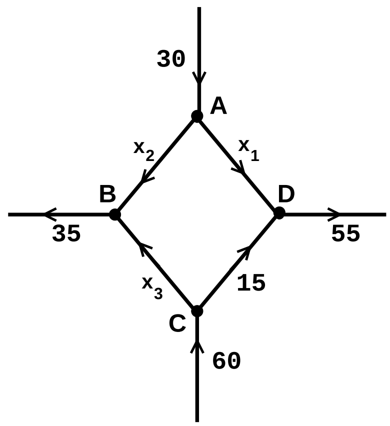
Some Applications of Linear Systems

I **Network Flow**: Suppose cars enter and leave intersections at certain rates per hour. For example 55 cars per hour leave the intersection D (see below). Find x_1, x_2, x_3, assuming that the net flow of cars into an intersection is equal to the net flow of cars out of the intersection:



Intersection **A** : $30 = x_2 + x_1$
Intersection **B** : $x_2 + x_3 = 35$
Intersection **C** : $60 = x_3 + 15$
Intersection **D** : $x_1 + 15 = 55$

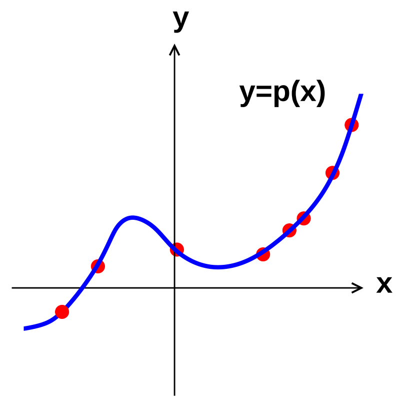
$$\Longrightarrow \begin{cases} x_1 + x_2 = 30 \\ x_2 + x_3 = 35 \\ x_3 = 45 \\ x_1 = 40 \end{cases} \qquad \Longrightarrow \left[A \,\middle|\, \mathbf{b}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 30 \\ 0 & 1 & 1 & 35 \\ 0 & 0 & 1 & 45 \\ 1 & 0 & 0 & 40 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 45 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus
$x_1 = 40$
$x_2 = -10$ (this means direction in figure should be in *opposite* direction in the figure above)
$x_3 = 45$

II **Polynomial Interpolation**: Given $n+1$ fixed points $(x_1, y_1),\ (x_2,\ y_2), \cdots, (x_{n+1},\ y_{n+1})$ in $\mathbb{R}^2$ that have distinct x coordinates, then there exists a unique polynomial of degree n of the form

$$\boxed{y = p(x) = a_0 + a_1 x + \cdots + a_n x^n}$$

such that $\quad p(x_1) = y_1,\ p(x_2) = y_2,\ \cdots\ p(x_{n+1}) = y_{n+1}$:



Thus we obtain the linear system in the unknowns variables $a_0, a_1, \cdots, a_{n+1}$:

$$\begin{cases} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n & = y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_n x_2^n & = y_2 \\ \quad\vdots & \quad\vdots \\ a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + \cdots + a_n x_{n+1}^n & = y_{n+1} \end{cases} \implies \left[A \,\middle|\, \mathbf{b} \right] = \left[\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots\cdots & x_1^n & y_1 \\ 1 & x_2 & x_2^2 & \cdots\cdots & x_2^n & y_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots\cdots & x_{n+1}^n & y_{n+1} \end{array} \right]$$

For example, find the (unique) quadratic polynomial that passes through the 3 points $(-1, 7),\ (1, 5),\ (2, 10)$.

<u>*Solution*</u>: Let $(x_1, y_1) = (-1, 7),\ (x_2, y_2) = (1, 5),\ (x_3, y_3) = (2, 10)$ and $y = p(x) = a_0 + a_1 x + a_2 x^2$.

Since $p(x_1) = y_1,\ p(x_2) = y_2,\ p(x_3) = y_3$ we get the linear system $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 10 \end{bmatrix}$.

