# THE CORE OF ZERO-DIMENSIONAL MONOMIAL IDEALS

CLAUDIA POLINI, BERND ULRICH AND MARIE A. VITULLI

ABSTRACT. The core of an ideal is the intersection of all its reductions. We describe the core of a zero-dimensional monomial ideal $I$ as the largest monomial ideal contained in a general reduction of $I$. This provides a new interpretation of the core in the monomial case as well as an efficient algorithm for computing it. We relate the core to adjoints and first coefficient ideals, and in dimension two and three we give explicit formulas.

## 1. INTRODUCTION

The purpose of this paper is to study the core of monomial ideals. According to Northcott and Rees [22], a subideal $J$ of an ideal $I$ is a *reduction* of $I$ provided $I^{r+1} = JI^r$ for some nonnegative integer $r$. In a Noetherian ring, $J$ is a reduction of $I$ if and only if $I$ is integral over $J$. Intuitively, a reduction of $I$ is a simplification of $I$ that shares essential properties with the original ideal. Reductions are highly non-unique, even minimal reductions (with respect to inclusion) that are known to exist for ideals in Noetherian local rings. Thus one considers the *core* of the ideal $I$, written core$(I)$, which is the intersection of all reductions of $I$.

The core, introduced by Rees and Sally [25], is in a sense the opposite of the integral closure: the integral closure $\bar{I}$ is the largest ideal integral over $I$, whereas core$(I)$ is the intersection of all ideals over which $I$ is integral. The core appears naturally in the context of Briançon-Skoda theorems that compare the integral closure filtration with the adic filtration of an ideal. It is also connected to adjoints, multiplier ideals and coefficient ideals.

Huneke-Swanson, Corso-Polini-Ulrich, Hyry-Smith, Polini-Ulrich, and Huneke-Trung [12, 4, 5, 16, 23, 13, 17] gave explicit formulas for cores in local rings (whose residue characteristic is zero or large enough) by expressing them as colon ideals. For certain classes of ideals, which include zero-dimensional ideals, they showed that core$(I) = J^{n+1} : I^n$, where $J$ is a minimal reduction of $I$ and $n$ is sufficiently large. Moreover, Hyry and Smith [16, 17] discovered an unforeseen relationship with Kawamata's conjecture on the non-vanishing of sections of line bundles. They proved that Kawamata's conjecture would follow from a formula that essentially amounts to a graded analogue of the above formula for the core.

The known formulas for the core usually require the ambient ring to be *local*. In contrast, in this paper we are primarily interested in the core of 0-dimensional monomial ideals in polynomial rings. Thus we start Section 2 by establishing the expected colon formula for the core in the global

setting, for 0-dimensional ideals. For this we prove that the core of 0-dimensional ideals commutes with localization.

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field $k$, write $\mathfrak{m} = (x_1, \ldots, x_d)$, and let $I$ be a monomial ideal, that is, an $R$-ideal generated by monomials. Even though there may not exist any proper reduction of $I$ which is monomial (or even homogeneous), the intersection of all reductions, the core, is again a monomial ideal (because of the torus action, see for instance [4, 5.1]). Lipman [19] and Huneke-Swanson [12] related the core to the adjoint ideal (see also [15, 16, 17, 23]). The integral closure and the adjoint of a monomial ideal are again monomial ideals and can be described in terms of the Newton polyhedron $\text{NP}(I)$ of $I$ [9, 10]. Such a description cannot exist for the core, since the Newton polyhedron only depends on the integral closure of the ideal, whereas the core may change when passing from $I$ to $\bar{I}$. When attempting to derive any kind of combinatorial description for the core of a monomial ideal from the known colon formulas, one faces the problem that the colon formulas involve non-monomial ideals, unless $I$ has a reduction $J$ generated by a monomial regular sequence. Instead, we exploit the existence of such non-monomial reductions to devise an interpretation of the core in terms of monomial operations. This is done in Section 3, where we prove that the core is the largest monomial ideal contained in a 'general locally minimal reduction' of $I$.

Let $I$ be a 0-dimensional monomial ideal in $k[x_1, \ldots, x_d]$ and $J$ an ideal generated by $d$ general $k$-linear combinations of minimal monomial generators of $I$. Unless $I$ is generated by monomials of the same degree, $J$ may not even be $\mathfrak{m}$–primary, but $J_{\mathfrak{m}}$ is a minimal reduction of $I_{\mathfrak{m}}$. Since $I$ is $\mathfrak{m}$-primary, there exist $n_i$ such that $x_i^{n_i} \in I$. The regular sequence $\underline{\alpha} = x_1^{dn_1}, \ldots, x_d^{dn_d}$ is contained in the core of $I_{\mathfrak{m}}$ by the Briançon–Skoda theorem. Hence $(J, \underline{\alpha})_{\mathfrak{m}} = J_{\mathfrak{m}}$. Because $K = (J, \underline{\alpha})$ is a reduction of $I$ with $K_{\mathfrak{m}} = J_{\mathfrak{m}}$, we call such $K$ a *general locally minimal reduction* of $I$. As $\text{core}(I)$ is a monomial ideal contained in $K$, it is contained in $\text{mono}(K)$, the largest monomial subideal of $K$. In Theorem 3.6 we actually show that $\text{core}(I) = \text{mono}(K)$. Notice that one cannot expect the inclusion $\text{core}(I) \subset \text{mono}(K)$ to be an equality unless $K$ is far from being monomial – which is guaranteed by our general choice of $K$.

