Finding Limits Analytically

We will examine three different cases.
When asked to find $\lim_{x \to c} f(x)$ analytically, **always** look at $f(c)$ first!

Case I

$f(c) =$ a number (even 0 is okay). That number is the answer to the limit.

Example 1: Evaluate the following limit analytically.

$$\lim_{x \to 8} (3x - 12)$$

$f(x) = 3x - 12$

$c = 8$

$3(8) - 12 = 24 - 12 = 12$ (case I)

Case II

$f(c) = \frac{\text{nonzero number}}{0}$. Case II tells us that $f(x)$ has a *vertical asymptote* at $x = c$. At a vertical asymptote, the value of the limit can be $+\infty$, $-\infty$, or DNE. We have to check right an left hand limits to decide which it is.

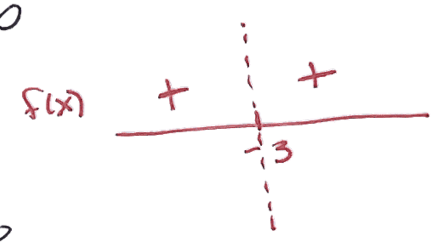
Example 2: Evaluate the following limit analytically.

$f(x) = \frac{7}{(x+3)^2}$ $\quad C = -3$

$$\lim_{x \to -3} \frac{7}{(x+3)^2}$$

$f(c) = \frac{7}{0}$ (case II)

$\lim_{x \to -3^-} \frac{7}{(x+3)^2} \Rightarrow \frac{7}{(-)^2} \Rightarrow \frac{7}{0^+} \to +\infty$



$\lim_{x \to -3^+} \frac{7}{(x+3)^2} \Rightarrow \frac{7}{(+)^2} \Rightarrow \frac{7}{0^+} \to +\infty$

So, $\boxed{\lim_{x \to -3} \frac{7}{(x+3)^2} = \infty}$

Example 3: Evaluate the following limit analytically. $f(x) = \frac{3}{x-2}$ $c = 2$

$$\lim_{x \to 2} \frac{3}{x-2}$$

$f(c) = \frac{3}{0} \Rightarrow$ case II

$\lim_{x \to 2^-} \frac{3}{x-2} = \frac{3}{(-)} \Rightarrow \frac{3}{0^-} \to -\infty$

$\lim_{x \to 2^+} \frac{3}{x-2} = \frac{3}{(+)} \Rightarrow \frac{3}{0^+} \to \infty$

So, $\lim_{x \to 2} \frac{3}{x-2} = \boxed{\text{DNE}}$

Case III

$f(c) = \frac{0}{0}$. In this case we need to do some algebra on $f(x)$ to end up with a "new" limit that falls into Case I or Case II.

Example 4: Evaluate the following limit analytically.

$$\lim_{x \to 0} \frac{x^2 - 4x}{x^2 + 2x}$$

$f(x) = \frac{x^2 - 4x}{x^2 + 2x}$ $c = 0$

$f(c) = \frac{0}{0}$ (case III)

$= \lim_{x \to 0} \frac{x(x-4)}{x(x+2)} = \lim_{x \to 0} \frac{x-4}{x+2} = \frac{-4}{2} = -2$

(case I)

Example 5: Evaluate the following limit analytically.

$$\lim_{x\to 1} \frac{x^2 - x}{(x-1)^2}$$

$f(x) = \frac{x^2-x}{(x-1)^2}$ $c=1$

$f(c) = \frac{0}{0}$ (case III)

$= \lim_{x\to 1} \frac{x(x-1)}{(x-1)^2} = \lim_{x\to 1} \frac{x}{x-1} \to \frac{1}{0}$ (case II)

$\lim_{x\to 1^-} \frac{x}{x-1} \Rightarrow \frac{1}{(-)} \Rightarrow \frac{1}{0^-} \to -\infty$

$\lim_{x\to 1^+} \frac{x}{x-1} \Rightarrow \frac{1}{(+)} \Rightarrow \frac{1}{0^+} \to \infty$

So, $\lim_{x\to 1} \frac{x^2-x}{(x-1)^2} = \boxed{DNE}$

We can also find limits of *piecewise functions* analytically.

Example 6: Evaluate the following limits analytically, where

$$f(x) = \begin{cases} 3x^2+2, & x \le 0 \\ 8x+2, & 0 < x < 1 \\ -8x+2, & x \ge 1 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = 2 \qquad \lim_{x\to 1^-} f(x) = 10$$

$$\lim_{x\to 0^+} f(x) = 2 \qquad \lim_{x\to 1^+} f(x) = -6$$

$$\lim_{x\to 0} f(x) = 2 \qquad \lim_{x\to 1} f(x) = \text{DNE}$$

Properties of Limits

Let c, k, L, and K be real numbers and n a positive integer. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$, then we have the following.

- $\lim_{x\to c} [kf(x)] = kL$
- $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$
- $\lim_{x\to c} [f(x)g(x)] = LK$
- $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, assuming $K \neq 0$
- $\lim_{x\to c} [f(x)]^n = L^n$

DIY

1. Find the following limit analytically.

$$\lim_{x\to 0} \frac{x^3+2x^2}{x^2-3x}$$

$f(c) = \frac{0}{0}$ (case III)

$$= \lim_{x\to 0} \frac{x^2(x+2)}{x(x-3)} = \lim_{x\to 0} \frac{x(x+2)}{x-3} = \frac{0}{-3} = \boxed{0}$$ (case I)

2. Find the following limit analytically.

$$\lim_{x\to 0} \frac{x^2-3x+2}{x^2+3x+2}$$

$f(c) = \frac{2}{2} = \boxed{1}$ (case I)

3. Find the following limit analytically.

$$\lim_{x\to 6} \frac{-1}{x-6}$$

$f(c) = \frac{-1}{0}$ (case II)

$$\lim_{x\to 6^-} \frac{-1}{x-6} \Rightarrow \frac{-1}{(-)} \Rightarrow \frac{-1}{0^-} \Rightarrow +\infty$$

$$\lim_{x\to 6^+} \frac{-1}{x-6} \Rightarrow \frac{-1}{(+)} \Rightarrow \frac{-1}{0^+} \Rightarrow -\infty$$

So, $\lim_{x\to 6} \frac{-1}{x-6} = \boxed{\text{DNE}}$