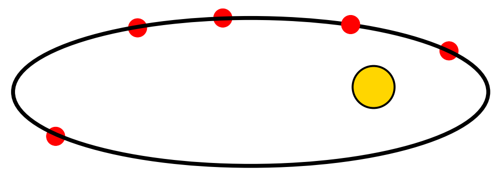
Use **GEM** or **GJEM** to solve the system to obtain $a_0 = 4,\ a_1 = -1,\ a_2 = 2$. Hence the unique polynomial of degree 2 is $\quad y = p(x) = 4 - x + 2x^2$.

III **Computing Planetary Orbits**: *Kepler's* 1^{st} *Law of Planetary Motion* says that a planet travels around the sun in an elliptical orbit in a plane with the sun at one focus of the ellipse. Hence the orbit of a planet can be described by the general formula for a conic section in the plane:

$$x^2 + axy + by^2 + cx + dy + e = 0$$

If a planet's position is known at just **five** (5) different points (x_1, y_1), $(x_2, y_2), \cdots, (x_5, y_5)$, then we can determine the equation of the planet's orbit by solving for the unknowns a, b, c, d, e :

$$\begin{cases} x_1^2 + ax_1y_1 + by_1^2 + cx_1 + dy_1 + e = 0 \\ x_2^2 + ax_2y_2 + by_2^2 + cx_2 + dy_2 + e = 0 \\ x_3^2 + ax_3y_3 + by_3^2 + cx_3 + dy_3 + e = 0 \\ x_4^2 + ax_4y_4 + by_4^2 + cx_4 + dy_4 + e = 0 \\ x_5^2 + ax_5y_5 + by_5^2 + cx_5 + dy_5 + e = 0 \end{cases} .$$

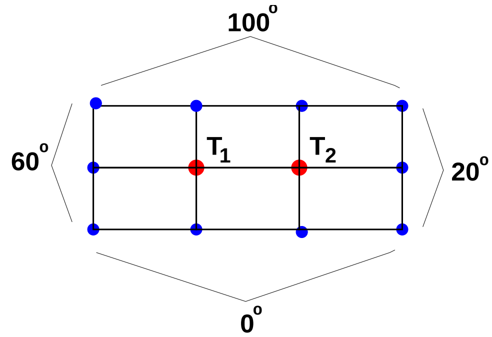


Or, equivalently: $\begin{cases} ax_1y_1 + by_1^2 + cx_1 + dy_1 + e = -x_1^2 \\ ax_2y_2 + by_2^2 + cx_2 + dy_2 + e = -x_2^2 \\ ax_3y_3 + by_3^2 + cx_3 + dy_3 + e = -x_3^2 \\ ax_4y_4 + by_4^2 + cx_4 + dy_4 + e = -x_4^2 \\ ax_5y_5 + by_5^2 + cx_5 + dy_5 + e = -x_5^2 \end{cases}$. The corresponding augmented matrix

of this linear system in the unknowns a, b, c, d, e is $\left[A \middle| \mathbf{b} \right] = \left[\begin{array}{ccccc|c} x_1y_1 & y_1^2 & x_1 & y_1 & 1 & -x_1^2 \\ x_2y_2 & y_2^2 & x_2 & y_2 & 1 & -x_2^2 \\ x_3y_3 & y_3^2 & x_3 & y_3 & 1 & -x_3^2 \\ x_4y_4 & y_4^2 & x_4 & y_4 & 1 & -x_4^2 \\ x_5y_5 & y_5^2 & x_5 & y_5 & 1 & -x_5^2 \end{array} \right]$.

Solve this system using **GEM** or **GJEM** to determine the unknowns a, b, c, d, e.