The idea behind the proof of Theorem 3.6 is to show that $\text{mono}(K)$ is independent of the general locally minimal reduction $K$. Using the inclusion reversing operation of linkage, we express $\text{mono}(K)$ in terms of $\text{Mono}((\underline{\alpha}) : K)$. Here $\text{Mono}(L)$ denotes the smallest monomial ideal containing an arbitrary ideal $L$, which can be easily computed as it is generated by the monomial supports of generators of $L$. We are able to show that $\text{Mono}((\underline{\alpha}) : K)$ does not depend on $K$, which together with the equality $\text{mono}(K) = (\underline{\alpha}) : \text{Mono}((\underline{\alpha}) : K)$ gives the independence of $\text{mono}(K)$. The last equality is also interesting as it establishes a link between mono and Mono, and because it yields an algorithm for computing mono. A different algorithm can be found in Saito-Sturmfels-Takayama [27]. Besides providing a new, combinatorial interpretation of the core, the formula $\text{core}(I) = \text{mono}(K)$ is in general more efficient computationally than the colon formula $\text{core}(I) = J^{n+1} : I^n$, as it only

requires taking colons of $d$-generated ideals. Furthermore the new formula holds without any restriction on the characteristic.

Another way to find a combinatorial description of the core of a monomial ideal is to express it as the adjoint of a power of the ideal and use the known description of adjoints in terms of Newton polyhedra. We pursue this approach in Section 4, where we show that $\mathrm{core}(I) = \mathrm{adj}(I^d)$ if $I$ is a 0-dimensional monomial ideal $I$ in a polynomial ring $k[x_1, \ldots, x_d]$ of characteristic zero and all large powers of $I$ are integrally closed or nearly integrally closed (see Theorem 4.12, which uses Boutot's Theorem [1], or Theorem 4.11 featuring a special case with an elementary proof). On the other hand, the assumption on the integral closedness is not always necessary, for in Sections 6 and 7 we present classes of ideals in dimension two and three for which this condition fails, whereas $\mathrm{core}(I) = \mathrm{adj}(I^d)$. Our results of Section 4 are based on the fact that both the core and the adjoint can be related to components of the graded canonical module $\omega_{R[It,t^{-1}]}$ of the extended Rees algebra $R[It, t^{-1}]$. This approach also led us to study the core by means of the first coefficient ideal $\check{I}$ of $I$. Let $D = \mathrm{End}(\omega_{R[It,t^{-1}]})$ denote the $S_2$-ification of the extended Rees algebra of $I$ and define $\check{I}$ to be the $R$-ideal with $D_1 = \check{I}t$; this ideal is also the first coefficient ideal of $I$, the largest ideal that has the same zeroth and first Hilbert coefficient as $I$ [28, 2]. As remarked before, the core may change as one passes from $I$ to its integral closure $\bar{I}$, however we show in Theorem 4.3 that one can replace $I$ by any ideal between $I$ and $\check{I}$ to compute the core, assuming that $I$ is a 0-dimensional monomial ideal in characteristic zero. If $I$ has a reduction generated by a monomial regular sequence we prove in fact that $\check{I}$ is the unique largest ideal integral over $I$ that shares the same core (see Corollary 4.9).

In Sections 6 and 7 we determine explicitly the core of ideals generated by monomials of the same degree, in a polynomial ring in $d \leq 3$ variables. For instance, consider the case $d = 2$ and write $I = \mu(x^n, y^n, \{x^{n-k_i}y^{k_i}\})$ with $\mu$ a monomial. We show that $\mathrm{core}(I) = \mu(x^\delta, y^\delta)^{2\frac{n}{\delta}-1}$ where $\delta = \gcd(\{k_i\}, n)$ (see Theorem 6.4). In particular if $\mu = 1$ and $\delta = 1$, then the core of $I$ is a power of the maximal ideal and $\mathrm{core}(I)$ equals $\mathrm{adj}(I^2)$ even though $I$ need not be integrally closed (see Corollary 6.6).

## 2. PRELIMINARIES

In this section we prove some general facts about cores in rings that are not necessarily local. First we deal with the behavior of cores under localization. This issue was addressed in [4] for local rings. Now instead we assume that the ideal be 0-dimensional in order to assure that the core is a finite intersection of reductions. We then use the results of [23, 13, 6] to obtain explicit formulas for the core in global rings.

**Proposition 2.1.** *Let $R$ be a Noetherian ring, $S$ a multiplicative subset of $R$, and $I$ a 0-dimensional ideal. Then*

$$\mathrm{core}(S^{-1}I) = S^{-1}\mathrm{core}(I).$$

*Proof.* Notice that there exists an integer $N \geq 0$ such that $I^N \subset J$ for every reduction $J$ of $I$ [34, 2.4]. From this it follows that $\text{core}(I)$ is 0-dimensional. Hence $R/\text{core}(I)$ is Artinian, which implies that $\text{core}(I)$ is a finite intersection of reductions. Say $\text{core}(I) = \bigcap_{i=1}^{t} J_i$. The inclusion $\text{core}(S^{-1}I) \subset S^{-1}\text{core}(I)$ follows from

$$\text{core}(S^{-1}I) \subset \bigcap_{i=1}^{t} S^{-1}J_i = S^{-1}\bigcap_{i=1}^{t} J_i = S^{-1}\text{core}(I).$$

