Implicit Differentiation

Up to this point, the functions that we have worked with have all been expressed in an explicit form. A function is in an *explicit form* if it is written in the format of $y = f(x)$. In words, a function is in an explicit form if y is isolated on one side of the equal sign by itself with everything else that does not involve y on the other side.

Example 1: The function $y = 3x + 4$ is in explicit form. Find $y' = \frac{dy}{dx}$.

$$y = 3x + 4 \Rightarrow \frac{dy}{dx} = y' = 3$$

When a function is not written in explicit form, we say it is in *implicit form* (independent and dependent variables, x and y, can be on the same side of the equation).

Example 1 revisited: The function $y - 3x = 4$ is in implicit form. Find $y' = \frac{dy}{dx}$. Note: since y is a function of x ($y = f(x)$), when we take the derivative of y with respect to x, its derivative is $y' = \frac{dy}{dx}$, **not 1**.

$$y - 3x = 4 \Rightarrow y' - 3 = 0 \Rightarrow y' = 3$$

Why is the derivative of y, with respect to x, $y' = \frac{dy}{dx}$ and not 1?

Since $y = f(x)$, the derivative of y depends on what $f(x)$ is. For example, if $y = x$, then $y' = \frac{dy}{dx} = 1$. However, if $y = x^2$, then $y' = \frac{dy}{dx} = 2x \neq 1$. If we are given a function in implicit form, we can't always put it in explicit form (solve for y). If we can't solve for y, then we don't know explicitly what $f(x)$ is and so we can't determine what $f'(x)$ is explicitly. So, if we can't explicitly solve for y (this is why we need implicit differentiation), we symbolically write that the derivative of y, with respect to x is $y' = \frac{dy}{dx}$.

Why is the derivative of x, with respect to x, 1?

We have seen, that $\frac{d}{dx}(x) = 1$. Notice that, as a fraction, $\frac{d}{dx}(x) = \frac{dx}{dx} = 1$.

How I Like to Think About It:

Whenever I have to take the derivative of something involving y, implicitly, I think to myself: what would the derivative be if y was the dependent variable, and then remember to multiply that by $y' = \frac{dy}{dx}$.

Example 2: Find $y' = \frac{dy}{dx}$ given that $3y^2 + 2xy = 4x + 3$.

Chain Rule $\frac{d}{dx} 3y^2$ outside: $3u^2$ Inside: y $\Rightarrow \frac{d}{dx} 3y^2 = 6yy'$

Product Rule $\frac{d}{dx}[2xy] = 2y + 2xy'$

$\Rightarrow 3y^2 + 2xy = 4x + 3 \Rightarrow$ taking derivatives $\Rightarrow 6yy' + 2y + 2xy' = 4$

$\Rightarrow y'(6y + 2x) = 4 - 2y \Rightarrow \boxed{y' = \frac{4-2y}{(6y+2x)}}$

Notice that we need the chain rule to take the derivative of $3y^2$ and the product rule to take the derivative of $2xy$. Again, this is because $y = f(x)$, and $f(x)$ could be something more complicated than just x.

Example 3: Find $y' = \frac{dy}{dx}$ given that

chain + quotient rules

$$3\sin\left(\frac{y}{x}\right) = 7x$$

$$3\cos\left(\frac{y}{x}\right)\left(\frac{y'x - y(1)}{x^2}\right) = 7$$

$$3\cos\left(\frac{y}{x}\right)\frac{y'}{x} - \frac{3y}{x^2}\cos\left(\frac{y}{x}\right) = 7$$

$$y' \, 3(1/x)\cos(y/x) = 7 + \frac{3y}{x^2}\cos\left(\frac{y}{x}\right)$$

$$\boxed{y' = \frac{7 + \frac{3y}{x^2}\cos\left(\frac{y}{x}\right)}{\frac{3}{x}\cos(y/x)}}$$

DIY

1. Find the equation of the tangent line to $2x^4 = 3y^2 - y$ at the point $(1, 1)$.

Point

Slope: $8x^3 = 6yy' - y'$

$8x^3 = y'(6y-1) \Rightarrow y' = \frac{8x^3}{6y-1}$

Plug in point $(1,1) \Rightarrow \frac{8(1)^3}{6(1)-1} = 8/5$ Slope

$y - 1 = 8/5(x-1)$

$y = 8/5 x - 8/5 + 1$

$\boxed{y = 8/5 x - 3/5}$

2. Find $y' = \frac{dy}{dx}$ given that $e^{5xy} = 2y + 1$.

chain + product rules

$e^{5xy}(5y + 5xy') = 2y'$

$5ye^{5xy} + 5xy'e^{5xy} = 2y'$

$5ye^{5xy} = 2y' - 5xy'e^{5xy}$

$5ye^{5xy} = y'(2 - 5xe^{5xy})$

$$\boxed{\frac{5ye^{5xy}}{2-5xe^{5xy}} = y'}$$