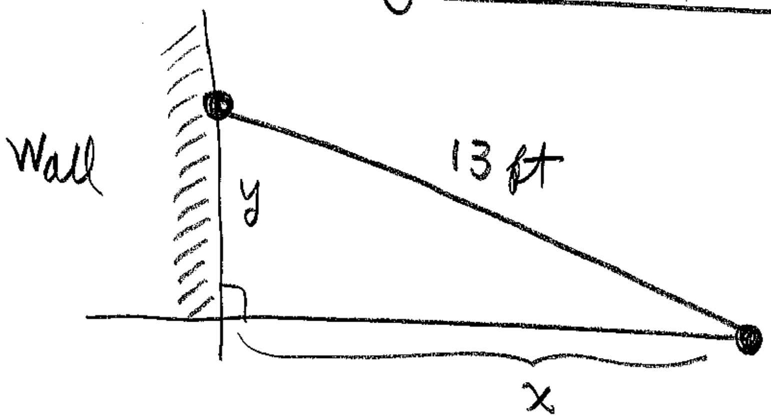
RR-01 Sliding Ladder Problem



given rate: $\frac{dx}{dt} = 3$ ft/sec

desired rate: $\frac{dy}{dt}$ when $x = 5$

Equation: $x^2 + y^2 = 13^2$ ⊛

$\therefore \quad \frac{d}{dt}\{x^2 + y^2\} = \frac{d}{dt}\{13^2\}$

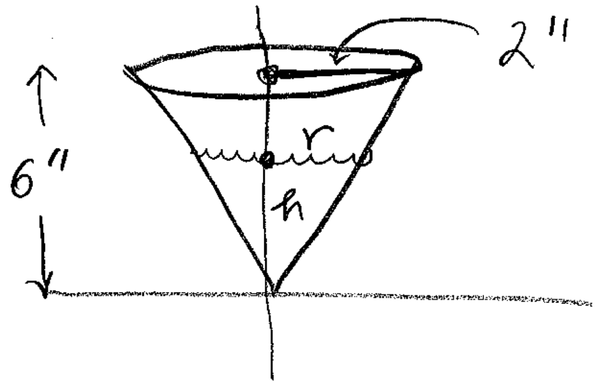
$\Rightarrow \quad 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt}$

When $x = 5$ ⊛ $\Rightarrow 5^2 + y^2 = 13^2 \Rightarrow y = 12$

$\therefore \quad \left.\frac{dy}{dt}\right|_{\substack{x=5\\y=12}} = \left(-\frac{5}{12}\right)(3) = -\frac{5}{4}$ ft/sec ✓

$\therefore$ Top of ladder slides down at rate of $\frac{5}{4}$ ft/sec ✓

$\boxed{\text{RR-02}}$ Conical Tank Problem

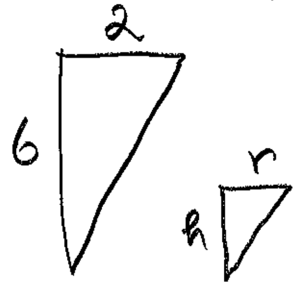


given rate: $\dfrac{dV}{dt} = \dfrac{2}{3}\ in^3/min$

desired rate: $\dfrac{dh}{dt}$ when $h = 4"$

Equation: $V = \frac{1}{3}\pi r^2 h$ ← this has too many variables so must eliminate some

Use Similar Triangles:



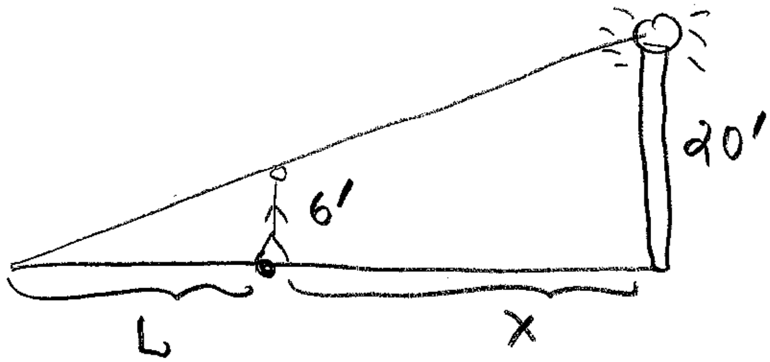
$\Rightarrow \dfrac{2}{6} = \dfrac{r}{h} \Rightarrow r = \dfrac{1}{3}h$ plug into

