Limits at Infinity

So far we have found limits of the form $\lim_{x\to c} f(x)$ where c is a finite number. Now we will look at limits of the form $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. So, we want to know what the function does as x gets bigger and bigger (goes off to infinity) and what happens to the function as x gets smaller and smaller (goes off to negative infinity).

Example 1: Find the following limits.

$$\lim_{x\to\infty} \frac{12}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12}{x} = 0$$

x	5	500	5000
f(x)	12/5	12/500	12/5000

x	-5	-500	-5000
f(x)	12/-5	12/-500	12/-5000

Example 2: Find the following limits.

$$\lim_{x\to\infty} 3x^3 = \infty \quad \text{and} \quad \lim_{x\to-\infty} 3x^3 = -\infty$$

x	10	100
f(x)	3,000	3,000,000

x	-10	-100
f(x)	-3,000	-3,000,000

Example 3: Find the following limit.

$$\lim_{x\to\infty} \frac{3}{x} - \frac{x}{2} = -\infty$$

$$\lim_{x\to\infty}\left(\frac{3}{x}\right) - \lim_{x\to\infty}\left(\frac{x}{2}\right) = 0 - \infty = -\infty$$

Fact: Finding the limit of a **rational function** at $\pm\infty$ is the same as taking the limit of the ratio of its leading terms (the terms with the highest power of x in the numerator and denominator).

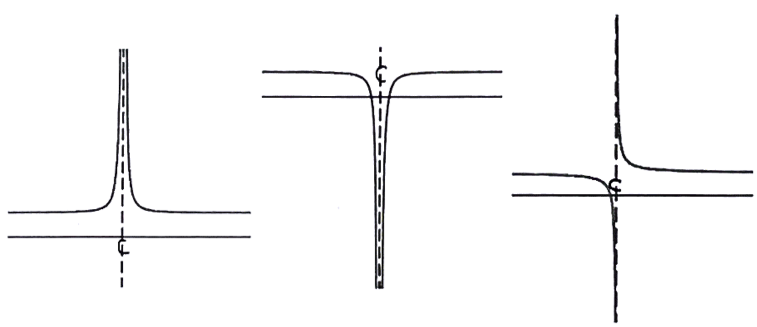
Example 4: Find the following limit.

$$\lim_{x\to\infty} \frac{3x^2+2}{x^2+2x}$$

$$= \lim_{x\to\infty} \frac{3x^2}{x^2} = \lim_{x\to\infty} 3 = 3$$

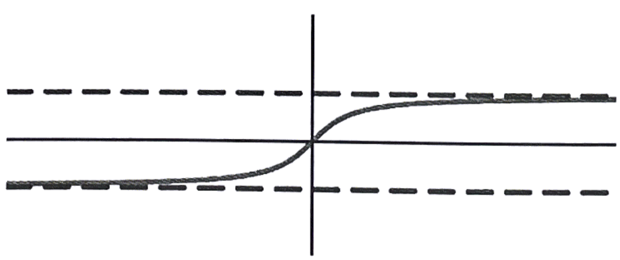
Vertical Asymptotes

Recall from lesson 4 that a function $f(x)$ has a *vertical asymptote* at $x = c$ if $\lim_{x\to c} f(x)$ is either a Case II limit or a Case III limit that becomes a Case II limit after we use algebra to simplify $f(x)$. In other words, $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$. Remember that, for rational functions, we only need to check the values for x that make the denominator zero when looking for vertical asymptotes.



Horizontal Asymptotes

The horizontal line $y = L$, where L is finite, is a *horizontal asymptote* for $f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.



Example 5: Find any horizontal asymptotes for the following functions.

$$f(x) = x^2 + 1 \qquad \text{and} \qquad g(x) = \frac{2x^3+1}{x^4}$$

$\lim_{x \to \infty} (x^2+1) \to (\infty)^2 + 1$
$= \infty$

$\lim_{x \to -\infty} (x^2+1) \to (-\infty)^2 + 1$
$= \infty$

$\boxed{\text{no H.A.}}$

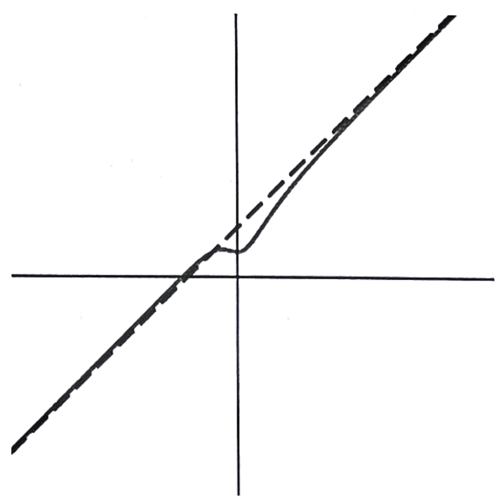
$\lim_{x \to \infty} \frac{2x^3+1}{x^4} = \lim_{x \to \infty} \frac{2x^3}{x^4}$
$= \lim_{x \to \infty} \frac{2}{x} = 0$ H.A. $y=0$

$\lim_{x \to -\infty} \frac{2x^3+1}{x^4} = \lim_{x \to -\infty} \frac{2x^3}{x^4}$
$= \lim_{x \to -\infty} \frac{2}{x} = 0$ H.A. $y=0$

So, the only H.A. is $y=0$.

Slant Asymptotes

A *slant asymptote* is an asymptote for a function $f(x)$ which is not vertical nor horizontal; it is a lime of the form $y = mx + b$, where m and b are constants.



We use polynomial long division to find slant asymptotes. The quotient (the answer ignoring any remainder) is the slant asymptote.

Fact: A rational function will have a slant asymptote **only** when the degree of the numerator is one higher than the degree of the denominator.

Example 6: Find the slant asymptote for the following function.

$$f(x) = \frac{x^3 + 2x^2 + 1}{x^2 + 1}$$

$$\begin{array}{r|l} & x + 2 \\ \hline x^2+1 & x^3 + 2x^2 + 1 \\ & -(x^3 + x) \\ \hline & 2x^2 - x + 1 \\ & -(2x^2 + 2) \\ \hline & -x - 1 \end{array}$$

$$\boxed{\text{S.A. } y = x+2}$$

DIY

1. Find any vertical, horizontal, and slant asymptotes for the following function.

$$h(x) = \frac{-2x^2}{x^2 - 9} = \frac{-2x^2}{(x-3)(x+3)}$$

Denominator is zero at $x = \pm 3$.

V.A.

$$\lim_{x \to -3} \frac{-2x^2}{x^2-9} \Rightarrow \frac{-18}{0} \text{ (case II limit) } \boxed{\text{V.A. at } x=-3}$$

$$\lim_{x \to 3} \frac{-2x^2}{x^2-9} \Rightarrow \frac{-18}{0} \text{ (case II limit) } \boxed{\text{V.A. at } x=3}$$

H.A.

$$\lim_{x \to \infty} \frac{-2x^2}{x^2-9} = \lim_{x \to \infty} \frac{-2x^2}{x^2} = \lim_{x \to \infty} -2 = -2$$

$$\lim_{x \to -\infty} \frac{-2x^2}{x^2-9} = \lim_{x \to \infty} \frac{-2x^2}{x^2} = \lim_{x \to -\infty} -2 = -2$$

$$\boxed{\text{H.A. } y = -2}$$

S.A.

Since the degrees of the numerator and denominator are the same, there is $\boxed{\text{no S.A.}}$