Optimization III

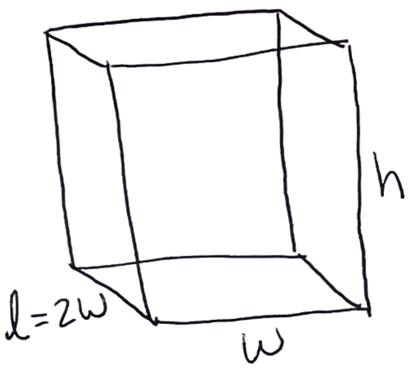
Optimization is all about finding the absolute maximum or minimum of a function, like we did in lesson 19. The only difference now is that we will have to come up with the function based on the scenario described in the question.

Strategy

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the *objective function* - the function we want to maximize or minimize. Write it in terms of the variables of the problem.
3. Identify the *constraint equation(s)* - equations that place constraints on the variables (if applicable). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function (if applicable).
5. With the objective function expressed in terms of a single variable, find the *interval of interest* for that variable.
6. Use methods from lesson 19 to find the absolute maximum or absolute minimum value of the objective function on the interval of interest. If necessary, check the endpoints of the interval of interest.

<u>Revenue:</u> $R =$ (unit price) $\times$ (quantity)
<u>Profit:</u> $P =$ (revenue) $-$ (costs)

Example 1: A company manufactures boxes. A certain type of box must have a volume of 534 cubic centimeters. The material used to make the sides of the box costs 4 cents per square centimeter and the material for the top and bottom of the box costs 8 cents per square centimeter. If the length of the box is twice the width, find the dimensions of the box that will minimize the cost of construction.



Objective:

$$C = 4(2wh + 2\cdot 2wh) + 8(2\cdot 2w^2)$$

$$C = 4(6wh) + 8(4w^2)$$

$$C = 24wh + 32w^2$$

$$C = 24w\left(\frac{267}{w^2}\right) + 32w^2, \ (0,\infty)$$

$$C = \frac{6408}{w} + 32w^2$$

Constraint:

$$2w^2h = 534$$

$$\Rightarrow h = \frac{267}{w^2}$$

$$C' = -\frac{6408}{w^2} + 64w$$

undefined at $w=0$, but $w=0$ is not in domain of C.

$$-\frac{6408}{w^2} + 64w = 0$$

$$\Rightarrow 64w^3 = 6408$$

$$\Rightarrow w = \sqrt[3]{\frac{801}{8}}$$

$$C'' = \frac{12816}{w^3} + 64$$

$$C''\left(\sqrt[3]{\frac{801}{8}}\right) > 0 \Rightarrow \text{min} \checkmark$$

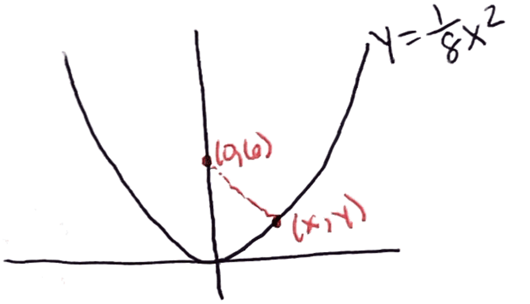
Dimensions:

$w = \sqrt[3]{\frac{801}{8}}$ cm

$l = 2w = 2\sqrt[3]{801/8}$ cm

$h = \frac{267}{w^2} = \frac{267}{\left(\sqrt[3]{\frac{801}{8}}\right)^2}$ cm

Example 2: Find the point(s) on the parabola $y = \frac{1}{8}x^2$ closest to the point $(0, 6)$.



Distance between Points (x_1,y_1) and (x_2,y_2) is $D=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Since $y=\frac{1}{8}x^2$ is symmetric about the y-axis, we only need to test on $(0,\infty)$ (Since $x=0$ will not give a min in this case) and get a symmetric answer in $(-\infty,0)$.

Minimize Distance!

$$D=\sqrt{(x-0)^2+(y-6)^2}$$

$$D=\sqrt{x^2+\left(\tfrac{1}{8}x^2-6\right)^2}$$

$$D=\sqrt{x^2+\tfrac{1}{64}x^4-\tfrac{3}{2}x^2+36}$$

$$D=\sqrt{\tfrac{1}{64}x^4-\tfrac{1}{2}x^2+36}$$

optimizing D is the same as optimizing what's inside the root.

$$d=\frac{1}{64}x^4-\frac{1}{2}x^2+36$$

on $(0,\infty)$

$$d'=\frac{1}{16}x^3-x=x\left(\tfrac{1}{16}x^2-1\right)$$

$$x\left(\tfrac{1}{16}x^2-1\right)=0$$

$x=0$ $\quad \Rightarrow x=\pm 4$

Test $x=4$ on $(0,\infty)$

d': $-$ on $(0,4)$, 0 at 4, $+$ on $(4,\infty)$

min ✓

By symmetry, $x=-4$ also gives a minimum distance.

Points: $(4,2)$ and $(4,-2)$

DIY

1. A company's marketing department has determined that if their product is sold at the price of p dollars per unit, they can sell $q = 3400 - 200p$ units. Each unit costs \$6 to make.

   (a) What price, p, should the company charge to maximize their revenue?

   (b) What price, p, should the company charge to maximize their profits?

q can't be negative

a) $R = (\text{price})(\text{quantity})$

$R = pq = p(3400 - 200p) = 3400p - 200p^2, \ [0, 17]$

$R' = 3400 - 400p \Rightarrow 3400 - 400p = 0$

$3400 = 400p$

$8.5 = p$

p	R
0	0
8.5	14,450 ← max
17	0

$\boxed{p = \$8.50}$

b) F = profit function

$F = R - 6q = 3400p - 200p^2 - 6(3400 - 200p)$

$F = -200p^2 + 4600p - 20400, \ [0, 17]$

$F' = -400p + 4600 \Rightarrow -400p + 4600 = 0$

$4600 = 400p$

$11.5 = p$

p	F
0	−20,400
11.5	6,050 ← max
17	0

$p = \$11.50$