$\therefore$ Equation becomes $V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

i.e. $V = \dfrac{\pi}{27}h^3 \Rightarrow \dfrac{dV}{dt} = \dfrac{\pi}{27}\left(3h^2\dfrac{dh}{dt}\right)$

so $\left.\dfrac{dh}{dt}\right|_{h=4} = \left.\dfrac{27}{3\pi h^2}\left(\dfrac{dV}{dt}\right)\right|_{h=4} = \dfrac{27}{3\pi(4)^2}\left(\dfrac{2}{3}\right) = \underline{\underline{\dfrac{3}{8\pi}\ in/min}}$ ✓

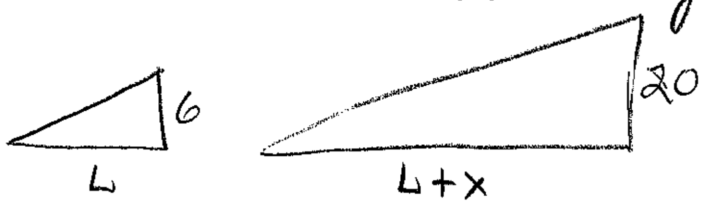
RR-03 Length of Shadow Problem



given rate: $\frac{dx}{dt} = -5$ ft/sec (minus because x is decreasing)

desired rate: $\frac{dL}{dt}$ when $x = 24'$

Equation: Need to figure out a relationship between x and L.
Use Similar Triangles:



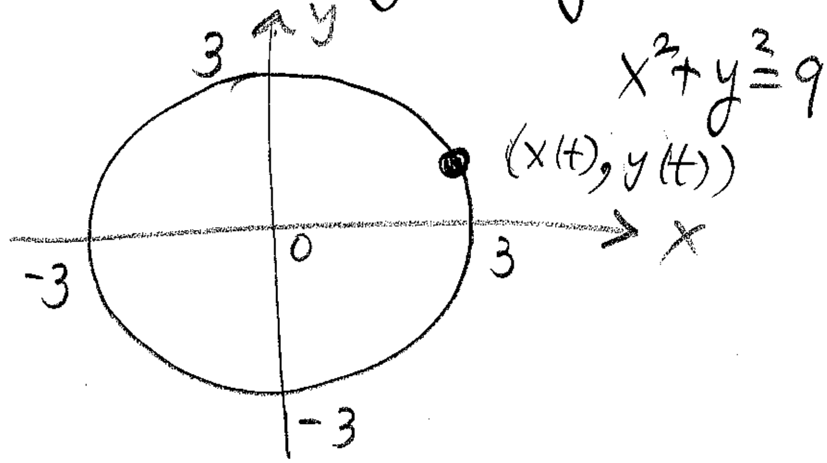
$\Rightarrow \frac{L}{6} = \frac{L+x}{20}$

Solve for L to get $L = \frac{3}{7}x$ ← this is the equation

$\therefore \quad \frac{dL}{dt} = \frac{3}{7}\frac{dx}{dt} \Rightarrow \left.\frac{dL}{dt}\right|_{x=24} = \left(\frac{3}{7}\right)(-5) = \frac{-15}{7}$ ft/sec ✓

