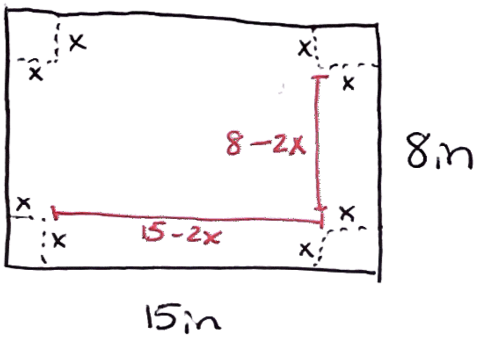
Optimization I

Optimization is all about finding the absolute maximum or minimum of a function, like we did in lesson 19. The only difference now is that we will have to come up with the function based on the scenario described in the question.

Strategy

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the *objective function* - the function we want to maximize or minimize. Write it in terms of the variables of the problem.
3. Identify the *constraint equation(s)* - equations that place constraints on the variables (if applicable). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function (if applicable).
5. With the objective function expressed in terms of a single variable, find the *interval of interest* for that variable.
6. Use methods from lesson 19 to find the absolute maximum or absolute minimum value of the objective function on the interval of interest. If necessary, check the endpoints of the interval of interest.

Example 1: A piece of cardboard is 15 inches by 8 inches. A square is to be cut from each corner and the sides folded up to make an open-top box. What size squares should be cut out so as to maximize the volume of the resulting box?



$V = lwh$

$l = 15-2x$ $\quad w = 8-2x$

$h = x$

$V = (15-2x)(8-2x)x$ $\qquad 0 \le x \le 4 \Leftrightarrow [0,4]$

$V = 4x^3 - 46x^2 + 120x$

$V' = 12x^2 - 92x + 120$

$= 4(3x-5)(x-6)$

$4(3x-5)(x-6) = 0$

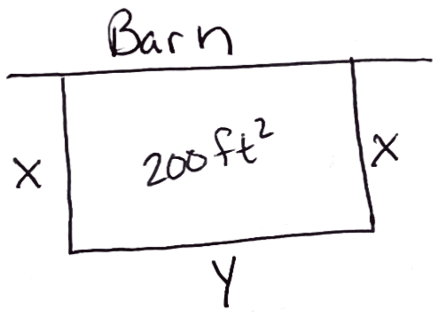
$x = 5/3 \quad x = 6$

not in interval

x	V
0	0
5/3	$\frac{2450}{27}$ ← max
4	0

$\boxed{\text{We should cut out squares with a side length of } 5/3 \text{ in.}}$

Example 2: A rectangular garden 200 square feet in area is to be fenced off against rabbits. Find the dimensions that will require the least amount of fencing given that one side of the garden is already protected by a barn.



$xy = 200$ (constraint)

$\Rightarrow y = \frac{200}{x}$

$P = 2x + y$ (objective function)

$P = 2x + \frac{200}{x}, \quad (0, \infty)$

$P' = 2 - \frac{200}{x^2}$ ← undefined when $x=0$, but zero is not in domain of P.

$2 - \frac{200}{x^2} = 0$

$\Rightarrow 2 = \frac{200}{x^2}$

$\Rightarrow x^2 = 100 \Rightarrow x = \pm 10$

$\Rightarrow x = 10$

Use 1st or 2nd derivative test to show $x=10$ is a minimum for P.

$P'' = \frac{400}{x^3}$

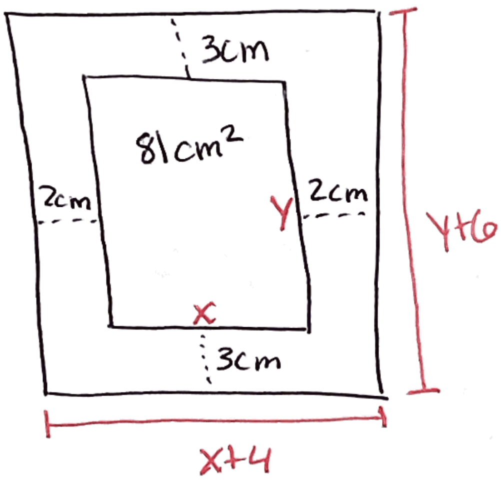
$P''(10) = \frac{400}{1000} > 0$

So $x=10$ is a min.

$x = 10, \quad y = \frac{200}{x} \Rightarrow y = \frac{200}{10} = 20$

$\boxed{\begin{array}{l} x = 10\text{ft} \\ y = 20\text{ft} \end{array}}$

Example 3: A page is to contain 81 square centimeters of print. The margins at the top and bottom are to be 3 centimeters each and, at the sides, 2 centimeters each. Find the dimensions of such a page that has the smallest possible total area.



$A = (x+4)(y+6)$ ← objective function

$A = (x+4)\left(\frac{81}{x} + 6\right)$

$A = 6x + \frac{324}{x} + 105, \quad (0, \infty)$

$A' = 6 - \frac{324}{x^2}$ ← undef. at $x=0$, but zero is not in domain of A.

$xy = 81$ ← constraint

$\Rightarrow y = \frac{81}{x}$

$6 - \frac{324}{x^2} = 0$

$\Rightarrow 6x^2 = 324$

$\Rightarrow x^2 = 54 \quad \Rightarrow x = \pm\sqrt{54} = \pm 3\sqrt{6}$

$\Rightarrow x = 3\sqrt{6}$.

Use 1st or 2nd derivative test to show $x = 3\sqrt{6}$ is a minimum.

A': − − − 0 + + + +

1 | $3\sqrt{6}$ | 9

min ✓

Dimensions of the page.

$x+4$ by $y+6$, $\quad y = \frac{81}{x} \Rightarrow y = \frac{27}{\sqrt{6}}$

$x+4 \rightarrow 3\sqrt{6}+4$

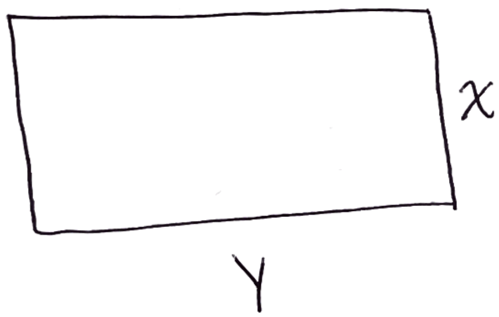
$y+6 \rightarrow \frac{27}{\sqrt{6}} + 6$

Dimensions:

$3\sqrt{6}+4$ by $\frac{27}{\sqrt{6}} + 6$

DIY

1. Find the dimensions of the rectangle, with perimeter 24, that has the largest area.



$A = xy$ objective

$A = x(12-x)$ $[0, 12]$

$A = 12x - x^2$

constraint

Perimeter: $2x + 2y = 24$

$y = 12 - x$

$A' = 12 - 2x$

$12 - 2x = 0$

$\Rightarrow 12 = 2x$

$\Rightarrow 6 = x$

x	A
0	0
6	36 ← max
12	0

Dimensions:

$x = 6$

$y = 6$ $(y = 12 - x)$