IV **Temperature Distribution**: Let D be a rectangular lamina in $\mathbb{R}^2$ (a thin plate) that is insulated so that heat flow can only occur across its 4 sides and its 4 corners are insulated. Suppose that the sides are kept at fixed temperatures as shown below. We want to estimate the temperature at interior points. One method is to partition D into small rectangular regions as shown below and assume that the temperature at a node is the average of its temperatures at its 4 nearest node (this is simplistic, but this is just to show the method). Find the temperature at the nodes shown below:



$$\Longrightarrow \begin{cases} T_1 = \dfrac{100+60+0+T_2}{4} \\ \\ T_2 = \dfrac{100+T_1+0+20}{4} \end{cases} \Longrightarrow \begin{cases} 4T_1 - T_2 = 160 \\ \\ -T_1 + 4T_2 = 120 \end{cases} \Longrightarrow T_1 = \frac{152}{3} \approx 50.67°,\ T_2 = \frac{128}{3} \approx 42.67°$$

$\boxed{\text{V}}$ **Difference Equations**: The famous *Fibonacci sequence*

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \cdots$$

is defined by the difference equation $\boxed{F_{k+1} = F_k + F_{k-1}}$ (*)

where $F_1 = 0$ and $F_2 = 1$ and $k = 2, 3, \cdots$. Thus $F_3 = 1,\ F_4 = 2,\ F_5 = 3,\ F_6 = 5, F_7 = 8, \cdots$. This sequence occurs in many places, even in nature - flowers, trees, honey bees, genetics, etc. This sequence is also associated with the **Golden Ratio** $\varphi = \dfrac{1+\sqrt{5}}{2}$. There are entire books written on the Fibonacci sequence and also on the Golden Ratio.

Question: How can we find the n^{th} Fibonacci number F_n when n is large?

Solution: Let $\mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ and note that the *single* equation (*) is equivalent to the *system*

$$\begin{cases} F_{k+1} = F_k + F_{k-1} \\ F_k = F_k \end{cases}$$

In matrix form, this system is $\mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = A\mathbf{u}_{k-1}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Hence for $k = 2, 3, 4 \cdots$ we get

$$\mathbf{u}_k = A\mathbf{u}_{k-1} \quad (*)$$

Since $\mathbf{u}_2 = A\mathbf{u}_1$, where $\mathbf{u}_1 = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we can now iterate $(*)$ as follows:

$$\begin{aligned} \mathbf{u}_3 &= A\mathbf{u}_2 = A(A\mathbf{u}_1) = A^2\mathbf{u}_1 \\ \mathbf{u}_4 &= A\mathbf{u}_3 = A(A^2\mathbf{u}_1) = A^3\mathbf{u}_1 \\ &\vdots \end{aligned}$$

We end up with a formula

$$\mathbf{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = A^{k-1}\,\mathbf{u}_1 \qquad (**)$$

and we can obtain any Fibonacci number F_k we wish simply by multiplying a power of A and the fixed column vector $\mathbf{u}_1$.

For example, suppose we needed the 11^{th} Fibonacci number F_{11}. Let $k = 10$ we can compute $A^9 = \begin{bmatrix} 55 & 34 \\ 34 & 21 \end{bmatrix}$ and we see that $(**)$ becomes

$$\mathbf{u}_{10} = \begin{bmatrix} F_{11} \\ F_{10} \end{bmatrix} = A^9\mathbf{u}_1 = \begin{bmatrix} 55 & 34 \\ 34 & 21 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 55 \\ 34 \end{bmatrix}, \text{ hence } \boldsymbol{F_{11} = 55} \text{ and } F_{10} = 34 \text{ (for free!).}$$

NOTE: An easy method for computing A^k will be given later.

$\boxed{\text{VI}}$ **Partial Fractions**: Find a *Partial Fraction Decomposition* for $\dfrac{3x^3+3x^2+3x-1}{x^2(x^2+1)}$.