To prove that $S^{-1}\text{core}(I) \subset \text{core}(S^{-1}I)$ we will show that every reduction of $S^{-1}I$ is the localization of a reduction of $I$. Let $\mathcal{J} \subset S^{-1}R$ be a reduction of $S^{-1}I$ and consider $J = \mathcal{J} \cap I$. Obviously $S^{-1}J = \mathcal{J}$. We claim that $J$ is a reduction of $I$. It suffices to prove this locally at every prime $\mathfrak{p}$ of $R$. If $(\mathcal{J} \cap R)_{\mathfrak{p}} = R_{\mathfrak{p}}$ then $J_{\mathfrak{p}} = I_{\mathfrak{p}}$. Now assume that $(\mathcal{J} \cap R)_{\mathfrak{p}} \neq R_{\mathfrak{p}}$. For every minimal prime $\mathfrak{q}$ of $\mathcal{J} \cap R$, the ideal $S^{-1}\mathfrak{q}$ is a minimal prime of $\mathcal{J}$, hence of $S^{-1}I$. Therefore $\mathfrak{q}$ is a minimal prime of $I$, showing that $\mathcal{J} \cap R$ is 0-dimensional. Hence $\mathfrak{p}$ is a minimal prime of $\mathcal{J} \cap R$. Therefore as before $S^{-1}\mathfrak{p}$ is a minimal prime of $\mathcal{J}$, which gives $R_{\mathfrak{p}} = (S^{-1}R)_{S^{-1}\mathfrak{p}}$. Hence $J_{\mathfrak{p}} = \mathcal{J}_{S^{-1}\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$. $\qquad\square$

Let $R$ be a ring. Recall that if $J$ is a reduction of an $R$-ideal $I$, then the *reduction number* $r_J(I)$ of $I$ with respect to $J$ is the smallest nonnegative integer $r$ with $I^{r+1} = JI^r$. For a sequence $\underline{\alpha} = \alpha_1, \ldots, \alpha_s$ of elements in $R$ and a positive integer $t$, we write $\underline{\alpha}^t$ for the sequence $\alpha_1^t, \ldots, \alpha_s^t$. If $L$ is a monomial ideal in a polynomial ring with minimal monomial generators $\underline{\alpha} = \alpha_1, \ldots, \alpha_s$, write $L^{\langle t \rangle} = (\underline{\alpha}^t)$.

**Lemma 2.2.** *Let $R$ be a Noetherian ring, and let $I$ be an ideal with $g = \text{ht}\, I > 0$ having a reduction generated by a regular sequence $\underline{\alpha}$. Then for $t \geq r_{(\underline{\alpha})}(I)$ and $i \geq 0$,*

$$(\underline{\alpha})^{t+i} : I^t = (\underline{\alpha}^{t+i}) : I^{gt+(g-1)(i-1)} = (\underline{\alpha}^{t+i}) : (I^{gt+(g-1)(i-1)}, \underline{\alpha}^{t+i}).$$

**Proof.** Since $\underline{\alpha}$ is a regular sequence we have

$$(\underline{\alpha}^{t+i}) : (\underline{\alpha})^{(g-1)(t+i-1)} = (\underline{\alpha})^{t+i}.$$

Hence for $t \geq r_{(\underline{\alpha})}(I)$,

$$
\begin{aligned}
(\underline{\alpha})^{t+i} : I^t &= ((\underline{\alpha}^{t+i}) : (\underline{\alpha})^{(g-1)(t+i-1)}) : I^t \\
&= (\underline{\alpha}^{t+i}) : (\underline{\alpha})^{(g-1)(t+i-1)} I^t \\
&= (\underline{\alpha}^{t+i}) : I^{gt+(g-1)(i-1)} \\
&= (\underline{\alpha}^{t+i}) : (I^{gt+(g-1)(i-1)}, \underline{\alpha}^{t+i}).
\end{aligned}
$$

$\qquad\square$

We are now ready to state the formulas for the core that we will use throughout:

**Theorem 2.3.** *Let $R$ be a Cohen-Macaulay ring containing an infinite field $k$ and $I$ a 0-dimensional ideal of height $d > 0$ having a reduction generated by a regular sequence $\underline{\alpha}$. Assume that $\text{char}\, k = 0$ or $\text{char}\, k > r_{(\underline{\alpha})}(I)$. Then for $t \geq r_{(\underline{\alpha})}(I)$,*

$$\text{core}(I) = (\underline{\alpha})^{t+1} : I^t = (\underline{\alpha}^{t+1}) : I^{dt} = (\underline{\alpha}^{t+1}) : (I^{dt}, \underline{\alpha}^{t+1}).$$

**Proof.** Proposition 2.1, [13, 3.7], and [23, 3.4] show that $\mathrm{core}(I) = (\underline{\alpha})^{t+1} : I^t$ for $t \geq r_{(\underline{\alpha})}(I)$. The last two equalities follow from Lemma 2.2. $\qquad\square$

**Remark 2.4.** If in Theorem 2.3 the ideal $I$ is unmixed then the assumption that $I$ has a reduction generated by a regular sequence is automatically satisfied, as can be seen from basic element theory. For a more general result we refer to [21, Theorem].

