The Chain Rule

Warm Up

Find functions $f(x)$ and $g(x)$ so that the following functions are equal to the composition $f(g(x))$.

1. $y = 7^{5x}$

$f(x) = 7^x$ $\Rightarrow y = f(g(x)) = 7^{5x}$

$g(x) = 5x$

2. $y = \sqrt[3]{\cot(x) + 1}$

$f(x) = \sqrt[3]{x}$ $\Rightarrow y = f(g(x)) = \sqrt[3]{\cot(x)+1}$

$g(x) = \cot(x)+1$

The Chain Rule

The *chain rule* is used to take the derivative of compositions of functions. If you need to take the derivative, and something more complicated than just x (or whatever your variable is) is being plugged into another function, then you probably need to use the chain rule.

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)}$$

Think: Take the derivative of the "outside" function, leave the "inside" function alone. Then, multiply by the derivative of the "inside" function.

Another way to write the chain rule, if we let $u = g(x)$, is the following.

$$\boxed{\frac{d}{dx}[f(g(x))] = \frac{df}{du}\frac{du}{dx}}$$

Example 1: Find the derivative of $y = 3(x^5 - 2)^7$.

$f(u) = 3u^7$ $g(x) = x^5 - 2$

$f'(u) = 21u^6$ $g'(x) = 5x^4$

$y' = f'(g(x))g'(x)$

$= 21(x^5-2)^6(5x^4)$

$= \boxed{105x^4(x^5-2)^6}$

Example 2: Find the derivative of $y = \sqrt[4]{2x^2 + 3x + 4}$

$f(u) = \sqrt[4]{u} = u^{1/4}$

$f'(u) = 1/4\, u^{-3/4}$

$g(x) = 2x^2 + 3x + 4$

$g'(x) = 4x + 3$

$$y' = f'(g(x))g'(x) = \frac{1}{4}(2x^2+3x+4)^{-3/4}(4x+3)$$

$$= \boxed{\frac{4x+3}{4(2x^2+3x+4)^{3/4}}}$$

Example 3: Find the derivative of the following function.

$$y = \left(\frac{4x}{3x+5}\right)^3$$

$f(u) = u^3 \Rightarrow f'(u) = 3u^2$

$g(x) = \frac{4x}{3x+5} \Rightarrow g'(x) = \frac{4(3x+5) - 4x(3)}{(3x+5)^2} = \frac{12x + 20 - 12x}{(3x+5)^2}$

$$= \frac{20}{(3x+5)^2}$$

$$y' = f'(g(x))g'(x) = 3\left(\frac{4x}{3x+5}\right)^2\left(\frac{20}{(3x+5)^2}\right)$$

$$= 3 \cdot \frac{16x^2}{(3x+5)^2} \cdot \frac{20}{(3x+5)^2}$$

$$= \boxed{\frac{960x^2}{(3x+5)^4}}$$

DIY

1. Find the derivative of $y = 2(\cos(x) + 4)^3$.

$f(u) = 2u^3 \Rightarrow f'(u) = 6u^2$

$g(x) = \cos x + 4 \Rightarrow g'(x) = -\sin x$

$y' = f'(g(x))g'(x) = 6(\cos x + 4)^2(-\sin x)$

$= \boxed{-6\sin x(\cos x + 4)^2}$

2. Find the derivative of the following function.

$f(u) = 5u^{-5/2} \Rightarrow f'(u) = \frac{-25}{2}u^{-7/2}$

$$\frac{5}{(x^4+2)^{5/2}} = 5(x^4+2)^{-5/2}$$

$g(x) = x^4 + 2 \Rightarrow g'(x) = 4x^3$

$y' = f'(g(x))g'(x) = \frac{-25}{2}(x^4+2)^{-7/2}(4x^3)$

$= \boxed{\dfrac{-50x^3}{(x^4+2)^{7/2}}}$

3. Find the equation of the tangent line to $y = e^{3x}$ at the point $x = 0$.

$f(u) = e^u$

$f'(u) = e^u$

$g(x) = 3x$

$g'(x) = 3$

Point: $(0, y(0)) = (0, e^{3(0)}) = (0, 1)$

Slope: $y' = f'(g(x))g'(x) = e^{3x}(3) = 3e^{3x}$

$y'(0) = 3e^{3(0)} = 3$

$y - 1 = 3(x - 0) \Rightarrow \boxed{y = 3x + 1}$