Thus, the length of his shadow is decreasing at a rate of $15/7$ ft/sec ✓

RR-04 Moving Along a Curve Problem



given rate: $\frac{dx}{dt} = 20$ units/sec

desired rate: $\left.\frac{dy}{dt}\right|_{(\sqrt{3},\, \sqrt{6})}$

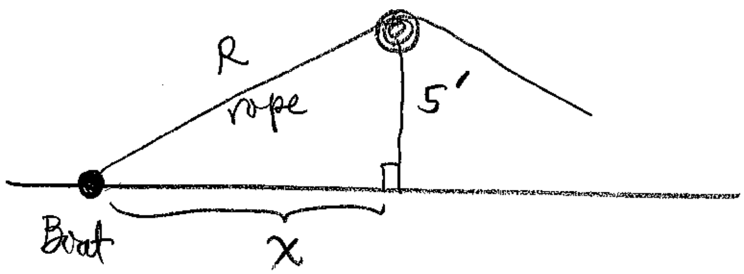
Equation: $x^2 + y^2 = 9$ (now differentiate w.r.t. t)

$\therefore\ 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\,\frac{dy}{dt}$

$\Rightarrow \left.\frac{dy}{dt}\right|_{(\sqrt{3},\, \sqrt{6})} = -\left(\frac{\sqrt{3}}{\sqrt{6}}\right)(20) = -10\sqrt{2}$ units/sec ✓

Hence the y-coordinate is decreasing a rate of $10\sqrt{2}$ units/sec (object is moving clockwise).

RR-05 Towing a Boat Problem



given rate: $\frac{dR}{dt} = -2$ ft/sec (rope pulled towards pulley)

desired rate: $\frac{dx}{dt}$ when $R = 13$ ft.

Equation: $x^2 + 5^2 = R^2$ ⊛ (differentiate w.r.t. t)

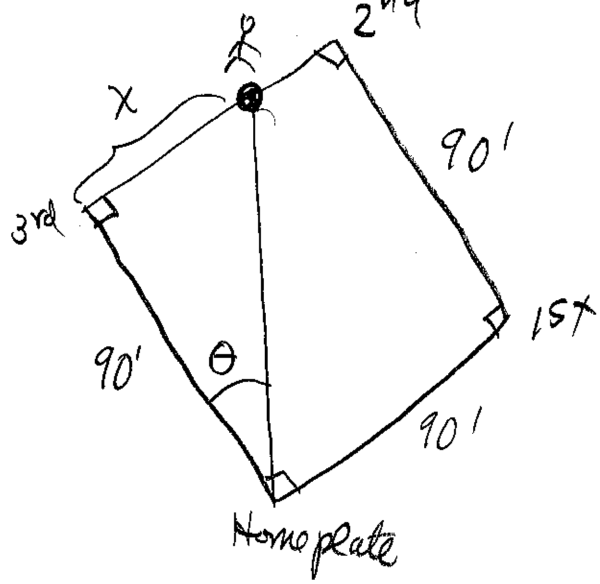
$\therefore \; 2x\frac{dx}{dt} + 0 = 2R\frac{dR}{dt} \Rightarrow \frac{dx}{dt} = \frac{R}{x}\frac{dR}{dt}$

When $R = 13$ ⊛ $\Rightarrow x^2 + 5^2 = 13^2 \Rightarrow x = 12$ ft.

$$\therefore \left.\frac{dx}{dt}\right|_{\substack{R=13 \\ x=12}} = \left(\frac{13}{12}\right)(-2) = -\frac{13}{6} \text{ ft/sec} \checkmark$$

i.e. the bow of the boat moves towards the shore at a rate of $\frac{13}{6}$ ft/sec ✓

RR-06 **Baseball Runner Problem**



given rate: $\frac{dx}{dt} = -24$ ft/sec (he runs towards 3rd base so x is decreasing, hence the − sign)

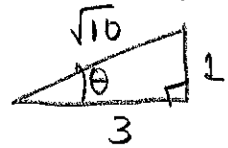
desired rate: $\frac{d\theta}{dt}$ when $x = 30'$

Equation: $\tan\theta = \frac{x}{90}$ ⊛

$\therefore \quad \frac{d(\tan\theta)}{dt} = \frac{d}{dt}\left(\frac{x}{90}\right) \Rightarrow (\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{90}\frac{dx}{dt}$