In the graded case, the assumption on the characteristic in Theorem 2.3 can be dropped:

**Theorem 2.5.** *Let $R$ be a Cohen-Macaulay geometrically reduced positively graded ring over an infinite field and $I$ a $0$-dimensional ideal of height $d > 0$ generated by forms of the same degree. Let $\underline{\alpha}$ be a homogeneous regular sequence generating a reduction of $I$. Then for $t \geq r_{(\underline{\alpha})}(I)$,*

$$\mathrm{core}(I) = (\underline{\alpha})^{t+1} : I^t = (\underline{\alpha}^{t+1}) : I^{dt} = (\underline{\alpha}^{t+1}) : (I^{dt}, \underline{\alpha}^{t+1}).$$

**Proof.** By [6, 4.1] we have $\mathrm{core}(I) = (\underline{\alpha})^{t+1} : I^t$ for $t \geq r_{(\underline{\alpha})}(I)$. The other two equalities follow from Lemma 2.2. $\qquad\square$

**Remark 2.6.** Notice that a regular sequence $\underline{\alpha}$ as in Theorem 2.5 always exists.

## 3. AN ALGORITHM

In this section we prove a formula for the core of $0$-dimensional monomial ideals. This formula gives a new interpretation of the core in terms of operations on monomial ideals and at the same time provides an algorithm that is more efficient in general than the formulas of Theorems 2.3 and 2.5. Furthermore the new approach does not require any restriction on the characteristic.

**Notation and Discussion 3.1.** Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$. For an $R$-ideal $L$ we let $\mathrm{mono}(L)$ denote the largest monomial ideal contained in $L$ and $\mathrm{Mono}(L)$ the smallest monomial ideal containing $L$. Note that $\mathrm{Mono}(L)$ is easy to compute, being the ideal generated by the monomial supports of generators of $L$. The computation of $\mathrm{mono}(L)$ is also accessible; the algorithm provided in [27, 4.4.2] computes $\mathrm{mono}(L)$ by multi-homogenizing $L$ with respect to $d$ new variables and then contracting back to the ring $R$. The ideal $\mathrm{mono}(L)$ can be computed in CoCoA with the built-in command *MonsInIdeal* .

From now on let $k$ be an infinite field and write $\mathfrak{m} = (x_1, \ldots, x_d)$ for the homogeneous maximal ideal of $R$. To begin we will use linkage to give a new algorithm to compute $\mathrm{mono}(L)$ for a class of ideals including $\mathfrak{m}$-primary ideals.

**Lemma 3.2.** *Let L be an unmixed R-ideal of height g and $\underline{\beta} \subset L$ a regular sequence consisting of g monomials. Then*

$$\mathrm{mono}(L) = (\underline{\beta}) : \mathrm{Mono}((\underline{\beta}) : L).$$

**Proof.** Notice that $(\underline{\beta}) : \mathrm{Mono}((\underline{\beta}) : L) \subset (\underline{\beta}) : ((\underline{\beta}) : L) \subset L$, where the last containment holds since $R/(\underline{\beta})$ is Gorenstein and $L$ is unmixed. Now observe that colons of monomial ideals are monomial. Hence $(\underline{\beta}) : \mathrm{Mono}((\underline{\beta}) : L) \subset \mathrm{mono}(L)$. The other inclusion follows from the following containments. First, $(\underline{\beta}) : L \subset (\underline{\beta}) : \mathrm{mono}(L)$. But $(\underline{\beta}) : \mathrm{mono}(L)$ is monomial and hence $\mathrm{Mono}((\underline{\beta}) : L) \subset (\underline{\beta}) : \mathrm{mono}(L)$. Therefore $\mathrm{mono}(L) \subset (\underline{\beta}) : \mathrm{Mono}((\underline{\beta}) : L)$. $\qquad\square$

**Notation and Discussion 3.3.** Now let $I$ denote an $\mathfrak{m}$-primary monomial ideal. For each $i$ let $n_i$ be the smallest power of $x_i$ in $I$; such $n_i$ exist since $I$ is $\mathfrak{m}$-primary. Write $\underline{\alpha} = x_1^{dn_1}, \ldots, x_d^{dn_d}$ and let $J$ be an ideal generated by $d$ general $k$-linear combinations of minimal monomial generators of $I$. If the ideal $I$ is generated by forms of the same degree, $J$ is a general minimal reduction of $I$ [22, 5.1]. In general however, $I$ and $J$ may not even have the same radical. Nevertheless, $J_\mathfrak{m}$ is a general minimal reduction of $I_\mathfrak{m}$ by [22, 5.1]. Consider the ideal $K = (J, \underline{\alpha})$. Observe that the $\mathfrak{m}$-primary ideal $K$ is a reduction of $I$. Thus $\mathrm{core}(I) \subset \mathrm{mono}(K)$ since the core is a monomial ideal. The Briançon-Skoda theorem implies $(\underline{\alpha})_\mathfrak{m} \subset \mathrm{core}(I_\mathfrak{m})$. Hence $K_\mathfrak{m} = J_\mathfrak{m}$, and whenever $I$ is generated by forms of the same degree then $K = J$. We call $K$ a *general locally minimal reduction* of $I$.

