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Several Simple Real-world Applications of Linear Algebra Tools

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Abstract. In this paper we provide several real-world motivated examples illustrating the power of the linear algebra tools as the product of matrices and matrix notation of systems of linear equations.

To explain some mathematical terms in a class or in a textbook it is often convenient to illustrate them by suitable examples having applications in our daily life. If the class or the textbook are on the basic level, the examples should be simple enough so that no special knowledge would be required. Four such examples, which illustrate the use of matrices, the efficiency of their products and an advantage of matrix notation for a system of linear equations, are presented.

Example 1

Three people denoted by P_1, P_2, P_3 intend to buy some rolls, buns, cakes and bread. Each of them needs these commodities in differing amounts and can buy them in two shops S_1, S_2. Which shop is the best for every person P_1, P_2, P_3 to pay as little as possible? The individual prices and desired quantities of the commodities are given in the following tables:

Demanded quantity of foodstuff:

	roll	bun	cake	bread
P_1	6	5	3	1
P_2	3	6	2	2
P_3	3	4	3	1

Prices in shops S_1 and S_2:

	S_1	S_2
roll	1.50	1.00
bun	2.00	2.50
cake	5.00	4.50
bread	16.00	17.00

For example, the amount spent by the person P_1 in the shop S_1 is:

$$6 \cdot 1.50 + 5 \cdot 2 + 3 \cdot 5 + 1 \cdot 16 = 50$$

and in the shop S_2 :

$$6 \cdot 1 + 5 \cdot 2.50 + 3 \cdot 4.50 + 1 \cdot 17 = 49,$$

for the other people similarly. These calculations can be written using a product of two matrices

$$\mathbf{P} = \begin{pmatrix} 6 & 5 & 3 & 1 \\ 3 & 6 & 2 & 2 \\ 3 & 4 & 3 & 1 \end{pmatrix}$$

(the demand matrix) and

$$\mathbf{Q} = \begin{pmatrix} 1.50 & 1 \\ 2 & 2.50 \\ 5 & 4.50 \\ 16 & 17 \end{pmatrix}$$

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(the price matrix). For example, the first row of the matrix

$$\mathbf{R} = \mathbf{PQ} = \begin{pmatrix} 50 & 49 \\ 58.50 & 61 \\ 43.50 & 43.50 \end{pmatrix}$$

expresses the amount spent by the person P_1 in the shop S_1 (the element r_{11}) and in the shop S_2 (the element r_{12}). Hence, it is optimal for the person P_1 to buy in the shop S_2, for the person P_2 in S_1 and the person P_3 will pay the same price in S_1 as in S_2.

Example 2

To encode a short message a number can be assigned to each letter of the alphabet according to a given table. The text as a sequence of numbers will be organized into a square matrix **A**; in the case that the number of letters is lower than the number of elements of the matrix **A**, the rest of the matrix can be filled with zero elements. Let a nonsingular square matrix **C** be given. To encode the text the matrix **A** can be multiplied by the matrix **C** for example on the left. Let the following table and the matrix **C** be given:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
8	7	5	13	9	16	18	22	4	23	11	3	21	1	6	15	12	19	2	14	17	20	25	24	10	26

$$\mathbf{C} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

We put the text "BILA KOCKA" (a white cat) into the matrix **A**:

$$\mathbf{A} = \begin{pmatrix} 7 & 4 & 3 \\ 8 & 11 & 6 \\ 5 & 11 & 8 \end{pmatrix}$$

and encode the text:

$$\mathbf{Z} = \mathbf{CA} = \begin{pmatrix} 19 & 19 & 14 \\ 12 & 15 & 11 \\ 8 & 11 & 6 \end{pmatrix}.$$

To decode the message we have to multiply the matrix **Z** by the matrix $\mathbf{C^{-1}}$ on the left:

$$\mathbf{C^{-1}Z} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 19 & 19 & 14 \\ 12 & 15 & 11 \\ 8 & 11 & 6 \end{pmatrix} = \mathbf{A}.$$

Since the matrix multiplication is not commutative, it is necessary to keep the order of the matrices in the product. If we multiply the matrices $\mathbf{C^{-1}}$ and **Z** in the opposite order, we obtain

$$\mathbf{ZC^{-1}} = \begin{pmatrix} 19 & 19 & 14 \\ 12 & 15 & 11 \\ 8 & 11 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 9 & 19 \\ 1 & 10 & 15 \\ 2 & 4 & 11 \end{pmatrix}$$

and it means "CERNY PSIK"(a black dog).

Example 3

Let us consider a group of people $P_1, ..., P_n$. We put $a_{ij} = 1$ if the person P_i can send some information to the person P_j, and $a_{ij} = 0$ otherwise (for convenience, we put $a_{ii} = 0$ for all $i = 1, ..., n$). Organizing this elements into a square matrix $\mathbf{A}$, we obtain a so called incidence matrix. Let $P_i \to P_j$ denotes the fact that P_i can send information to P_j. Thus, for example, the elements of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

tell us that $P_1 \to P_2, P_1 \to P_4, P_2 \to P_3, P_3 \to P_1, P_3 \to P_4, P_4 \to P_1, P_4 \to P_2$ (since $P_1 \to P_4$ and $P_4 \to P_1$, it is obvious that P_1 and P_4 can send information to each other). How may we interpret the matrix

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}?$$

Denoting the elements of $\mathbf{A}^2$ by $(a^2)_{ij}$, we obtain for example

$$(a^2)_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} + a_{34}a_{42} = 1 + 0 + 0 + 1 = 2$$

and this result shows that the person P_3 can send information to P_2 in two stages by two ways: $P_3 \to P_1 \wedge P_1 \to P_2$ (because $a_{31}a_{12} = 1$) and $P_3 \to P_4 \wedge P_4 \to P_2$ (because $a_{34}a_{42} = 1$). Similarly, since $(a^2)_{14} = 0$, there is no possibility to send information from P_1 to P_4 in two stages (but it is possible directly, because $a_{14} = 1$).

