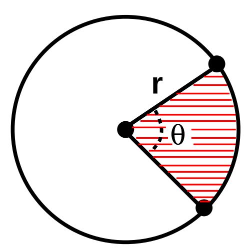
(2) A kite at a constant altitude 0f 100 ft above the ground moves horizontally at an unknown but constant speed. When 200 ft of string has been let out, the angle between the string and the ground is decreasing at $\frac{1}{40}$ rad/s. How fast, in ft/s, is the kite moving?

A. 5 **B.** 10 **C.** 15 **D.** 20 **E.** 25

(3) The minimum value of the function $f(x) = \frac{3}{8}x^4 + x^3$ is

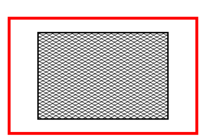
A. -4 **B.** $-\frac{3}{2}$ **C.** -2 **D.** 0 **E.** $-\frac{5}{8}$

(4) In this picture of a circular sector the area $\frac{1}{2}r^2\theta$ and the length of the circular arc is $r\theta$, where r is the radius and θ is the opening angle. What is the maximum area among circular sectors whose perimeter is 6?



A. 3.5 **B.** 4 **C.** 1.75 **D.** 3 **E.** 2.25

(5) The top and bottom margins of a rectangular poster are each 1 inch and the side margins are each 2 inches. If the area of printed material on the poster is fixed at 32 square inches, find the smallest possible area of the entire poster.



A. 72 square inches **B.** 80 square inches **C.** 64 square inches **D.** 96 square inches
E. 76 square inches

(6) Let $f(x)$ be a polynomial with $f(2) = 1$. Assume that $f'(x) \geq 3$ for every x in $[2, 4]$. What is the smallest possible value of $f(4)$?

Hint: Apply the Mean Value Theorem

A. 1 **B.** 3 **C.** 4 **D.** 6 **E.** 7

(7) Let f be a function whose derivative is given by $f'(x) = (x-1)(x+3)(x-4)$. The function f has

A. Local maxima at $x = -3$ and $x = 1$; and a local minimum at $x = 4$

B. Local maxima at $x = 1$ and $x = 4$; and a local minimum at $x = -3$

C. Local maxima at $x = -3$ and $x = 4$; and a local minimum at $x = 1$

D. A local maximum at $x = 4$; and local minima at $x = -3$ and $x = 1$

E. A local maximum at $x = 1$; and local minima at $x = -3$ and $x = 4$

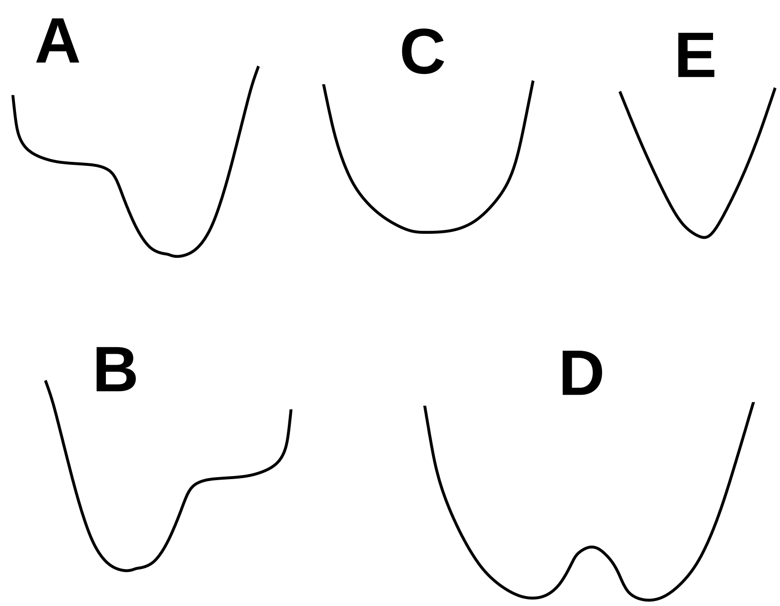
(8) On the open interval $(2, 3)$, the function $f(x) = x^3 - 6x^2 + 9x + 30$ is

A. Increasing and concave up **B.** Decreasing and concave down

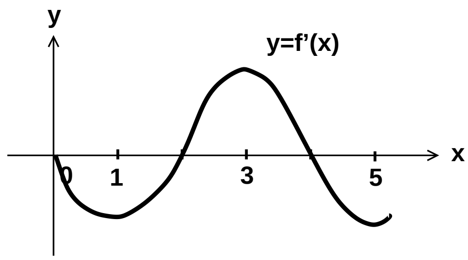
C. Increasing and concave down **D.** Decreasing and concave up

E. None of the above

(9) Find the shape of the graph of $y = 3x^4 - 8x^3$.



(10) The graph of $y = f'(x)$, the derivative of f, is shown below.



Which of the following statements about f are TRUE?

