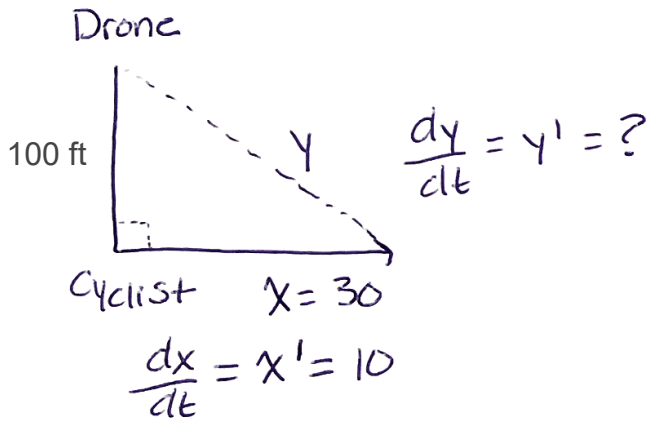
Related Rates II

Strategy

1. Read the problem carefully; underline given numerical information.
2. Draw a diagram.
3. Assign variables to functions of time (what changes with respect to time?).
4. In terms of your variables, write out what you know and what you want to find.
5. Relate what we know and what we want to find using an equation with the variables assigned in step 3.
6. Use implicit differentiation to differentiate both sides of the equation with respect to time, t.
7. Substitute the given information and solve for the rate we want to find. **Do not substitute too early!**

Example 1: A drone is hovering at a constant altitude of 100 ft while filming cyclists in a race. A cyclist passes directly under the drone traveling at 10 ft/s. How fast is the distance between the cyclist and the drone increasing when the cyclist is 30 ft from the point where she passed under the drone?



$\frac{dy}{dt} = y' = ?$

$\frac{dx}{dt} = x' = 10$

$\sqrt{x^2 + (100)^2} = y$

$(x^2 + (100)^2)^{1/2} = y$

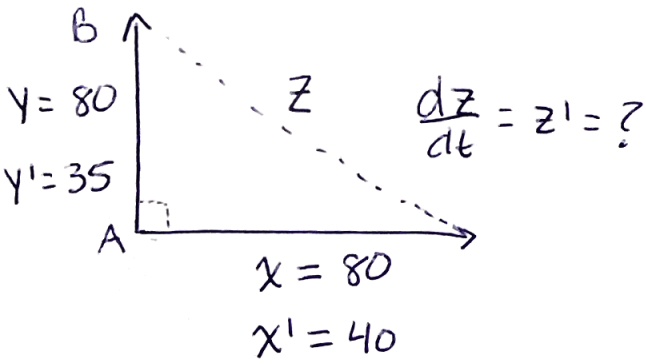
$\frac{1}{2}(x^2 + (100)^2)^{-1/2}(2x \cdot x') = y'$

$\Rightarrow \frac{x x'}{\sqrt{x^2 + (100)^2}} = y'$

$\Rightarrow \frac{30(10)}{\sqrt{(30)^2 + (100)^2}} = y'$

$\Rightarrow \boxed{\frac{30}{\sqrt{109}} \text{ ft/s} = y'}$

Example 2: At noon, ship A is 10 km south of ship B. Ship A is sailing east at 40 km/h and ship B is sailing north at 35 km/h. How fast is the distance between the ships changing at 2:00 pm?



At noon, $y = 10$ km

$\Rightarrow$ at 2:00

$y = 10 + 35(2) = 80$ km

At noon, $x = 0$ km

$\Rightarrow$ at 2:00 $x = 0 + 40(2) = 80$ km

$$z = \sqrt{x^2 + y^2} \quad \Rightarrow \quad z = (x^2 + y^2)^{1/2}$$

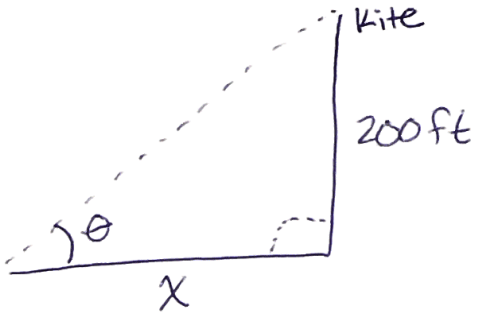
$$\Rightarrow z' = \frac{1}{2}(x^2 + y^2)^{-1/2}(2xx' + 2yy')$$

$$\Rightarrow z' = \frac{xx' + yy'}{\sqrt{x^2 + y^2}}$$

$$\Rightarrow z' = \frac{(80)(40) + (80)(35)}{\sqrt{(80)^2 + (80)^2}}$$

$$\Rightarrow \boxed{z' = \frac{75}{\sqrt{2}} \text{ km/h}}$$

Example 3: A kite 200 ft above the ground moves horizontally at a speed of 5 ft/s. At what rate is the angle (in radians) of elevation changing when the angle of elevation is $\frac{\pi}{6}$ radians?



x $\quad \theta = \pi/6$

$x' = 5$ $\quad \theta' = ?$

$$\frac{x}{200} = \cot(\theta) \Rightarrow \frac{1}{200}x = \cot(\theta)$$

$$\Rightarrow \frac{x'}{200} = -\csc^2(\theta)\cdot\theta'$$

$$\Rightarrow \frac{5}{200} = -\csc^2\left(\frac{\pi}{6}\right)\theta'$$

$\sin(\pi/6) = \frac{1}{2}$

$\Rightarrow \csc(\pi/6) = 2$

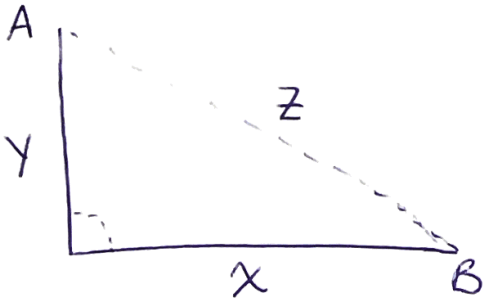
$\Rightarrow \csc^2(\pi/6) = 4$

$$\Rightarrow \frac{5}{200} = -4\theta' \Rightarrow \boxed{\frac{-1}{160}\ \text{rad/s} = \theta'}$$

Since the question asks, "How fast is the angle of elevation changing?" this means that the question wants us to tell them whether the angle is increasing or decreasing by keeping the appropriate sign in our answer.

DIY

1. Two runners leave from the same point at the same time. Runner A is traveling north and runner B is traveling east. After one hour, runner A is 10 miles away from the starting point and is running at a speed of 12 mph. Runner B is 9 miles east of the starting point and is running at a speed of 11 mph. At what rate is the distance between the two runners changing at this moment?



$x = 9 \quad y = 10 \quad z =$

$x' = 11 \quad y' = 12 \quad z' = ?$

$$z = \sqrt{x^2 + y^2} \Rightarrow z = (x^2 + y^2)^{1/2}$$

$$\Rightarrow z' = \frac{1}{2}(x^2 + y^2)^{-1/2}(2xx' + 2yy')$$

$$\Rightarrow z' = \frac{xx' + yy'}{\sqrt{x^2 + y^2}}$$

$$z' = \frac{(11)(9) + (10)(12)}{\sqrt{81 + 100}}$$

$$\Rightarrow \boxed{z' = \frac{219}{\sqrt{181}} \text{ mph}}$$