Limits at Infinity

So far we have found limits of the form $\lim_{x\to c} f(x)$ where c is a finite number. Now we will look at limits of the form $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. So, we want to know what the function does as x gets bigger and bigger (goes off to infinity) and what happens to the function as x gets smaller and smaller (goes off to negative infinity).

Example 1: Find the following limits.

$$\lim_{x\to\infty} \frac{12}{x} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{12}{x}$$

Example 2: Find the following limits.

$$\lim_{x\to\infty} 3x^3 \qquad \text{and} \qquad \lim_{x\to-\infty} 3x^3$$

Example 3: Find the following limit.

$$\lim_{x\to\infty} \frac{3}{x} - \frac{x}{2}$$

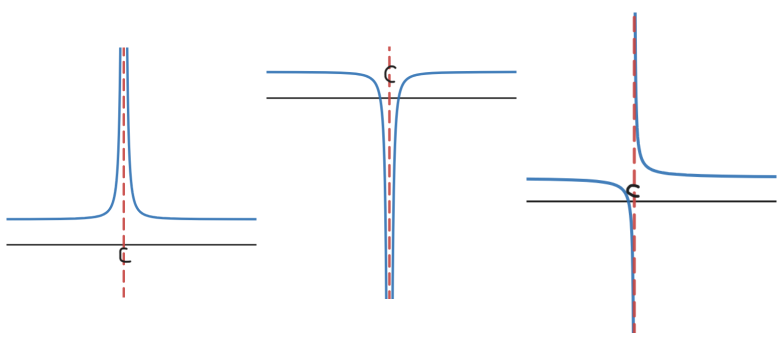
Fact: Finding the limit of a **rational function** at $\pm\infty$ is the same as taking the limit of the ratio of its leading terms (the terms with the highest power of x in the numerator and denominator).

Example 4: Find the following limit.

$$\lim_{x\to\infty} \frac{3x^2+2}{x^2+2x}$$

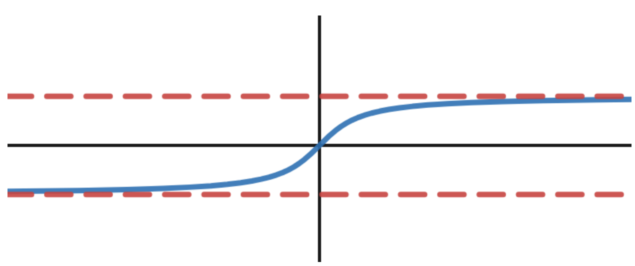
Vertical Asymptotes

Recall from lesson 4 that a function $f(x)$ has a *vertical asymptote* at $x = c$ if $\lim_{x\to c} f(x)$ is either a Case II limit or a Case III limit that becomes a Case II limit after we use algebra to simplify $f(x)$. In other words, $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$. Remember that, for rational functions, we only need to check the values for x that make the denominator zero when looking for vertical asymptotes.



Horizontal Asymptotes

The horizontal line $y = L$, where L is finite, is a *horizontal asymptote* for $f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

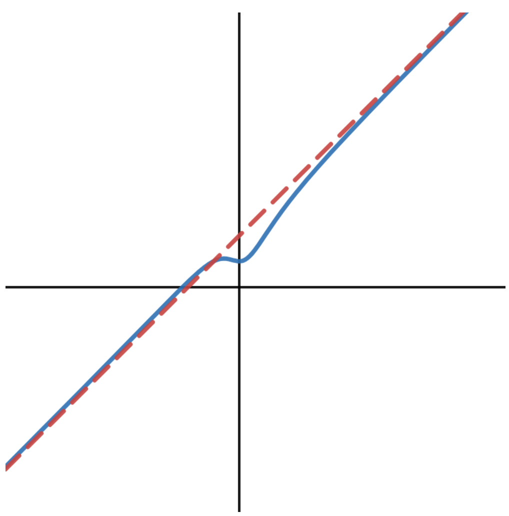


Example 5: Find any horizontal asymptotes for the following functions.

$$f(x) = x^2 + 1 \qquad \text{and} \qquad g(x) = \frac{2x^3 + 1}{x^4}$$

Slant Asymptotes

A *slant asymptote* is an asymptote for a function $f(x)$ which is not vertical nor horizontal; it is a lime of the form $y = mx + b$, where m and b are constants.



We use polynomial long division to find slant asymptotes. The quotient (the answer ignoring any remainder) is the slant asymptote.
Fact: A rational function will have a slant asymptote **only** when the degree of the numerator is one higher than the degree of the denominator.

Example 6: Find the slant asymptote for the following function.

$$f(x) = \frac{x^3 + 2x^2 + 1}{x^2 + 1}$$

DIY

1. Find any vertical, horizontal, and slant asymptotes for the following function.

$$h(x) = \frac{-2x^2}{x^2 - 9}$$