The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

If we need to take the derivative of two functions being divided, we cannot simply divide the derivative of the numerator by the derivative of the denominator;

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}.$$

Bad news.

Example 1: Compute the derivative of the following function.

$$y = \frac{\sin(x) + x}{2x + 1}$$

$$y' = \frac{(\cos(x)+1)(2x+1) - (\sin x + x)(2)}{(2x+1)^2}$$

$$= \frac{2x\cos x + \cos x + 2x + 1 - 2\sin x - 2x}{4x^2 + 4x + 1}$$

$$= \boxed{\frac{2x\cos x + \cos x + 1 - 2\sin x}{4x^2 + 4x + 1}}$$

Example 2: Compute the derivative of the following function.

$$y = \frac{ae^x}{(a^2 + \sqrt{x})}, \quad \text{where } a \text{ is a constant}$$

→ a is just a number.

$$y' = \frac{ae^x(a^2 + \sqrt{x}) - ae^x\left(\frac{1}{2}x^{-1/2}\right)}{(a^2 + \sqrt{x})^2}$$

$$= \boxed{\frac{a^3e^x + ae^x\sqrt{x} - \frac{ae^x}{2\sqrt{x}}}{a^4 + 2a^2\sqrt{x} + x}}$$

Derivatives of Other Trigonometric Functions

- $\frac{d}{dx}\tan(x) = \sec^2(x)$
- $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
- $\frac{d}{dx}\cot(x) = -\csc^2(x)$
- $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

Example 3: Find the derivative of the function $y = 5\cos(x)\cot(x)$.

$y' = -5\sin x \cot x + 5\cos x(-\csc^2 x)$

$= \boxed{-5\sin x \cot x - 5\cos x \csc^2 x}$

DIY

1. Find the derivative of the following function at $t = 7$.

$$y = \frac{2+e^t}{3-e^t}$$

$y' = \dfrac{e^t(3-e^t) - (2+e^t)(-e^t)}{(3-e^t)^2}$

$= \dfrac{3e^t - e^{2t} + 2e^t + e^{2t}}{(3-e^t)^2}$

$= \dfrac{5e^t}{(3-e^t)^2}$ Plug in $t=7$ $\Rightarrow \boxed{\dfrac{5e^7}{(3-e^7)^2}}$

2. Find the equation of the tangent line to the graph of $y = 3e^x \sec(x)$ at $x = 0$.

$\cos(0) = 1$
$\Rightarrow \sec(0) = \frac{1}{\cos(0)} = \frac{1}{1} = 1$

$\tan(0) = \frac{\sin(0)}{\cos(0)} = \frac{0}{1} = 0$

Point: $(0, y(0)) = (0, 3e^0 \sec(0)) = (0, 3)$

Slope: $y' = 3e^x \sec x + 3e^x \sec x \tan x$

$y'(0) = 3e^0 \sec(0) + 3e^0 \sec(0) \tan(0)$

$= 3 + 0 = 3$

$y - 3 = 3(x - 0) \Rightarrow \boxed{y = 3x + 3}$

3. Find the derivative of the following function.

$$y = \frac{\cos(x)}{\cot(x) + \sec(x)}$$

$$y' = \frac{-\sin x(\cot x + \sec x) - \cos(x)(-\csc^2 x + \sec x \tan x)}{(\cot x + \sec x)^2}$$

$$= \frac{-\sin x\left(\frac{\cos x}{\sin x} + \frac{1}{\cos x}\right) - \cos(x)\left(\frac{-1}{\sin^2 x} + \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\right)}{(\cot x + \sec x)^2}$$

$$= \frac{-\cos x - \tan x + \frac{\cos x}{\sin^2 x} - \frac{\sin x}{\cos x}}{(\cot x + \sec x)^2}$$

$$= \boxed{\frac{-\cos x - 2\tan x + (\cos x)(\csc^2 x)}{(\cot x + \sec x)^2}}$$