In order to prove the equality $\mathrm{core}(I) = \mathrm{mono}(K)$ we need to show that $\mathrm{mono}(K)$ is independent of $K$; by this we mean that $\mathrm{mono}(K)$ is constant as the coefficient matrix defining $J$ varies in a suitable dense open set of an affine $k$-space:

**Lemma 3.4.** *With assumptions as in 3.1 and in 3.3, the ideal $\mathrm{Mono}((\underline{\alpha}) : K)$ does not depend on the general locally minimal reduction K.*

**Proof.** Let $f_1, \ldots, f_n$ be minimal monomial generators of $I$. Let $\underline{z} = z_{ij}$, $1 \le i \le d$, $1 \le j \le n$, be variables and write $T = R[\underline{z}]$. Let $\mathcal{J}$ denote the $T$-ideal generated by the $d$ generic linear combinations $\sum_{j=1}^{n} z_{ij} f_j$, $1 \le i \le d$, and let $\mathcal{K}$ be the $T$-ideal $(\mathcal{J}, \underline{\alpha})$. For $\underline{\lambda} = \lambda_{ij}$, $1 \le i \le d$, $1 \le j \le n$, any elements in $k$, we consider the maximal ideal $\mathcal{M} = (\mathfrak{m}, \underline{z} - \underline{\lambda}) = (\mathfrak{m}, \{z_{ij} - \lambda_{ij}\})$ of $T$. We identify the set $\mathcal{A} = \{\mathcal{M} = (\mathfrak{m}, \underline{z} - \underline{\lambda}) \,|\, \underline{\lambda} \in k^{dn}\}$ with the set of $k$-rational points of the affine space $\mathbb{A}_k^{dn}$. Write $\pi_{\underline{\lambda}} : T \to R$ for the homomorphism of $R$-algebras with $\pi_{\underline{\lambda}}(z_{ij}) = \lambda_{ij}$. This map induces a local homomorphism $T_{\mathcal{M}} \to R_\mathfrak{m}$, which we still denote by $\pi_{\underline{\lambda}}$.

Notice that $\pi_{\underline{\lambda}}(\mathcal{K}) = K$ for $\underline{\lambda}$ in a dense open subset $U_1 \subset \mathbb{A}_k^{dn}$.

Now we claim that there is a dense open subset $U_2 \subset \mathbb{A}_k^{dn}$ such that $\mathcal{K}_\mathcal{M}$ is Cohen-Macaulay. Indeed, let $N$ be a $(d-1)^{\mathrm{st}}$ syzygy of the $T$-ideal $\mathcal{K}$. The free locus of $N$ is a dense open subset $\mathcal{U}$ of $\mathrm{Spec}(T)$. It contains $\mathfrak{m}T$ since $N_{\mathfrak{m}T}$ is a $(d-1)^{\mathrm{st}}$ syzygy of the ideal $\mathcal{K}_{\mathfrak{m}T}$ over the $d$-dimensional regular local ring $T_{\mathfrak{m}T}$. Thus intersecting $\mathcal{U}$ with $\mathcal{A}$ we obtain a dense open subset $U_2 \subset \mathbb{A}_k^{dn}$ where $N_\mathcal{M}$ is free. Since the ideal $\mathcal{K}_\mathcal{M}$ has height at least $d$ it is Cohen-Macaulay.

For every $\underline{\lambda} \in U_2$ the ideal $\mathcal{K}_{\mathcal{M}}$ is Cohen-Macaulay. Therefore $(\underline{\alpha}) : \mathcal{K}_{\mathcal{M}}$ specializes according to [14, 2.13], that is, $\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K}_{\mathcal{M}}) = (\underline{\alpha}) : \pi_{\underline{\lambda}}(\mathcal{K}_{\mathcal{M}})$. Thus $\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K})_{\mathfrak{m}} = ((\underline{\alpha}) : \pi_{\underline{\lambda}}(\mathcal{K}))_{\mathfrak{m}}$ because $\pi_{\underline{\lambda}}(T_{\mathcal{M}}) = R_{\mathfrak{m}}$. On the other hand, $\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K})$ is $\mathfrak{m}$-primary since $\underline{\alpha} = \pi_{\underline{\lambda}}(\underline{\alpha}) \subset \pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K})$. Therefore $\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K}) = (\underline{\alpha}) : \pi_{\underline{\lambda}}(\mathcal{K})$ for every $\underline{\lambda} \in U_2$.

We think of $T$ as a polynomial ring in $x_1, \ldots, x_d$ over $k[\underline{z}]$. Write the generators of $(\underline{\alpha}) : \mathcal{K}$ as sums of monomials in the $x$'s with coefficients $g_1(z), \ldots, g_t(z)$. The $R$-ideal $\mathrm{Mono}(\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K}))$ is independent of $\underline{\lambda}$ for $\underline{\lambda} \in U_3 = D_{g_1 \cdots g_t}$.

For $\underline{\lambda} \in U_1 \cap U_2 \cap U_3$ the $R$-ideal $K = \pi_{\underline{\lambda}}(\mathcal{K})$ is a general locally minimal reduction of $I$ and $\mathrm{Mono}((\underline{\alpha}) : K) = \mathrm{Mono}((\underline{\alpha}) : \pi_{\underline{\lambda}}(\mathcal{K})) = \mathrm{Mono}(\pi_{\underline{\lambda}}((\underline{\alpha}) : \mathcal{K}))$ does not depend on $\underline{\lambda}$. $\qquad\square$

**Corollary 3.5.** *With assumptions as in 3.1 and in 3.3, the ideal* $\mathrm{mono}(K)$ *does not depend on the general locally minimal reduction $K$.*

**Proof.** The claim follows from Lemmas 3.2 and 3.4. $\qquad\square$

We are now ready to prove the main result of this section.

