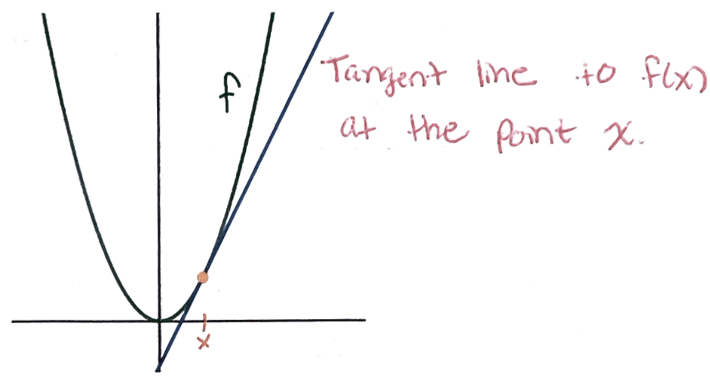
The Derivative

Tangent Lines

The *tangent line* to a function $f(x)$ at a point c is a line that touches the graph of $f(x)$ at the point $(c, f(c))$. Note, this means that the point $(c, f(c))$ is always on the tangent line.

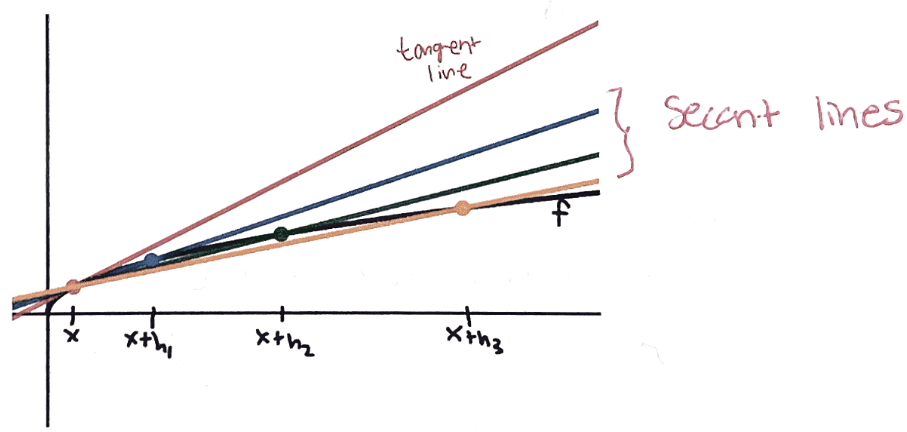


We can think of this as taking the slope of the graph at the point $(x, f(x))$ and extending it into a line with the same slope.

Slope of the Tangent Line to $f(x)$ at $x =$ Slope of the graph of $f(x)$ at the point $(x, f(x))$.

Tangent Lines as Limits of Secant Lines

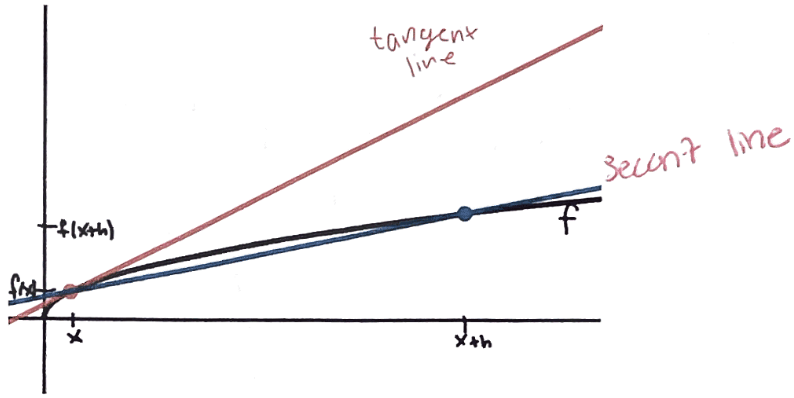
A *secant line* is a line that passes through the graph of $f(x)$ in two points.



What happens to points $x + h_1$, $x + h_2$, and $x + h_3$ as the secant lines approach the tangent line?

