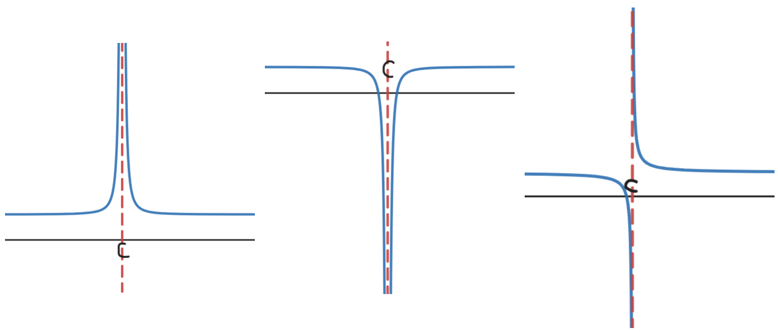
Continuity

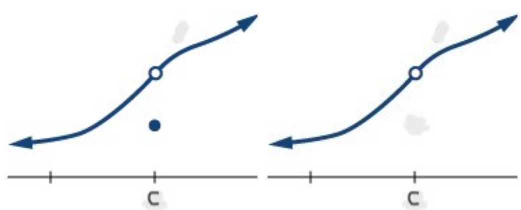
A function is *continuous* if it has no *discontinuities*. Think: we can draw the graph of the function without ever lifting our pen.

Types of Discontinuities (at a point c)

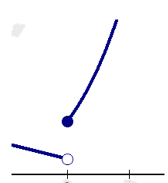
- Vertical Asymptote: A function $f(x)$ will have a vertical asympotote at $x = c$ if $\lim_{x\to c} f(x)$ is either a Case II limit or a Case III limit that becomes a Case II limit after we use algebra to simplify $f(x)$. In other words, $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$.



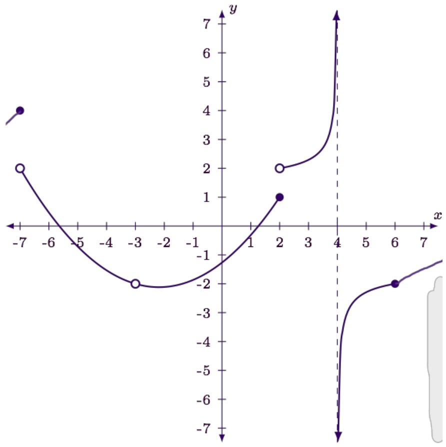
- Hole: A function $f(x)$ will have a hole at $x = c$ if the left- and right-sided limits at c are both finite and equal (so the two-sided limit exists and is finite), but this limit value is **not** equal to the function value at c ($f(c)$ may even be undefined).
  Symbolically, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c} f(x) = L < \infty$, but $f(c) \neq L$ ($f(c)$ could even be undefined).
  Holes can show up as the Case III limits that turn into Case I limits after we use algebra to simplify $f(x)$.



- Jump: The left- and right-sided limits of $f(x)$ at c are both finite, but not equal.
  Symbolically, $\lim_{x\to c^-} f(x) = L$, $\lim_{x\to c^+} f(x) = M$, with $L, M < \infty$ and $L \neq M$.



Example 1: Classify the discontinuities in the following graph.



Continuity and Limits

A function $f(x)$ is continuous at the point $x = c$ if and only if $\lim_{x\to c} f(x)$ exists and is finite, $f(c)$ is defined, and $\lim_{x\to c} f(x) = f(c)$. Symbolically,

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c),$$

where all the above limits and function value are finite.

Where to Check for Discontinuities

- *Polynomials* are continuous everywhere (for any value of x), so they have no discontinuities.
- *Root Functions* (square root, cubed root, etc.) are continuous on their domains.
- *Rational Functions* (a polynomial divided by a polynomial) only have discontinuities where their denominators are zero. They are continuous everywhere else.
- *Piecewise Functions*: As long as each "piece" of a piecewise function is continuous, we only need to check the x-values where the function changes definition (switches from one piece to another).

Example 2: Classify the discontinuities, if any, of the following function.

$$f(x) = \frac{x^2 - 7x}{x^2 + 3x}$$

Example 3: Classify the discontinuities, if any, of the following function.

$$g(x) = x^{25} - 4x^{17} + 56x + 8$$

Example 4: Classify the discontinuities, if any, of the following function.

$$f(x) = \begin{cases} 2x + 6, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Example 5: Classify the discontinuities, if any, of the following function.

$$f(x) = \begin{cases} 8x^2 + 4, & x \leq 0 \\ 3x + 4, & 0 < x < 1 \\ 2x + 9, & x \geq 1 \end{cases}$$

DIY

1. Classify the discontinuities, if any, of the following function.

$$g(x) = \frac{x^2 - 11x + 18}{x^2 - 6x + 8}$$

2. Classify the discontinuities, if any, of the following function.

$$h(x) = \begin{cases} x - 6, & x < 8 \\ \sqrt[3]{x}, & x \geq 8 \end{cases}$$