**Theorem 3.6.** *With assumptions as in 3.1 and in 3.3,*
$$\mathrm{core}(I) = \mathrm{mono}(K) = (\underline{\alpha}) : \mathrm{Mono}((\underline{\alpha}) : K).$$

**Proof.** We already know that $\mathrm{core}(I) \subset \mathrm{mono}(K)$. Furthermore $\mathrm{mono}(K) = (\underline{\alpha}) : \mathrm{Mono}((\underline{\alpha}) : K)$ by Lemma 3.2. Thus it suffices to show that $\mathrm{mono}(K) \subset \mathrm{core}(I)$. From [4, 4.5] it follows that
$$\mathrm{core}(I_{\mathfrak{m}}) = (K_1)_{\mathfrak{m}} \cap \ldots \cap (K_t)_{\mathfrak{m}}$$
for general locally minimal reductions $K_1, \ldots, K_t$ of $I$. According to Corollary 3.5 we may assume that $\mathrm{mono}(K) = \mathrm{mono}(K_i)$ for $1 \leq i \leq t$. Therefore $\mathrm{mono}(K) \subset K_1 \cap \ldots \cap K_t$ and thus $\mathrm{mono}(K)_{\mathfrak{m}} \subset \mathrm{core}(I_{\mathfrak{m}}) = \mathrm{core}(I)_{\mathfrak{m}}$, where the last equality holds by Proposition 2.1. Hence $\mathrm{mono}(K) \subset \mathrm{core}(I)$ as $\mathrm{core}(I)$ is $\mathfrak{m}$-primary. $\qquad\square$

**Remark 3.7.** The above theorem gives a new interpretation of the core of a monomial ideal $I$ as the largest monomial ideal contained in a general locally minimal reduction of $I$. This idea can be easily implemented in CoCoA using a script to obtain $d$ general elements in the ideal $I$ and the built-in command *MonsInIdeal* to compute $\mathrm{mono}(K)$.

**Remark 3.8.** The formula of Theorem 2.3 does not hold in arbitrary characteristic (see [23, 4.9]). However, if $J$ and $I$ are monomial ideals, $J^{n+1} : I^n$ is obviously independent of the characteristic. On the other hand, the algorithm based on Theorem 3.6 works in any characteristic, but its output, $\mathrm{mono}(K)$, is characteristic dependent. In fact we are now going to exhibit a zero-dimensional monomial ideal $I$ for which $\mathrm{core}(I) = \mathrm{mono}(K)$ varies with the characteristic. As $I$ has a reduction $J$ generated by a monomial regular sequence this shows that the formula of Theorem 2.3 fails to hold in arbitrary characteristic even for 0-dimensional monomial ideals.

**Example 3.9.** Let $R = k[x,y]$ be a polynomial ring over an infinite field $k$, consider the ideal $I = (x^6, x^5 y^3, x^4 y^4, x^2 y^8, y^9)$, and write $J = (x^6, y^9)$. One has $r_J(I) = 2$. If $\operatorname{char} k \neq 2$ then the formula of Theorem 2.3 as well as the algorithm of Theorem 3.6 give $\operatorname{core}(I) = J^3 : I^2 = J(x^4, x^3 y, x^2 y^2, xy^5, y^6) = (x^{10}, x^9 y, x^8 y^2, x^7 y^5, x^6 y^6, x^4 y^9, x^3 y^{10}, x^2 y^{11}, xy^{14}, y^{15})$. On the other hand, if $\operatorname{char} k = 2$ then Theorem 3.6 shows that $\operatorname{core}(I) = (x^{10}, x^8 y, x^7 y^5, x^6 y^6, x^4 y^9, x^3 y^{10}, x^2 y^{11}, xy^{14}, y^{15}) \supsetneq J^3 : I^2$.

## 4. THE CORE, THE FIRST COEFFICIENT IDEAL AND THE ADJOINT

**Notation and Discussion 4.1.** Let $R$ be a Gorenstein ring, let $I$ be an $R$-ideal with $g = \operatorname{ht} I > 0$, and assume that $I$ has a reduction $J$ which is locally a complete intersection of height $g$. Consider the inclusions

$$A = R[Jt, t^{-1}] \subset B = R[It, t^{-1}] \subset R[t, t^{-1}].$$

Notice that $A$ is a Gorenstein ring. We define $\omega_A = At^{g-1} \subset R[t, t^{-1}]$ and write $-^{\vee} = \operatorname{Hom}_A(-, \omega_A)$, $F = \operatorname{Quot}(R[t])$. We may choose $\omega_B = \omega_A :_{R[t,t^{-1}]} B = \omega_A :_F B \simeq B^{\vee}$ as a graded canonical module of $B$. According to [23, 2.2.2],

$$(1) \qquad\qquad \omega_B = \oplus_i (J^{s+i-g+1} : I^s) t^i$$

for every $s \geq r_J(I)$. Observe that $[\omega_B]_i = Rt^i$ for $i \ll 0$. Write

$$
\begin{aligned}
D &= \omega_B :_{R[t,t^{-1}]} \omega_B \\
&= \omega_B :_F \omega_B \\
&= \omega_A :_F \omega_B \\
&= A :_F (A :_F B) \\
&= A :_{R[t,t^{-1}]} (A :_{R[t,t^{-1}]} B).
\end{aligned}
$$