So $\frac{d\theta}{dt} = \left(\frac{1}{90\sec^2\theta}\right)\frac{dx}{dt}$

Now when $x = 30$ ⊛ $\Rightarrow \tan\theta = \frac{30}{90} = \frac{1}{3}$



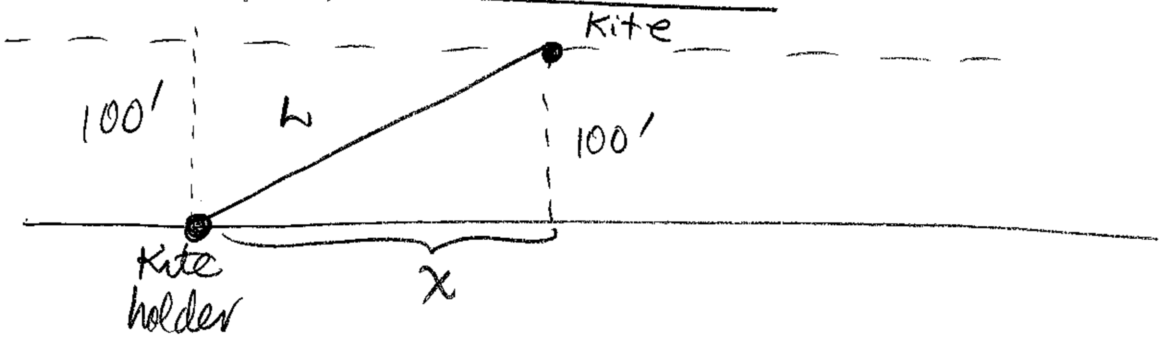
$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$

$\Rightarrow \sec\theta = \frac{\sqrt{10}}{3}$

$$\therefore \left.\frac{d\theta}{dt}\right|_{x=30} = \frac{1}{90\left(\frac{\sqrt{10}}{3}\right)^2}(-24) = -\frac{6}{25} \text{ radians/sec} \checkmark$$

i.e. the angle θ is <u>decreasing</u> at a rate of 6/25 radians/sec

RR-07 Kite Problem



given rate: $\frac{dx}{dt} = 10$ ft/sec

desired rate: $\frac{dL}{dt}$ when $L = 200$ ft

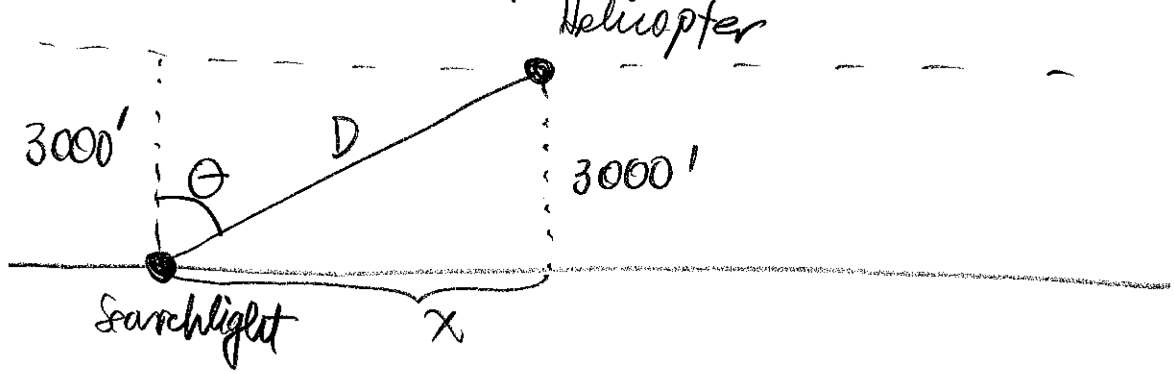
Equation: $x^2 + 100^2 = L^2$ ⊛ (now differentiate w.r.t. t)

$\Rightarrow \quad 2x \frac{dx}{dt} + 0 = 2L \frac{dL}{dt} \Rightarrow \frac{dL}{dt} = \frac{x}{L} \frac{dx}{dt}$

