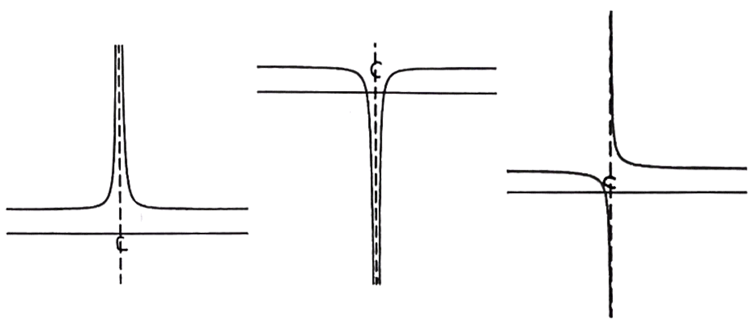
# Continuity

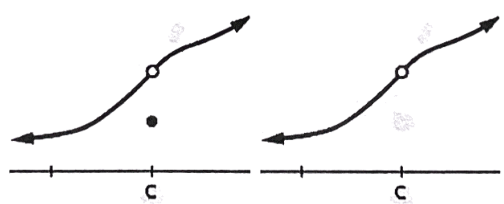
A function is *continuous* if it has no *discontinuities*. Think: we can draw the graph of the function without ever lifting our pen.

## Types of Discontinuities (at a point $c$)

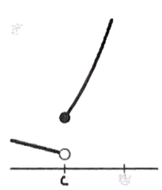
- Vertical Asymptote: A function $f(x)$ will have a vertical asympotote at $x = c$ if $\lim_{x\to c} f(x)$ is either a Case II limit or a Case III limit that becomes a Case II limit after we use algebra to simplify $f(x)$. In other words, $\lim_{x\to c^-} f(x) = \pm\infty$ and/or $\lim_{x\to c^+} f(x) = \pm\infty$.



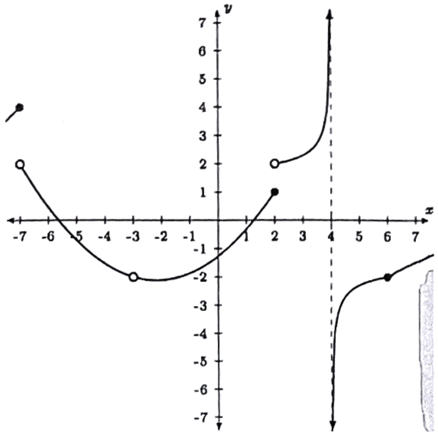
- Hole: A function $f(x)$ will have a hole at $x = c$ if the left- and right-sided limits at $c$ are both finite and equal (so the two-sided limit exists and is finite), but this limit value is **not** equal to the function value at $c$ ($f(c)$ may even be undefined).
  Symbolically, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c} f(x) = L < \infty$, but $f(c) \neq L$ ($f(c)$ could even be undefined).
  Holes can show up as the Case III limits that turn into Case I limits after we use algebra to simplify $f(x)$.




- Jump: The left- and right-sided limits of $f(x)$ at $c$ are both finite, but not equal.
  Symbolically, $\lim_{x\to c^-} f(x) = L$, $\lim_{x\to c^+} f(x) = M$, with $L, M < \infty$ and $L \neq M$.



Example 1: Classify the discontinuities in the following graph.



Jump at $x = -7$

Hole at $x = -3$

Jump at $x = 2$

Vertical Asymptote (VA) at $x = 4$.

## Continuity and Limits

A function $f(x)$ is continuous at the point $x = c$ if and only if $\lim_{x \to c} f(x)$ exists and is finite, $f(c)$ is defined, and $\lim_{x \to c} f(x) = f(c)$. Symbolically,

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c),$$

where all the above limits and function value are finite.

## Where to Check for Discontinuities

- *Polynomials* are continuous everywhere (for any value of $x$), so they have no discontinuities.
- *Root Functions* (square root, cubed root, etc.) are continuous on their domains.
- *Rational Functions* (a polynomial divided by a polynomial) only have discontinuities where their denominators are zero. They are continuous everywhere else.
- *Piecewise Functions*: As long as each "piece" of a piecewise function is continuous, we only need to check the $x$-values where the function changes definition (switches from one piece to another).