As the secant lines get closer to the tangent line at x, the points $x+h_1$, $x+h_2$, and $x+h_3$, get closer to x. In other words, if we form the secant line intersecting $f(x)$ at x and $x+h$, as the secant line approaches the tangent line at x, $x+h$ gets closer to x, so h gets smaller $(h \to 0)$.

Finding the Slope of the Tangent Line Using Slopes of Secant Lines



What is the slope of the secant line in the picture above?

Recall that slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$ for the line formed by points (x_1, y_1) and (x_2, y_2).

$\Rightarrow$ Slope of secant line $= m_{sec} = \dfrac{f(x+h) - f(x)}{x+h-x} = \dfrac{f(x+h) - f(x)}{h}$ ← Difference Quotient

Take the limit as $h \to 0$ to get the slope of the tangent line in the picture above.

Slope of tangent line $= m_{tan} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ ← Limit of the difference quotient.

The Derivative

We define the *derivative* of a function $f(x)$ at the point x to be the slope of the tangent line at any point x.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Other Notation: If $y = f(x)$, then $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x)$.

Geometric Interpretation: The slope of the tangent line to $f(x)$ at the point $x = c$ is equal to the derivative of f at the point c ($f'(c) =$ slope of the tangent line at c).

Super Important!

Example 1: Given $f(x) = 3x^2 + 1$, use the limit of the difference quotient to find $f'(x)$.

$$f(x+h) = 3(x+h)^2 + 1 = 3x^2 + 6xh + 3h^2 + 1$$

$$f(x+h) - f(x) = 3x^2 + 6xh + 3h^2 + 1 - (3x^2 + 1)$$
$$= 6xh + 3h^2$$

$$\frac{f(x+h) - f(x)}{h} = \frac{6xh + 3h^2}{h} = \frac{h(6x+3h)}{h} = 6x + 3h$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (6x + 3h) = 6x.$$

So, $\boxed{f'(x) = 6x}$

Example 2: Find the equation of the tangent line to the graph of $f(x) = 3x^2 + 1$ at the point $x = 2$.

↳ to find the equation of a line, we need a point and a slope.

From example 1, $f'(x) = 6x$. So the slope of the tangent line to $f(x)$ at $x=2$ is $f'(2)$.

$f'(2) = 6(2) = 12 \leftarrow$ slope.

Know the x-coordinate of the point is 2, so the y-coordinate is $f(2) = 3(2)^2 + 1 = 13$. Point: $(2, 13)$.

Recall the point-slope form of a line with slope m and point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

So we have, $y - 13 = 12(x-2) \Rightarrow y = 12x - 24 + 13$

$$\boxed{y = 12x - 11}$$

Example 3: The derivative of a function $g(x)$ is found by computing

$$g'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}.$$

What could $g(x)$ be?

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - \boxed{g(x)}}{h} = \lim_{h \to 0} \frac{e^{x+h} - \boxed{e^x}}{h}$$

By comparing equations, we see that $\boxed{g(x) = e^x}$

can check answer by forming the limit of the difference quotient using $g(x) = e^x$.

DIY

1. Find the slope of the tangent line to the graph of $f(x) = \frac{2}{3x}$ at the point $x = c$.

Slope $= f'(c)$ Need to find $f'(x)$ and plug in c.

$$f(x+h) = \frac{2}{3(x+h)}$$

common denominator.

$$f(x+h) - f(x) = \frac{2}{3(x+h)} - \frac{2}{3x} = \frac{6x - 6x - 6h}{3(x+h)\,3x} = \frac{-6h}{3(x+h)(3x)}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{-6h}{3(x+h)(3x)}}{h} = \frac{-6\cancel{h}}{3(x+h)3x} \cdot \frac{1}{\cancel{h}} = \frac{-6}{3(x+h)3x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{-6}{3(x+h)3x} = \frac{-6}{3(x)3x} = \frac{-6}{9x^2} = \frac{-2}{3x^2}$$

$$\Rightarrow \boxed{f'(c) = \text{Slope} = \frac{-2}{3c^2}}$$