When $L = 200$ ⊛ $\Rightarrow x = 100\sqrt{3}$

$$\therefore \left.\frac{dL}{dt}\right|_{\substack{L=200 \\ x=100\sqrt{3}}} = \left(\frac{100\sqrt{3}}{200}\right)(10) = 5\sqrt{3} \text{ ft/sec } \checkmark$$

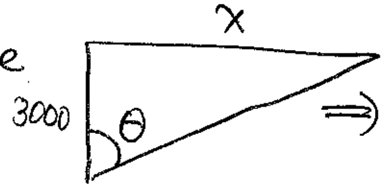
RR-08 Searchlight Problem



<u>given rate:</u> $\frac{dx}{dt} = 100$ ft/sec

<u>desired rate:</u> $\frac{d\theta}{dt}$ when $D = 5000$ ft

<u>Equation:</u> Since

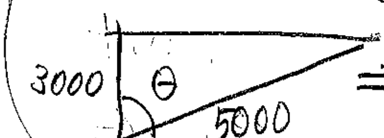


$\Rightarrow \quad \tan\theta = \frac{x}{3000}$

$$\Rightarrow \frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{x}{3000}\right) \Rightarrow (\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{3000}\frac{dx}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{3000\sec^2\theta}\frac{dx}{dt}.$$

Now when $D = 5000$, we get

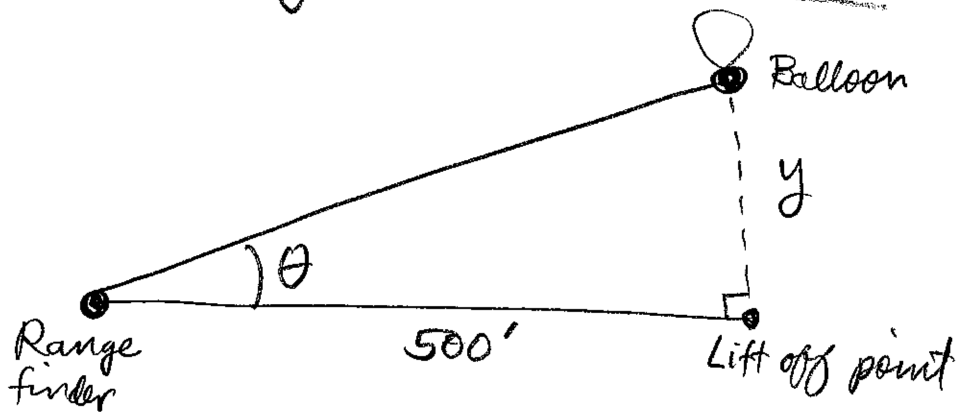


$\Rightarrow \cos\theta = \frac{3000}{5000} = \frac{3}{5}$

$\therefore \sec\theta = \frac{5}{3}$

$$\therefore \left.\frac{d\theta}{dt}\right|_{D=5000} = \frac{1}{3000\left(\frac{5}{3}\right)^2}(100) = \frac{3}{250} \text{ radians/sec} \checkmark$$

RR-09 Rising Balloon Problem



given rate: $\frac{d\theta}{dt} = 0.14$ radians/min (when $\theta = \pi/4$)

