Exponential Decay

We use the same exponential model for exponential decay as we did for exponential growth.

$$\frac{dy}{dt} = ky \quad \Rightarrow \quad y = Ce^{kt}$$

$$C = \text{initial amount}$$

$$k = \text{rate of decay}$$

The only difference is that, in exponential growth, $k > 0$, and, in exponential decay, $k < 0$.

Example 1: The population P of a species of bird is decreasing at a rate that is proportional to the population itself. If $P = 5000$ when $t = 3$ and $P = 4000$ when $t = 4$, what is the population when $t = 9$?

$P = Ce^{kt}$

$5000 = Ce^{3k} \Rightarrow 5000e^{-3k} = C$

$4000 = Ce^{4k} \Rightarrow 4000 = 5000e^{-3k}e^{4k}$

$\Rightarrow 4/5 = e^{k} \Rightarrow k = \ln(4/5)$

$5000 = Ce^{3\ln(4/5)} \Rightarrow C = 5000e^{-3\ln(4/5)}$

$P(t) = 5000e^{-3\ln(4/5)}e^{\ln(4/5)t}$

$P(9) \approx$ 1311 birds

Half-Life Formula

The *half-life* of a substance is the amount of time it takes for half of the initial amount to decay. The following relationship holds between the half-life and the rate of decay, k.

$$k = \frac{\ln\left(\frac{1}{2}\right)}{\text{half-life}}$$

Example 2: The radioactive isotope ^{239}Pu has a half-life of approximately 24,100 years. After 2,000 years there are 5 grams of ^{239}Pu left.

(a) What was the initial quantity?

Half-life $\Rightarrow k = \frac{\ln(1/2)}{24100}$

$P = Ce^{kt}$

$5 = Ce^{\frac{\ln(1/2)}{24100}(2000)}$

$\Rightarrow C = 5e^{-\frac{\ln(1/2)}{24100}(2000)} \approx 5.296\text{ g}$

(b) How much remains after 5,000 years?

$-\ln(1/2) = \ln(2)$

$P = Ce^{kt} \Rightarrow P(t) = 5e^{\ln(2)\left(\frac{2000}{24100}\right)} e^{\frac{\ln(1/2)}{24100}t}$

$P(5000) \approx 4.587\text{g}$

DIY

1. The radioactive isotope ^{14}C has a half-life of approximately 5,715 years. A piece of charcoal contains only 25% as much of the radioactive carbon as a piece of modern charcoal. How old is this sample of charcoal?

25% of initial amount = 25% of C = 0.25C.

Half-life $\Rightarrow k = \frac{\ln(1/2)}{5715}$

$0.25C = Ce^{\frac{\ln(1/2)}{5715}t} \Rightarrow 0.25 = e^{\frac{\ln(1/2)}{5715}t}$

$\Rightarrow \ln(0.25) = \frac{\ln(1/2)}{5715}t \Rightarrow \frac{5715\ln(0.25)}{\ln(1/2)} = t$

$\Rightarrow t = 11,430$ Years