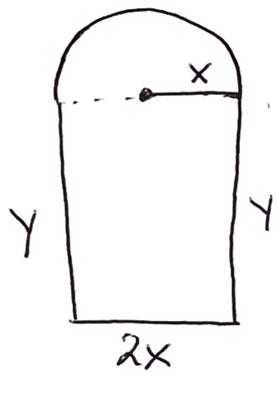
Optimization II

Optimization is all about finding the absolute maximum or minimum of a function, like we did in lesson 19. The only difference now is that we will have to come up with the function based on the scenario described in the question.

Strategy

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the *objective function* - the function we want to maximize or minimize. Write it in terms of the variables of the problem.
3. Identify the *constraint equation(s)* - equations that place constraints on the variables (if applicable). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function (if applicable).
5. With the objective function expressed in terms of a single variable, find the *interval of interest* for that variable.
6. Use methods from lesson 19 to find the absolute maximum or absolute minimum value of the objective function on the interval of interest. If necessary, check the endpoints of the interval of interest.

Example 1: A window in the shape of a rectangle capped by a semicircle is to have a perimeter of 44 inches. Choose the radius of the semicircular part so that the window admits the most light.



Maximize Area

$$A = 2xy + \frac{\pi x^2}{2}$$

objective

$P = 44\text{in}$ constraint

$$P = 2x + 2y + \pi x$$ Half the circumference of a circle of radius x.

$$P = (2+\pi)x + 2y \Rightarrow 44 = (2+\pi)x + 2y$$

$$\Rightarrow 44 - (2+\pi)x = 2y$$

$$\Rightarrow y = 22 - (1 + \pi/2)x$$

$$A = 2xy + \frac{\pi x^2}{2} \quad \left[0, \frac{22}{1+\pi/2}\right]$$

$$A = 2x(22 - (1+\pi/2)x) + \frac{\pi}{2}x^2$$

$$A = 44x - (2+\pi)x^2 + \frac{\pi}{2}x^2$$

$$A' = 44 - (4+2\pi)x + \pi x$$

$$A' = 44 - (4+\pi)x$$

$$44 - (4+\pi)x = 0$$

$$\Rightarrow \frac{44}{4+\pi} = x$$

x	A
0	0
max $\frac{44}{4+\pi}$	$\frac{968\pi}{(4+\pi)^2} + \frac{88\left(22 - \frac{44(1+\frac{\pi}{2})}{4+\pi}\right)}{(4+\pi)}$ ≈ 135.54
$\frac{22}{1+\pi/2}$	$\frac{242\pi}{(1+\frac{\pi}{2})^2} \approx 115.03$

$$\boxed{\text{radius} = x = \frac{44}{4+\pi} \text{ in}}$$

Example 2: A cylindrical paint can is to have a volume of 12 fluid ounces, which is approximately 22 cubic inches. Find the dimensions of the can that will require the least amount of material to manufacture. The volume of a cylinder is $V = \pi r^2 h$ and the surface area is $A = 2\pi rh + 2\pi r^2$, where r is the radius and h is the height.

Objective:

$$A = 2\pi r h + 2\pi r^2$$

$$A = 2\pi r\left(\frac{22}{\pi r^2}\right) + 2\pi r^2$$

$$A = \frac{44}{r} + 2\pi r^2 \qquad (0, \infty)$$

Constraint:

$$V = \pi r^2 h = 22 \text{ in}^3$$

$$\Rightarrow h = \frac{22}{\pi r^2}$$

$$A' = \frac{-44}{r^2} + 4\pi r$$

undefined when $r=0$, but $r=0$ is not in the domain of A.

$$0 = \frac{-44}{r^2} + 4\pi r$$

$$\Rightarrow 4\pi r = \frac{44}{r^2} \Rightarrow \pi r^3 = 11$$

$$\Rightarrow r = \sqrt[3]{11/\pi}$$

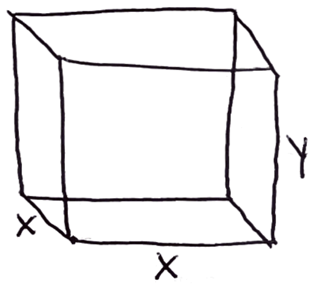
$$A'' = \frac{88}{r^3} + 4\pi \qquad A''\left(\sqrt[3]{11/\pi}\right) = \frac{88}{11/\pi} + 4\pi > 0 \quad \text{min } \checkmark$$

Dimensions:

$$\boxed{r = \sqrt[3]{11/\pi} \text{ in}, \quad h = \frac{22}{\pi (11/\pi)^{2/3}} \text{ in}}$$

DIY

1. A box with a square base and open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimize the amount of material used.



Objective:

$$A = x^2 + 4xy$$

$$A = x^2 + 4x\left(\frac{32000}{x^2}\right)$$

$$A = x^2 + \frac{128000}{x} \qquad (0, \infty)$$

Constraint:

$$x^2 y = 32000$$

$$y = \frac{32000}{x^2}$$

$$A' = 2x - \frac{128000}{x^2}$$

undefined at zero, but $x=0$ is not in domain of A.

$$2x - \frac{128000}{x^2} = 0$$

$$\Rightarrow \quad 2x^3 = 128000$$

$$x^3 = 64000$$

$$x = 40$$

$$A'' = 2 + \frac{256000}{x^3}$$

$$A''(40) > 0 \Rightarrow \text{min} \checkmark$$

$$y = \frac{32000}{(40)^2} = 20$$

length: 40 cm
width: 40 cm
height: 20cm