Example 2: Classify the discontinuities, if any, of the following function.

$$f(x) = \frac{x^2 - 7x}{x^2 + 3x}$$

Rational function $\Rightarrow$ Check $x$-values where denominator is zero.

$x^2 + 3x = 0 \Rightarrow x(x+3) = 0 \Rightarrow$ check $x=0$ and $x=-3$

Determine the type of limit f has at 0 and -3 to determine the type of discontinuity at each point.

$x=-3$: $\lim\limits_{x \to -3} \frac{x^2-7x}{x^2+3x} \Rightarrow \frac{30}{0}$ Case II limit $\Rightarrow$ V.A.

$\boxed{f(x) \text{ has a V.A. at } x=-3}$

$x=0$: $\lim\limits_{x \to 0} \frac{x^2-7x}{x^2+3x} \Rightarrow \frac{0}{0}$ Case III limit

$= \lim\limits_{x \to 0} \frac{x(x-7)}{x(x+3)} = \lim\limits_{x \to 0} \frac{x-7}{x+3} \Rightarrow \frac{-7}{3}$ Case I limit

$\boxed{f(x) \text{ has a hole at } x=0.}$

Example 3: Classify the discontinuities, if any, of the following function.

$$g(x) = x^{25} - 4x^{17} + 56x + 8$$

Polynomials are continuous everywhere (no discontinuities)!

Example 4: Classify the discontinuities, if any, of the following function.

$2x+6$ and $4$ are both continuous, so we just check $x$-values where $f$ changes definition, $x=2$.

$$f(x) = \begin{cases} 2x+6, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

For piecewise functions, look at the left- and right-sided limits, and function value to classify discontinuities.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+6) = 10$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x+6) = 10$$

$$f(2) = 4$$

The left- and right-sided limits are finite & equal, but they're not equal to $f(2)$. So, there's a hole at $x=2$.

Example 5: Classify the discontinuities, if any, of the following function.

$8x^2+4$, $3x+4$, and $2x+9$ are all continuous, so we just check where $f$ changes definition, $x=0$ and $x=1$.

$$f(x) = \begin{cases} 8x^2+4, & x \leq 0 \\ 3x+4, & 0 < x < 1 \\ 2x+9, & x \geq 1 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (8x^2+4) = 4$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3x+4) = 4$$

$$\lim_{x\to 0} f(x) = 4$$

$$f(0) = 4$$

$f$ is continuous at $x=0$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x+4) = 7$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+9) = 11$$

Since the left- and right-sided limits are finite, but not equal, $f$ has a jump at $x=1$.

# DIY

1. Classify the discontinuities, if any, of the following function.

$$g(x) = \frac{x^2 - 11x + 18}{x^2 - 6x + 8}$$

g(x) is a rational function.

$x^2 - 6x + 8 = 0 \Rightarrow (x-2)(x-4) = 0 \Rightarrow$ check $x=2$ and $x=4$.

$x=2$: $\lim_{x\to 2} \frac{x^2-11x+18}{x^2-6x+8} \Rightarrow \frac{0}{0}$ case III limit.

$= \lim_{x\to 2} \frac{(x-2)(x-9)}{(x-2)(x-4)} = \lim_{x\to 2} \frac{x-9}{x-4} \Rightarrow \frac{7}{2}$ case I

g has a hole at $x=2$.

$x=4$: $\lim_{x\to 4} \frac{x^2-11x+18}{x^2-6x+8} \Rightarrow \frac{-10}{0}$ case II limit.

g has a V.A. at $x=4$.

2. Classify the discontinuities, if any, of the following function.

$$h(x) = \begin{cases} x-6, & x < 8 \\ \sqrt[3]{x}, & x \geq 8 \end{cases}$$

$x-6$ and $\sqrt[3]{x}$ are both continuous, so just need to check $x=8$.

$\lim_{x\to 8^-} h(x) = \lim_{x\to 8^-} (x-6) = 2$

$\lim_{x\to 8^+} h(x) = \lim_{x\to 8^+} \sqrt[3]{x} = 2$

$\lim_{x\to 8} h(x) = 2$

$h(8) = 2$.

Since h is continuous at 8, it is continuous everywhere (no discontinuities).