Notice that $D \simeq \operatorname{End}_B(\omega_B) \simeq B^{\vee\vee}$ is an $S_2$-fication of $B$. Define $\check{I}$ to be the $R$-ideal with $[D]_1 = \check{I} t$. One has $I \subset \check{I} \subset \bar{I}$, and $\check{I}$ is the first coefficient ideal of $I$ in the sense of [28, 2, 3]. Finally, write $C = R[\check{I} t, t^{-1}]$. The inclusions $B \subset C \subset D$ are equalities locally in codimension one in $A$, and hence upon applying $\omega_A :_F - \simeq -^{\vee}$ yield equalities

$$(2) \qquad\qquad \omega_B = \omega_C = \omega_D.$$

We first give a formula expressing $D$ and $\check{I}$ in terms of colon ideals. For this we need to consider an integer $u \geq 0$ such that the graded canonical module of $B = R[It, t^{-1}]$ is generated in degrees at most $g - 1 + u$ as a module over $A = R[Jt, t^{-1}]$. Whenever $I$ is a monomial ideal one can take $u = 0$, as we will see in Theorem 4.6. However, this is not longer true if $I$ is not monomial and $B$ is not Cohen-Macaulay, see [23, 4.13].

**Theorem 4.2.** *In addition to the assumptions of 4.1 suppose that $R$ is regular. Let $s \geq r_J(I)$ be an integer and $u \geq 0$ an integer such that $J^{s+u+i} : I^s = J^i(J^{s+u} : I^s)$ for every $i \geq 0$. One has*

$$D = \oplus_i (J^{i+u} : (J^{s+u} : I^s))t^i.$$

*In particular*

$$\check{I} = J^{1+u} : (J^{s+u} : I^s).$$

**Proof.** We need to prove that $D = A :_{R[t,t^{-1}]} (J^{s+u} :_R I^s)t^u$. The Briançon-Skoda Theorem [20, Theorem 1] gives $I^{s+i} \subset J^{s+i-g+1}$ for every integer $i$, hence $J^i \subset J^{s+i-g+1} :_R I^s$. Now Equation (1) shows that $A \subset \omega_B$. The same equation and our assumption also give $[\omega_B]_i = (J^{s+u} :_R I^s)[t^u\omega_A]_i$ for $i \geq g - 1 + u$. Hence writing $L = A + (J^{s+u} :_R I^s)t^u\omega_A$ we obtain an exact sequence of graded $A$-modules

$$0 \longrightarrow L \longrightarrow \omega_B \longrightarrow N \longrightarrow 0,$$

with $N$ concentrated in finitely many degrees. It follows that $N$ has grade $\geq 2$.

Thus applying $\omega_A :_F - \simeq -^\vee$ yields

$$
\begin{aligned}
D &= \omega_A :_F \omega_B \\
&= \omega_A :_F L \\
&= (\omega_A :_F A) \cap (\omega_A :_F (J^{s+u} :_R I^s)t^u\omega_A) \\
&= \omega_A \cap (A :_F (J^{s+u} :_R I^s)t^u) \\
&= A :_{\omega_A} (J^{s+u} :_R I^s)t^u.
\end{aligned}
$$

As $J^{g-1+u} \subset J^{s+u} :_R I^s$ we obtain

$$J^{i+u} :_R (J^{s+u} :_R I^s) \subset J^{i+u} :_R J^{g-1+u} = J^{i-g+1},$$

where the last equality holds because $\mathrm{gr}_J(R)$ is Cohen-Macaulay and $\mathrm{ht}\, J > 0$. Thus $A :_{R[t,t^{-1}]} (J^s :_R I^s)t^u \subset \omega_A$, showing that

$$A :_{R[t,t^{-1}]} (J^{s+u} :_R I^s)t^u = A :_{\omega_A} (J^{s+u} :_R I^s)t^u = D.$$

$\square$

In many cases all ideals between $I$ and $\check{I}$ have the same core:

**Theorem 4.3.** *In addition to the assumptions of 4.1 suppose that $R$ contains an infinite field $k$ with $\mathrm{char}\, k = 0$ or $\mathrm{char}\, k > r_J(I)$. Further assume that $R$ is local or $I$ is 0-dimensional. Then $\mathrm{core}(I) = \mathrm{core}(\check{I})$.*

**Proof.** By Proposition 2.1 and [23, 4.8] we have $J^{s+1} : \check{I}^s \subset \mathrm{core}(\check{I})$ for $s \gg 0$. On the other hand $\mathrm{core}(\check{I}) \subset \mathrm{core}(I)$ since $\check{I}$ is integral over $I$. From Proposition 2.1 and [23, 4.5] we obtain $\mathrm{core}(I) = J^{s+1} : I^s$. Finally, Equations (1) and (2) show that

$$(J^{s+1} : I^s)t^g = [\omega_B]_g = [\omega_C]_g = (J^{s+1} : \check{I}^s)t^g.$$

$\square$

**Theorem 4.4.** *Let $R$ be a Gorenstein geometrically reduced positively graded ring over an infinite field and $I$ a 0-dimensional ideal generated by forms of the same degree. Then $\operatorname{core}(I) = \operatorname{core}(\check{I})$.*