I. The graph of f is concave up on the interval $(2, 4)$

II. $f(x)$ has a local minimum at $x = 2$

III. $(1, f(1))$ is an inflection point for f

A. None are TRUE **B.** I and III **C.** II and III **D.** I and II **E.** I, II, and III

(11) Determine the value of b so that $f(x) = x^2 + \dfrac{b}{x^3}$ has an inflection point at $x = 1$

A. 1 **B.** $-\dfrac{1}{6}$ **C.** $\dfrac{1}{3}$ **D.** $-\dfrac{1}{3}$ **E.** $\dfrac{1}{6}$

(12) Find the limit $\displaystyle\lim_{x\to\infty}\left(1 + \frac{3}{x}\right)^{4x}$. **A.** e^4 **B.** 4 **C.** e^{12} **D.** 12 **E.** $e^{\frac{3}{4}}$

(13) Which of these statements are TRUE?

(I) $\displaystyle\lim_{x\to 0^+} x^2 \ln(3x) = 0$

(II) $\displaystyle\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) = 0$

(III) $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x^2+2x} + x}{x} = 2$

A. (I) and (II) only **B.** (I) and (III) only **C.** (I) only **D.** (II) and (III)
E. All are TRUE

(14) Use a linear approximation to estimate the value of $e^{-0.01}$

A. 1.001 **B.** 1.01 **C.** 0.9 **D.** 0.99 **E.** 0.999

(15) Find an antiderivative of the function $\dfrac{(x-1)^2}{x}$.

A. $\dfrac{2(x-1)^3}{3x^2}$ **B.** $\dfrac{x^2-1}{x^2}$ **C.** $\dfrac{1}{2}x^2 - 2x + \ln|x|$ **D.** $\dfrac{x(x-1)^3}{3}$ **E.** $\dfrac{(x-1)^2\,(2x+1)}{6}$

(16) If a particle is moving at the speed $v(t) = \dfrac{1}{t^2+1}$, then the distance it covers from time $t = 1$ to $t = \sqrt{3}$ is: **A.** $\dfrac{\pi}{3}$ **B.** $\dfrac{\pi}{4}$ **C.** $\dfrac{\pi}{2}$ **D.** π **E.** $\dfrac{\pi}{12}$

(17) If $F(x)$ is the antiderivative of $f(x) = 2x + \dfrac{4}{\sqrt{x}} + 1$ such that $F(1) = 6$, then $F(4) =?$

A. 17 **B.** 20 **C.** 26 **D.** 29 **E.** 32

(18) If $g(x)$ is the unique solution to the Initial Value Problem (**IVP**) $\begin{cases} g''(x) = 6x \\ g(0) = -4 \\ g'(0) = 1 \end{cases}$,

then $g(1) = ?$ **A.** -2 **B.** -1 **C.** 0 **D.** 4 **E.** 6

(19) Evaluate these indefinite integrals:

(a) $\displaystyle\int \left(\frac{4}{\sqrt{x}} + x\sqrt{x}\right)\, dx$ (b) $\displaystyle\int (x^2-3)^2\, dx$ (c) $\displaystyle\int \left(\frac{6u-1}{2u}\right)\, du$

(d) $\displaystyle\int 1 + 2\sin x - 3\cos x\, dx$ (e) $\displaystyle\int \left(\frac{\theta^2}{2} - \sec^2\theta\right)\, d\theta$

(20) Given that $F'(x) = \dfrac{1}{x^2+3x}$, express $\displaystyle\int \left(\frac{100}{x^2+3x} - 1\right)\, dx$ in terms of $F(x)$.

(21) Compute the differential: (a) $y = \sin^{-1} 3x + \big(\sin 3x\big)^{-1}$ (b) $w = p2^p + 4$

(22) Given the table of values

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-3	2	2	5
2	4	0	-1	π

, use linear approximations to estimate

(a) $f(1.8)$ (b) $h(1.1)$, where $h = f \circ g$.

Answers

1. C **2.** B **3.** C **4.** E **5.** A **6.** E **7.** E **8.** D
9. A **10.** C **11.** B **12.** C **13.** E **14.** D **15.** C
16. E **17.** E **18.** A

19. (a) $8\sqrt{x} + \dfrac{2}{5}x^{\frac{5}{2}} + C$ (b) $\dfrac{x^5}{5} - 2x^3 + 9x + C$ (c) $3u - \dfrac{1}{2}\ln|u| + C$

(d) $x - 2\cos x - 3\sin x + C$ (e) $\dfrac{\theta^3}{6} - \tan\theta + C$

20. $100F(x) - x + C$

21. (a) $dy = \left(\frac{3}{\sqrt{1-9x^2}} - 3\csc 3x \cot 3x\right) dx$ (b) $dw = 2^p\left(p \ln 2 + 1\right) dp$ **22.** (a) 4.2 (b) 3.5