desired rate: $\frac{dy}{dt}$ when $\theta = \pi/4$

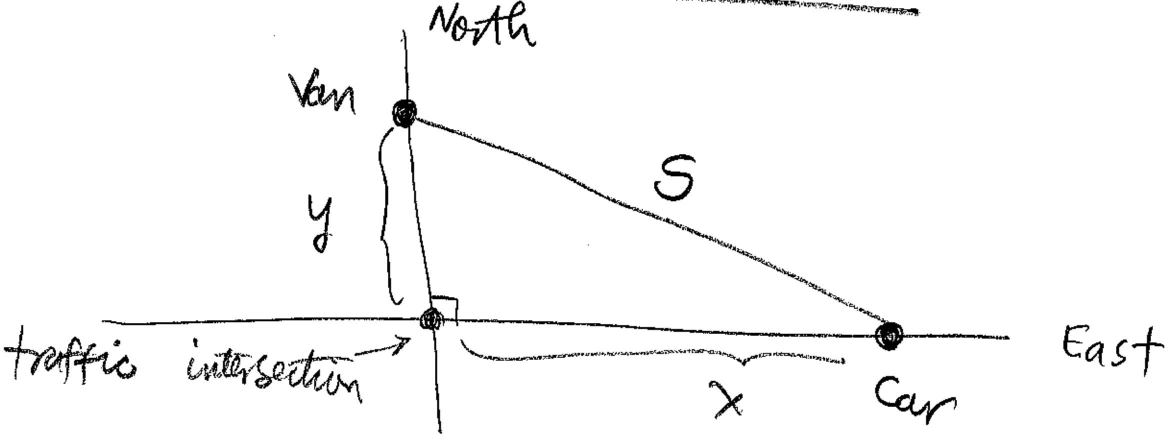
Equation: $\tan\theta = \frac{y}{500}$

$$\therefore \frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{500}\right) \Rightarrow (\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{500}\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$

so $$\left.\frac{dy}{dt}\right|_{\theta=\pi/4} = 500\left(\frac{2}{\sqrt{2}}\right)^2(0.14) = 140 \text{ ft/sec}$$ ✓

RR-10 Two Cars Problem



given rate: $\frac{ds}{dt} = 20$ mph when $y = 0.6$ mi
$x = 0.8$ mi

desired rate: $\frac{dx}{dt}$ when $\frac{dy}{dt} = -60$ mph (minus sign because Van moving towards intersection, so y is decreasing)

Equation: $x^2 + y^2 = s^2$ ⊛

$$\Rightarrow \frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(s^2) \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

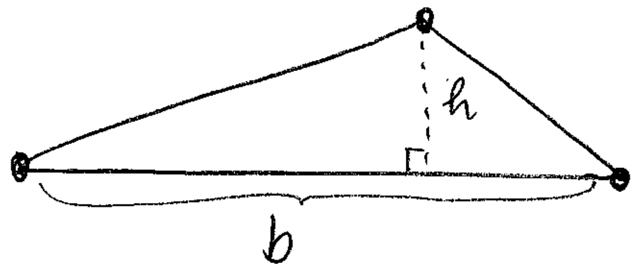
$$\Rightarrow \frac{dx}{dt} = \frac{s\frac{ds}{dt} - y\frac{dy}{dt}}{x}$$

Now when $x = 0.8$, $y = 0.6$
⊛ $\Rightarrow s = 1$

$$\therefore \frac{dx}{dt} = \frac{(1)(20) - (0.6)(-60)}{0.8} = 70 \text{ mph}$$

RR-11 Area of Triangle Problem

not necessarily a right angle



given rate: $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm²/min

desired rate: $\frac{db}{dt}$ when $h = 10$ cm and $A = 100$ cm²

Equation: $A = \frac{1}{2} bh$ ⊛

$\therefore \frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2} bh\right) \Rightarrow \frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$

$\Rightarrow \frac{db}{dt} = \frac{2\frac{dA}{dt} - b\frac{dh}{dt}}{h}$,

when $h=10$ and $A=100$
⊛ $\Rightarrow 100 = \frac{1}{2} b(10)$
$\Rightarrow b = 20$

$\therefore$ At the desired instant in time,

$$\frac{db}{dt} = \frac{2(2) - (20)(1)}{10} = -\frac{8}{5} \text{ cm/min}$$ ✓

i.e. the length of the base of the triangle is <u>decreasing</u> at rate of 8/5 cm/min ✓

