Area and Riemann Sums

Sigma Notation

The sigma symbol, $\sum$, means to add (sum) things up.

Example 1: Evaluate the following sum.

$$\sum_{i=1}^{4}(2i+3)$$

$= 5 + 7 + 9 + 11 = 32$

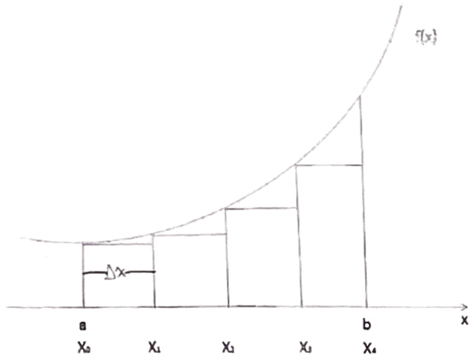
Example 2: Rewrite the following sum using sigma notation.

$$\sqrt{3-2}+\sqrt{4-2}+\sqrt{5-2}+\sqrt{6-2}+\cdots+\sqrt{n-2}$$

$= \sum_{i=3}^{n} \sqrt{i-2}$

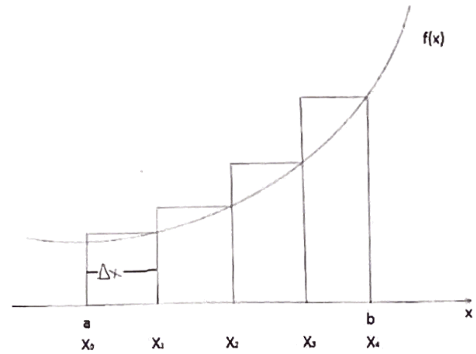
Area and Riemann Sums

Left and Right Riemann Sums use rectangles to estimate the area between a curve and the x-axis. If we want to use Riemann sums to approximate the area between a curve and the x-axis on the interval $[a, b]$ with n rectangles, then we partition the x-axis between a and b into n subintervals of width $\boxed{\Delta x = \frac{b-a}{n}}$. When using the *Left Riemann Sum*, L_n, we create rectangles from the left endpoint of each subinterval. When using the *Right Riemann Sum*, R_n, we create rectangles from the right endpoint of each subinterval. Examples using 4 rectangles are pictured below.



Left Riemann Sum (form rectangles at the left endpoints)

$$L_4 = \sum_{i=0}^{3} f(x_i)\Delta x$$



Right Riemann Sum (form rectangles at the right endpoints)

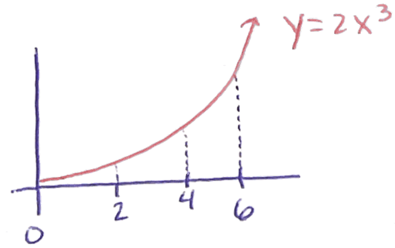
$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

Example 3: Use the left and right Riemann sums with 3 rectangles to estimate the (signed) area under the curve of $y = 2x^3$ on the interval $[0, 6]$.

$$\Delta x = \frac{6-0}{3} = 2$$

$x_0 = 0$
$x_1 = 2$
$x_2 = 4$
$x_3 = 6$

Intervals:
$[0,2]$
$[2,4]$
$[4,6]$
L R



$$L_3 = 2\left(2(0)^3 + 2(2)^3 + 2(4)^3\right) = 288$$

$$R_3 = 2\left(2(2)^3 + 2(4)^3 + 2(6)^3\right) = 1152$$

General Riemann Sum Formulas

In general, the formulas for the left and right Riemann sums for $f(x)$ on $[a, b]$ using n rectangles are:

$$L_n = \sum_{i=0}^{n-1} f\left(i\Delta x + a\right)\Delta x$$

and

$$R_n = \sum_{i=1}^{n} f\left(i\Delta x + a\right)\Delta x.$$

Example 4: Use the left and right Riemann sums with 90 rectangles to estimate the signed area under the curve of $y = 2x^5 + 3$ on the interval $[10, 20]$.

$n=90$, $a=10$, $b=20$, $\Delta x = \frac{20-10}{90} = \frac{10}{90} = \frac{1}{9}$

$$L_{90} = \sum_{i=0}^{89} \left(2\left(\tfrac{1}{9}i + 10\right)^5 + 3\right)\frac{1}{9}$$

$$R_{90} = \sum_{i=1}^{90} \left(2\left(\tfrac{1}{9}i + 10\right)^5 + 3\right)\frac{1}{9}$$

DIY

1. Evaluate the following sum.

$$\sum_{i=2}^{5} (i^2 + 1)$$

$$= 5 + 10 + 17 + 26 = 58$$

2. Use sigma notation to rewrite the following sum.

$$(2(0)^5 + 1) + (2(1)^5 + 1) + (2(2)^5 + 1) + \cdots + (2(n)^5 + 1)$$

$$= \sum_{i=0}^{n} (2(i)^5 + 1)$$