Curve Sketching

We have seen how limits and derivatives can give us information about what the graph of a function looks like: where it has asymptotes, extrema, where it is increasing/decrasing, concave up/concave down, etc.. Now we are going to put all this information together to sketch the graph of a function.

Strategy

1. Find any x- and y-intercepts of the function.
2. Find the intervals on which the function is increasing and the intervals on which it is decreasing.
3. Find any relative extrema for the function.
4. Find the intervals on which the function is concave up and the intervals on which it is concave down.
5. Find any inflection points.
6. Find any asymptotes for the function.
7. Use the information in steps 1-6 to sketch a graph of the function.

Example 1: Use the strategy above to sketch a graph of $f(x) = \frac{2x^2}{x+1}$.

$f'(x) = \frac{2x^2+4x}{(x+1)^2}$ $\quad$ $f''(x) = \frac{4}{(x+1)^3}$ $\quad$ $x=-1$ not in domain of f.

X-int: $\frac{2x^2}{x+1} = 0 \Rightarrow 2x^2 = 0 \Rightarrow x=0$

Y-int: $f(0) = \frac{0}{1} = 0$

$\boxed{(0,0) \text{ on graph}}$

Inc/Dec: f' undefined at $x=-1$

$\frac{2x^2+4x}{(x+1)^2} \Rightarrow 2x(x+2) = 0 \Rightarrow x=0,\ x=-2$

$f'(x)$: + 0 − u − 0 +

at -2, -1, 0

max (at -2), min (at 0)

$\boxed{\begin{array}{l}\text{Inc: } (-\infty,-2),\ (0,\infty) \\ \text{Dec: } (-2,-1),\ (-1,0)\end{array}}$

CU/CD: f'' undefined at $x=-1$, never zero

$f''(x)$ $\overset{-}{\text{———}}\ \underset{-1}{\overset{u}{|}}\ \overset{+}{\text{———}}$

CU: $(-1, \infty)$

CD: $(-\infty, -1)$

No I.P. since f is undefined at $x=-1$.

Asymptotes:

V.A. the denominator of f is zero when $x=-1$. Since we can't cancel the factor of $x+1$, this means there is a $\boxed{\text{V.A. at } x=-1.}$

H.A. $\lim\limits_{x\to\infty} \frac{2x^2}{x+1} = \lim\limits_{x\to\infty} \frac{2x^2}{x} = \lim\limits_{x\to\infty} 2x = \infty$

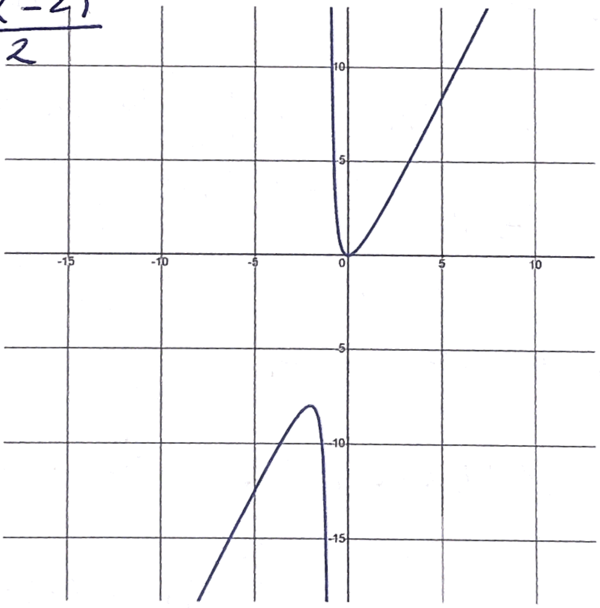
$\lim\limits_{x\to-\infty} \frac{2x^2}{x+1} = \lim\limits_{x\to-\infty} \frac{2x^2}{x} = \lim\limits_{x\to-\infty} 2x = -\infty$

$\boxed{\text{NO H.A.}}$

S.A.

$$\begin{array}{r|l} & 2x-2 \\ \hline x+1 & 2x^2 \\ & -(2x^2+2x) \\ \hline & -2x \\ & -(-2x-2) \\ \hline & 2 \end{array}$$

$\boxed{\text{S.A. } y=2x-2}$



Another way to look at V.A.

$\lim\limits_{x\to-1} \frac{2x^2}{x+1} \Rightarrow \frac{2}{0}$

This is a case II limit, $\boxed{\text{So we have a V.A. at } x=-1.}$

DIY

1. Use the following information to sketch a graph of $f(x)$.

   - The point $(-2, 0)$ is on the graph of f.
   - f has a horizontal asymptote at $y = 0$, vertical asymptotes at $x = \pm 1$, and no slant asymptotes.
   - $f'(x) > 0$ on the intervals $(-\infty, -1)$ and $(-1, 0)$.
   - $f'(x) < 0$ on the intervals $(0, 1)$ and $(1, \infty)$.
   - $f''(x) > 0$ on the intervals $(-\infty, -1)$ and $(1, \infty)$.
   - $f''(x) < 0$ on the interval $(-1, 1)$.