# Concavity

A function is *concave up* on an interval $I$, if it looks like either of the figures below on that interval (think of a right-side-up bowl).

Let's look at the slopes of the tangent lines to these curves. Notice that the slopes get bigger and bigger as we move form left to right on the graphs above. The slope of the tangent line getting bigger implies that the derivative is getting bigger (increasing), so $f'(x)$ is increasing when $f(x)$ is concave up.

We know, from last time, that the sign of the derivative tells us whether a function is increasing or decreasing. If we know that $f''(x)$, which is the derivative of $f'(x)$, is positive, then we know that $f'(x)$ is increasing. We then know that $f(x)$ is concave up. Symbolically: $f''(x) > 0 \Leftrightarrow f'(x)$ increasing $\Leftrightarrow f(x)$ concave up.

Big Takeaway: $f''(x) > 0 \Leftrightarrow f(x)$ is concave up.

A function is *concave down* on an interval $I$, if it looks like either of the figures below on that interval (think of an upside-down bowl).

Let's again look at the slopes of the tangent lines to these curves. Notice that the slopes get smaller and smaller as we move form left to right on the graphs above. The slope of the tangent line getting smaller implies that the derivative is getting smaller (decreasing), so $f'(x)$ is decreasing when $f(x)$ is concave down.

If we know that $f''(x)$, which is the derivative of $f'(x)$, is negative, then we know that $f'(x)$ is decreasing. We then know that $f(x)$ is concave down. Symbolically: $f''(x) < 0 \Leftrightarrow f'(x)$ decreasing $\Leftrightarrow f(x)$ concave down.

Big Takeaway: $f''(x) < 0 \Leftrightarrow f(x)$ is concave down.

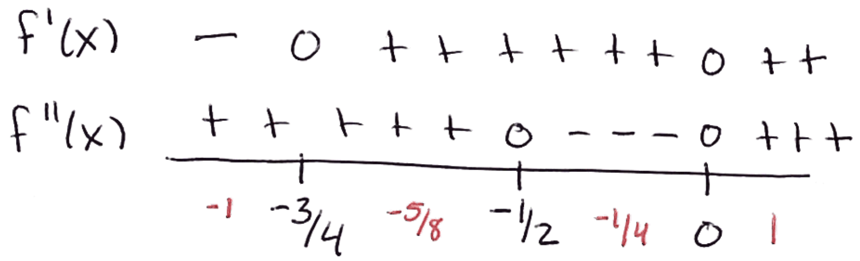
# Inflection Points

*Inflection points* are points where $f''(x) = 0$ or where $f''(x) = \text{DNE}$, **AND** where $f(x)$ changes concavity.

Example 1: Find the intervals on which $f(x) = x^4 + x^3 + 2$ is concave up, concave down, and find its inflection points. On what interval is $f(x)$ both concave up and decreasing?

$f'(x) = 4x^3 + 3x^2 \rightarrow x^2(4x+3) = 0 \rightarrow x = 0, \ x = -3/4$

$f''(x) = 12x^2 + 6x \rightarrow 6x(2x+1) = 0 \rightarrow x = 0, \ x = -1/2$



CU: $(-\infty, -1/2) \cup (0, \infty)$

CD: $(-1/2, 0)$

I.P.: $(-1/2, 31/16)$

$(0, 2)$

CU and Dec: $(-\infty, -3/4)$

## The Second Derivative Test

Let $f(x)$ be a function so that $f'(c) = 0$.

- If $f''(c) > 0$, then $f$ has a relative minimum at $x = c$.
- If $f''(c) < 0$, then $f$ has a relative maximum at $x = c$.
- If $f''(c) = 0$, then the second derivative test is inconclusive and we have to use the first derivative test to determine if $f$ has a max, min, or neither at $x = c$.

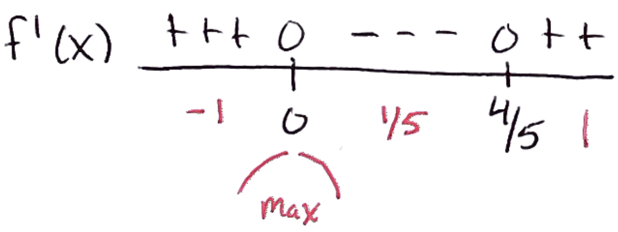
Example 2: Use the second derivative test (and the first derivative test, if necessary) to classify the relative extrema of the function $f(x) = x^5 - x^4 + 7$.

$f'(x) = 5x^4 - 4x^3 \rightarrow x^3(5x-4) = 0, \quad x = 0, \quad x = 4/5$

$f''(x) = 20x^3 - 12x^2$

$f''(4/5) = 64/25 > 0 \rightarrow$ min.

$f''(0) = 0 \rightarrow$ inconclusive, need 1st derivative test.



max: $(0, 7)$

min: $\left(4/5, \frac{21619}{3125}\right)$

## DIY

1. Find the intervals on which $y = 2\ln(x^2+4)$ is concave up, concave down, and find its inflection points.

$y' = \frac{4x}{x^2+4}$  $y'' = \frac{4(x^2+4) - 4x(2x)}{(x^2+4)^2} = \frac{4x^2+16-8x^2}{(x^2+4)^2}$

$\Rightarrow y'' = \frac{16-4x^2}{(x^2+4)^2}$

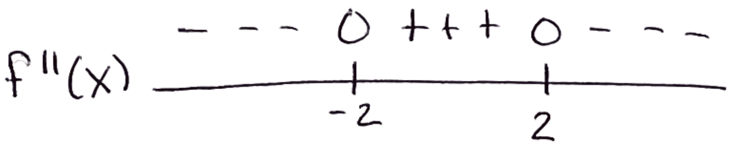
Denominator is never zero!

$\frac{16-4x^2}{(x^2+4)^2} = 0 \Rightarrow 16-4x^2 = 0$

$\Rightarrow 16 = 4x^2$

$\Rightarrow 4 = x^2$

$\Rightarrow \pm 2 = x$



$$\boxed{\begin{array}{l} \text{CU: } (-2, 2) \\ \text{CD: } (-\infty, -2) \cup (2, \infty) \\ \text{IP: } (-2, 2\ln(8)) \text{ and } (2, 2\ln(8)) \end{array}}$$

2. Use the second derivative test (and the first derivative test, if necessary) to classify the relative extrema of the function $f(x) = 2 + 3x - 2x^3$.

$f'(x) = 3-6x^2 \rightarrow 3-6x^2 = 0 \Rightarrow 3 = 6x^2 \Rightarrow x^2 = 1/2$

$\Rightarrow x = \pm 1/\sqrt{2}$

$f''(x) = -12x$

$f''(-1/\sqrt{2}) = 12/\sqrt{2} > 0 \Rightarrow$ minimum at $x = -1/\sqrt{2}$

$f''(1/\sqrt{2}) = -12/\sqrt{2} < 0 \Rightarrow$ maximum at $x = 1/\sqrt{2}$.