Hence, the element $(a^2)_{ij}$ gives the number of ways in which the person P_i can send information to P_j in two stages.

Similarly, $(a^3)_{ij}$ represents the number of ways in which the person P_i can send information to P_j in three stages:

$$\mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

and thus for example

$$(a^3)_{32} = (a^2)_{31}a_{12} + (a^2)_{32}a_{22} + (a^2)_{33}a_{32} + (a^2)_{34}a_{42} = 1 + 0 + 0 + 1 = 2.$$

Hence, there are two ways to send information from P_3 to P_2 in three stages: $P_3 \to P_4 \wedge P_4 \to P_1 \wedge P_1 \to P_2$(because $(a^2)_{31}a_{12} = (a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} + a_{34}a_{41})a_{12} = (0+0+0+1) \cdot 1 = 1$) and $P_3 \to P_1 \wedge P_1 \to P_4 \wedge P_4 \to P_2$ (because $(a^2)_{34}a_{42} = (a_{31}a_{14} + a_{32}a_{24} + a_{33}a_{34} + a_{34}a_{44})a_{42} = (1 + 0 + 0 + 0) \cdot 1 = 1$).

In general, the number of ways in which P_i can send information to P_j in at most k stages is given by the element in the i-th row and j-th column of the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + ... + \mathbf{A}^k$. Thus, in the above example we deduce from the matrix

$$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 2 & 3 & 2 & 3 \\ 2 & 2 & 1 & 1 \\ 3 & 4 & 2 & 3 \\ 3 & 3 & 2 & 2 \end{pmatrix}$$

that for example there are four ways in which P_3 can send information to P_2 in at most three stages.

Example 4

Three people (denoted by P_1, P_2, P_3) organized in a simple closed society produce three commodities z_1, z_2, z_3. Each person sells and buys from each other. All their products are consumed by them, no other commodities enter the system (the "closed model"). The proportions of the products consumed by each of P_1, P_2, P_3 are given in the following table:

	z_1	z_2	z_3
P_1	0.6	0.2	0.3
P_2	0.1	0.7	0.2
P_3	0.3	0.1	0.5

For example, the first column lists that 60% of the produced commodity z_1 are consumed by P_1, 10% by P_2 and 30% by P_3. Thus, it is obvious that the sum of elements in each column is equal to 1.

Let us denote x_1, x_2, x_3 the incomes of the persons P_1, P_2, P_3. Then the amount spent by P_1 on z_1, z_2, z_3 is $0.6x_1 + 0.2x_2 + 0.3x_3$. The assumption that the consumption of each person equals his income leads to the equation $0.6x_1 + 0.2x_2 + 0.3x_3 = x_1$, similarly for the other persons. We obtain the system of linear equations:

$$\begin{array}{ccccccc} 0.6x_1 & + & 0.2x_2 & + & 0.3x_3 & = & x_1 \\ 0.1x_1 & + & 0.7x_2 & + & 0.2x_3 & = & x_2 \\ 0.3x_1 & + & 0.1x_2 & + & 0.5x_3 & = & x_3. \end{array}$$

This system can be rewritten as the equation $\mathbf{Ax} = \mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 0.6 & 0.2 & 0.3 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.1 & 0.5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x} = (x_1, x_2, x_3)^T.$$

Moreover, we assume the income to be nonnegative, i.e. $x_i \geq 0$ for $i = 1, 2, 3$ (we denote it $\mathbf{x} \geq \mathbf{o}$). We can rewrite this equation into the equivalent form $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{o}$:

$$\left(\begin{array}{ccc|c} -0.4 & 0.2 & 0.3 & 0 \\ 0.1 & -0.3 & 0.2 & 0 \\ 0.3 & 0.1 & -0.5 & 0 \end{array}\right)$$

An arbitrary solution of the system has the form $\mathbf{x} = t(13, 11, 10)^T$ and it is $\mathbf{x} \geq \mathbf{o}$ for $t \geq 0$.

Thus, to ensure that this society survives, the persons P_1, P_2, P_3 have to have their incomes in the proportions 13:11:10.

Note: Let us consider a closed model, let $\mathbf{A} = (a_{ij})$ be an $n \times n$ coefficient matrix as above. Instead of the condition that the consumption is equal to the income we can consider that the consumption does not exceed the income, i.e. $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \leq x_i$ for all $i = 1, ..., n$. We will show that the equation $\mathbf{Ax} = \mathbf{x}$ has to hold in this case, too. For otherwise, there exists a p such that $a_{p1}x_1 + a_{p2}x_2 + ... + a_{pn}x_n < x_p$ and then $\sum_{i=1}^{n}(a_{i1}x_1 + ...a_{in}x_n) < \sum_{i=1}^{n} x_i$. Since the sum of elements in each column of the matrix $\mathbf{A}$ is equal to 1, the left side of the previous unequality can be rewritten: $\sum_{i=1}^{n}(a_{i1}x_1 + ... + a_{in}x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n}(\sum_{i=1}^{n} a_{ij})x_j = \sum_{j=1}^{n} 1 \cdot x_j = \sum_{j=1}^{n} x_j$. Thus $\sum_{j=1}^{n} x_j < \sum_{i=1}^{n} x_i$ and it is a contradiction.

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