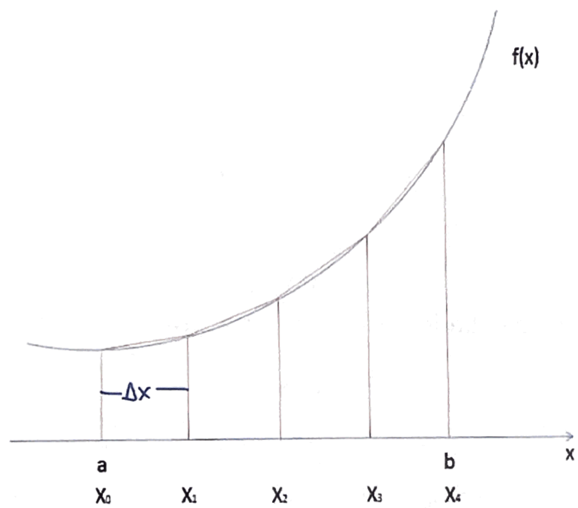
Numerical Integration

We have seen examples of many types of functions that we can integrate by hand. However, there are functions that we cannot integrate by hand, so we rely on computers. One method computers use to compute definite integrals is the *Trapezoidal Rule.* This method uses the same ideas as Riemann sums, but instead of using rectangles to estimate the area between the function and the x-axis, it uses trapezoids. If we want to use the trapezoidal rule to approximate the area between a curve and the x-axis on the interval $[a, b]$ with n trapezoids, then we partition the x-axis between a and b into n subintervals of width $\Delta x = \frac{b-a}{n}$. We then form trapezoids under the curve on each of these subintervals and compute their areas. An example using 4 trapezoids is pictured below. Recall that the area of a trapezoid is given by $A = \frac{1}{2}(\text{height})(\text{base } 1 + \text{base } 2)$.



For this example, we have that

$$\begin{aligned} T_4 &= \left[\frac{1}{2}\Delta x(f(x_0) + f(x_1))\right] + \left[\frac{1}{2}\Delta x(f(x_1) + f(x_2))\right] + \left[\frac{1}{2}\Delta x(f(x_2) + f(x_3))\right] + \left[\frac{1}{2}\Delta x(f(x_3) + f(x_4))\right] \\ &= \frac{1}{2}\Delta x(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)). \end{aligned}$$

General Trapezoidal Rule Formula

The general formula for the trapezoidal rule using n trapezoids on the interval $[a, b]$ is

$$T_n = \frac{1}{2}\Delta x \left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right),$$

where $\Delta x = \frac{b-a}{n}$.

Example 1: Use the trapezoidal rule with $n = 4$ trapezoids to estimate $\int_0^4 (3x^2+1)\,dx$. Compute the integral to find the exact area and compare the exact area with the trapezoidal rule estimate.

$f(x) = 3x^2+1, \quad a=0, \quad b=4$

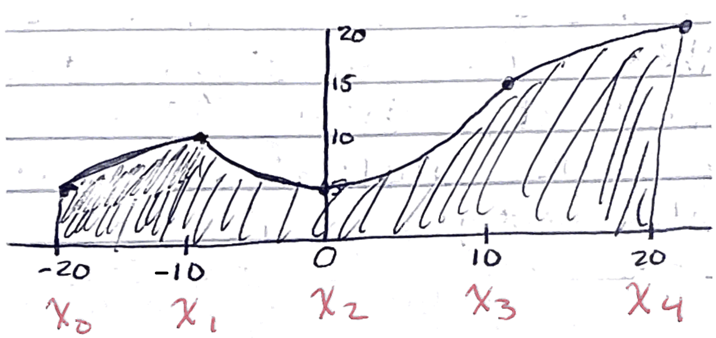
$\Delta x = \frac{4-0}{4} = 1, \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4$

$T_4 = (\tfrac{1}{2})(1)(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4))$

$= (\tfrac{1}{2})(1)(1 + 2(4) + 2(13) + 2(28) + 49) = 70$

$\int_0^4 (3x^2+1)\,dx = x^3 + x \Big|_0^4 = 68$

Example 2: Approximate the shaded area using the trapezoidal rule with $n = 4$ trapezoids.



$\Delta x = \frac{20 - -20}{4} = \frac{40}{4} = 10$

$T_4 = (\tfrac{1}{2})(10)(f(-20) + 2f(-10) + 2f(0) + 2f(10) + f(20))$

$= (\tfrac{1}{2})(10)(5 + 2(10) + 2(5) + 2(15) + 20)$

$= 425$

Example 3: Use the trapezoidal rule with $n = 3$ trapezoids to approximate $\int_2^6 \ln(x)\,dx$.

$f(x) = \ln(x)$ $\quad a = 2, \quad b = 6, \quad \Delta x = \frac{6-2}{3} = \frac{4}{3}$

$x_0 = 2, \quad x_1 = \frac{10}{3}, \quad x_2 = \frac{14}{3}, \quad x_3 = 6$

$$T_3 = \left(\tfrac{1}{2}\right)\left(\tfrac{4}{3}\right)\left(f(2) + 2f\left(\tfrac{10}{3}\right) + 2f\left(\tfrac{14}{3}\right) + f(6)\right)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{4}{3}\right)\left(\ln(2) + 2\ln\left(\tfrac{10}{3}\right) + 2\ln\left(\tfrac{14}{3}\right) + \ln(6)\right)$$

$$\approx 5.32$$

DIY

1. Use the trapezoidal rule with $n = 3$ trapezoids to approximate the following integral.

$$\int_{0.5}^{1.5} \frac{\sin(x)}{x}\,dx$$

$f(x) = \frac{\sin x}{x}$ $\quad \Delta x = \frac{1.5 - 0.5}{3} = \frac{1}{3}$ $\quad a = 0.5$, $b = 1.5$

$x_0 = \frac{1}{2}, \quad x_1 = 5/6, \quad x_2 = 7/6, \quad x_3 = 1.5$

$$T_3 = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)\left(\frac{\sin(1/2)}{1/2} + 2\,\frac{\sin(5/6)}{5/6} + 2\,\frac{\sin(7/6)}{7/6} + \frac{\sin(3/2)}{3/2}\right)$$

$$\approx \boxed{0.829}$$