Solution:

$$\begin{aligned}\frac{3x^3+3x^2+3x-1}{x^2(x^2+1)} &= \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}\\ &= \frac{Ax(x^2+1)+B(x^2+1)+x^2(Cx+D)}{x^2(x^2+1)}\end{aligned}$$

$$\Longrightarrow\ Ax(x^2+1)+B(x^2+1)+x^2(Cx+D)=3x^3+3x^2+3x-1$$

$$\Longrightarrow\ x^3(A+C)+x^2(B+D)+x(A)+(B)=3x^3+3x^2+3x-1$$

$$\Longrightarrow \begin{cases} A+C &= 3\\ B+D &= 3\\ A &= 3\\ B &= -1\end{cases} \quad\Longrightarrow\quad \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}3\\3\\3\\-1\end{bmatrix}$$

Thus the augmented matrix becomes

$$\left[\begin{array}{cccc|c}1&0&1&0&3\\0&1&0&1&3\\1&0&0&0&3\\0&1&0&0&-1\end{array}\right]\sim\cdots\cdots\sim\left[\begin{array}{cccc|c}\boxed{1}&0&0&0&3\\0&\boxed{1}&0&0&-1\\0&0&\boxed{1}&0&0\\0&0&0&\boxed{1}&4\end{array}\right]\Longrightarrow A=3,\ B=-1,\ C=0,\ D=4$$

Hence

$$\frac{3x^3+3x^2+3x-1}{x^2(x^2+1)}=\frac{3}{x}-\frac{1}{x^2}+\frac{4}{x^2+1}$$

VII **Leontief Model - Nobel Prize**: Our economy has 3 industries: coal, electricity, and auto. To produce \$1 of each we get the data:

Requires $\longrightarrow$	**Coal**	**Elect**	**Auto**
s_1 = Coal Industry	\$0.10	\$0.25	\$0.20
s_2 = Electricity Industry	\$0.30	\$0.40	\$0.50
s_3 = Auto Industry	\$0.10	\$0.15	\$0.10

Suppose one week demand is \$50 K for Coal; \$75 K for Elect; \$125 K for Auto.

Thus $D = \begin{bmatrix} 50,000 \\ 75,000 \\ 125,000 \end{bmatrix}$. Find production levels to meet internal and external demands for each of the three industries.

Solution: Define a_{ij} and p_j as:

$a_{ij} =$ # units (dollars) produced by Industry s_i to produce 1 unit (\$1) of Industry s_j

$p_j =$ production level of Industry s_j

$\Longrightarrow a_{ij}\, p_j =$ # units (dollars) produced by s_i and consumed by s_j

Hence total number of units (dollars) produced by s_i is

$$a_{i1}p_1 + a_{i2}p_2 + a_{i3}p_3 \qquad (\textbf{internal demand})$$

Now $p_i + d_i$ is the **external demand** for Industry s_i. Hence in order to meet internal and external demands for Industry s_i, we need $a_{i1}p_1 + a_{i2}p_2 + a_{i3}p_3 = p_i + d_i$. Thus for all three industries we get the linear system:

$$\begin{cases} a_{11}p_1 + a_{12}p_2 + a_{13}p_3 = p_1 + d_1 \\ a_{21}p_1 + a_{22}p_2 + a_{23}p_3 = p_2 + d_2 \\ a_{31}p_1 + a_{32}p_2 + a_{33}p_3 = p_3 + d_3 \end{cases}$$

Thus, $AP = P + D$ where $A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} 0.10 & 0.25 & 0.20 \\ 0.30 & 0.40 & 0.50 \\ 0.10 & 0.15 & 0.10 \end{bmatrix}$, $D = \begin{bmatrix} 50,000 \\ 75,000 \\ 125,000 \end{bmatrix}$, $P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$.

Solving for P:

$$\begin{aligned} AP &= P + D \\ AP - P &= D \\ AP - IP &= D \\ (A - I)P &= D \\ (A - I)^{-1}(A - I)P &= (A - I)^{-1} D \\ P &= (A - I)^{-1} D \end{aligned}$$

Applying this to our 3 industry economy, we get

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \$229,921.59 \\ \$437,795.27 \\ \$237,401.57 \end{bmatrix}$$