RR-12 Air Pressure Problem

$$PV^{1.4} = C \leftarrow \text{constant}$$

given rate: $\frac{dP}{dt} = -10 \text{ kPa/min}$, when $V = 400 \text{cm}^3$ and $P = 80 \text{ kPa}$

desired rate: $\frac{dV}{dt}$

Equation: $PV^{1.4} = C$

$$\Rightarrow \frac{d}{dt}\{PV^{1.4}\} = \frac{d}{dt}\{C\}$$

$$\Rightarrow P\left(1.4V^{0.4}\frac{dV}{dt}\right) + V^{1.4}\left(\frac{dP}{dt}\right) = 0$$

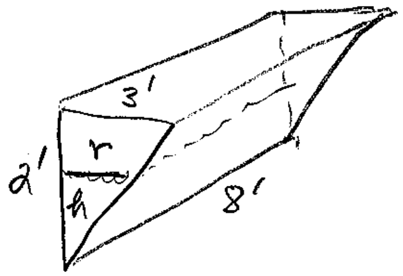
$$\Rightarrow \frac{dV}{dt} = -\frac{V}{1.4P}\left(\frac{dP}{dt}\right)$$

Hence at the desired instant of time

$$\frac{dV}{dt} = \frac{-(400)}{(1.4)(80)}(-10) = \frac{250}{7} \text{ cm}^3/\text{min}$$ ✓

(the volume of gas is increasing)

RR-13 Water Trough Problem



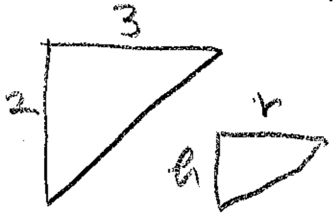
<u>given rate:</u> $\frac{dV}{dt} = 5 \text{ ft}^3/\text{min}$

<u>desired rate:</u> $\frac{dh}{dt}$ when $h = \frac{3}{2}$ ft

<u>Equation:</u> $V = (\text{area of base})(\text{height}) = \left(\frac{1}{2} r h\right)(8) = 4rh$

too many variables

Use Similar Triangles



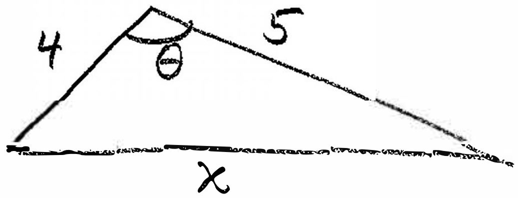
$\Rightarrow \frac{3}{2} = \frac{r}{h} \quad \therefore r = \frac{3}{2} h$

Hence $V = 4\left(\frac{3}{2} h\right) h = 6h^2$

$\therefore \frac{dV}{dt} = 6\left(2h \frac{dh}{dt}\right) \Rightarrow \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt}$

$$\therefore \left.\frac{dh}{dt}\right|_{h=\frac{3}{2}} = \frac{1}{12\left(\frac{3}{2}\right)}(5) = \frac{5}{18} \text{ ft/min}$$ ✓

RR-14 Law of Cosines Problem



given rate: $\frac{d\theta}{dt} = \frac{1}{10}$ radians/min

desired rate: $\frac{dx}{dt}$ when $\theta = \frac{\pi}{3}$ (60°)

Equation: Law of Cosines $\Rightarrow x^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta$

i.e. $x^2 = 41 - 40\cos\theta$ $(*)$

$\therefore \frac{d}{dt}\{x^2\} = \frac{d}{dt}\{41 - 40\cos\theta\} \Rightarrow 2x\frac{dx}{dt} = (40\sin\theta)\frac{d\theta}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{20\sin\theta}{x}\frac{d\theta}{dt}$ Now when $\theta = \frac{\pi}{3}$ $(*) \Rightarrow x = \sqrt{21}$

$$\therefore \left.\frac{dx}{dt}\right|_{\theta=\frac{\pi}{3}} = \left(\frac{20\sin\frac{\pi}{3}}{\sqrt{21}}\right)\left(\frac{1}{10}\right) = \frac{20\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{21}}\left(\frac{1}{10}\right) = \frac{\sqrt{3}}{\sqrt{21}}$$

$$= \sqrt{\frac{3}{21}} = \frac{1}{\sqrt{7}} \text{ cm/min}$$