Related Rates I

Strategy

1. Read the problem carefully; underline given numerical information.
2. Draw a diagram.
3. Assign variables to functions of time (what changes with respect to time?).
4. In terms of your variables, write out what you know and what you want to find.
5. Relate what we know and what we want to find using an equation with the variables assigned in step 3.
6. Use implicit differentiation to differentiate both sides of the equation with respect to time, t.
7. Substitute the given information and solve for the rate we want to find. **Do not substitute too early!**

Recall that rates of change correspond to derivatives of functions.

Example 1: The radius of a spherical balloon is increasing at a rate of 5 mm/s. How fast is the volume increasing when the radius is 50 mm? Note that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

What is changing wrt time? Volume, radius
V r

$V \qquad V' = \frac{dV}{dt} = ?$

$r = 50 \qquad r' = \frac{dr}{dt} = 5$

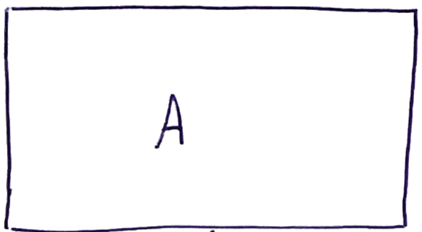
$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi (50)^2 (5)$$

$$\Rightarrow \boxed{\frac{dV}{dt} = 50{,}000\pi \ mm^3/s}$$

Example 2: The length of a rectangle is decreasing at a rate of 3 in/s and its width is decreasing at a rate of 2 in/s. When the length is 10 in and the width is 8 in, how fast is the area of the rectangle decreasing?

Decreasing $\Rightarrow$ negative rate of change



$w = 8$

$w' = \dfrac{dw}{dt} = -2$

$A' = \dfrac{dA}{dt} = ?$

$\ell = 10$

$\ell' = \dfrac{d\ell}{dt} = -3$

$$A = \ell w \Rightarrow \frac{dA}{dt} = \frac{d\ell}{dt} w + \ell \frac{dw}{dt}$$

$$\Rightarrow \frac{dA}{dt} = (-3)(8) + (10)(-2)$$

$$\Rightarrow \frac{dA}{dt} = -24 - 20 = -44$$

Since the question asks, "How fast is the area of the rectangle decreasing?" this means the question already knows that our rate is negative, so we give the positive version of our answer: 44 in^2/s.

The area is decreasing at a rate of 44 in^2/s.

Example 3: Gravel is dumped out of a dump truck onto the ground at 4 ft^3/s, forming a conical pile whose base diameter is always equal to its height. How fast is the height of the pile increasing when the pile is 5 ft high? Note that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

V $\qquad h = 5$

$\frac{dV}{dt} = 4 \qquad \frac{dh}{dt} = ?$

$d = h \Rightarrow 2r = h \Rightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

$\Rightarrow V = \frac{1}{3}\pi \frac{h^2}{4} h \Rightarrow V = \frac{1}{12}\pi h^3$

$\Rightarrow \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$\Rightarrow 4 = \frac{1}{4}\pi (25) \frac{dh}{dt}$

$\Rightarrow 16 = 25\pi \frac{dh}{dt}$

$\Rightarrow \boxed{\frac{16}{25\pi} \text{ ft/s} = \frac{dh}{dt}}$

Example 4: Assume x and y are both differentiable functions of t and $2x^4y = 10$. Find $\frac{dy}{dt}$ if $\frac{dx}{dt} = 2$ and $x = 1$.

If $x=1 \Rightarrow 2(1)^4 y = 10 \Rightarrow 2y = 10 \Rightarrow y = 5.$

$x = 1 \qquad y = 5$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = ?$

$$2x^4 y = 10$$

$$\Rightarrow 8x^3 \frac{dx}{dt} y + 2x^4 \frac{dy}{dt} = 0$$

$$\Rightarrow 8(1)(2)(5) + 2(1) \frac{dy}{dt} = 0$$

$$\Rightarrow 2\frac{dy}{dt} = -80$$

$$\Rightarrow \boxed{\frac{dy}{dt} = -40}$$