**Proof.** Let $J$ be a reduction of $I$ generated by a homogeneous regular sequence and $s \gg 0$ an integer. As in the proof of Theorem 4.3 one sees that $J^{s+1} : I^s = J^{s+1} : \check{I}^s \subset \operatorname{core}(\check{I}) \subset \operatorname{core}(I)$. Furthermore from Theorem 2.5 we obtain $\operatorname{core}(I) = J^{s+1} : I^s$. $\qquad\square$

**Assumptions 4.5.** Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field $k$ and write $\mathfrak{m} = (x_1, \ldots, x_d)$ for the homogeneous maximal ideal of $R$. Let $I \neq 0$ be a monomial ideal of height $g$ and let $\mathfrak{a}$ be an ideal generated by $g$ $k$-linear combinations of the minimal monomial generators of $I$. We assume that $I$ has a reduction $J$ generated by a regular sequence of monomials, and we write $r$ for the reduction number of $I$ with respect to $J$.

Now our goal is to express $\check{I}$ as a colon ideal and to prove that under certain conditions, $\check{I}$ is the unique largest ideal in $\bar{I}$ having the same core as $I$. For this we need the next theorem, which says that we may take $u = 0$ in Theorem 4.2 provided we are in the setting of 4.5.

**Theorem 4.6.** *With assumptions as in 4.5 one has for every $s \geq r$ and every $i \geq 0$,*

$$J^{s+i} : I^s = J^i(J^s : I^s)$$

*and*

$$(\mathfrak{a}^{s+i} : I^s)_{\mathfrak{m}} = \mathfrak{a}^i(\mathfrak{a}^s : I^s)_{\mathfrak{m}}.$$

**Proof.** To prove the first equality write $f_1, \ldots, f_g$ for the monomial generators of $J$. Clearly $J^i(J^s : I^s) \subset J^{s+i} : I^s$. Notice also that $J^{s+i} : I^s \subset J^{s+i} : J^s \subset J^i$ since $J$ is generated by a regular sequence. Let $f$ be a monomial contained in $J^{s+i} : I^s$, and write $f = f_{j_1} \cdots f_{j_i} \cdot h$. Observe that $f_{j_1} \cdots f_{j_i} \cdot hI^s = fI^s \subset J^{s+i}$. Therefore $hI^s \subset J^{s+i} : (f_{j_1} \cdots f_{j_i}) = J^s$. Hence $h \in J^s : I^s$, which gives $f \in J^i(J^s : I^s)$.

To prove the second equality notice that $r_{\mathfrak{a}_{\mathfrak{m}}}(I_{\mathfrak{m}}) \leq r$ [29, 3.4] and hence $(\mathfrak{a}^{s+i} : I^s)_{\mathfrak{m}} = (J^{s+i} : I^s)_{\mathfrak{m}}$ by Equation (1). Also observe that $(J^{s+i+1} : I^s)_{\mathfrak{m}} = \mathfrak{a}(J^{s+i} : I^s)_{\mathfrak{m}}$ whenever $i \geq i_0$ for some fixed integer $i_0$, because $\omega_B \otimes_R R_{\mathfrak{m}}$ is finitely generated as a graded module over $R_{\mathfrak{m}}[\mathfrak{a}t, t^{-1}]$. Hence it suffices to prove that $(J^{s+i} : I^s)_{\mathfrak{m}} = \mathfrak{a}^i(J^s : I^s)_{\mathfrak{m}}$ for each of the finitely many $i$ in the range $0 \leq i \leq i_0$. We write $H = (J^{s+i} : I^s)_{\mathfrak{m}}$ and $K = (J^s : I^s)_{\mathfrak{m}}$. Notice that $I^i K \subset H$ by Equation (1) since $\omega_B$ is a $B$-module.

We complete $f_1, \ldots, f_g$ to monomial generators $f_1, \ldots, f_n$ of $I$. Let $\underline{z} = z_{ij}$, $1 \leq i \leq g$, $1 \leq j \leq n$, be variables and write $T = R_{\mathfrak{m}}[\underline{z}]$. Let $\mathcal{J}$ denote the $T$-ideal generated by the $g$ generic linear combinations $\sum_{j=1}^{n} z_{ij} f_j$, $1 \leq i \leq g$. Notice that $\mathcal{J}^i KT \subset HT$ as $\mathcal{J} \subset IT$. Since $H = J^i K$ and $\mathcal{J}$ specializes to $J_{\mathfrak{m}}$ modulo $(\{z_{ij} - \delta_{ij}\})$, it follows that $HT = \mathcal{J}^i KT + [(\{z_{ij} - \delta_{ij}\}) \cap HT]$. Consider the maximal ideal $\mathcal{M} = (\mathfrak{m}, \underline{z} - \underline{\delta}) = (\mathfrak{m}, \{z_{ij} - \delta_{ij}\})$ of $T$. As $\underline{z} - \underline{\delta}$ form a regular sequence on $T_{\mathcal{M}}$ and $T_{\mathcal{M}}/HT_{\mathcal{M}}$, we conclude that $HT_{\mathcal{M}} = \mathcal{J}^i KT_{\mathcal{M}}$ according to Nakayama's Lemma. For $\underline{\lambda} = \lambda_{ij}$, $1 \leq i \leq g$, $1 \leq j \leq n$, any elements in $k$, we consider the maximal ideal $\mathcal{M}_\lambda = (\mathfrak{m}, \underline{z} - \underline{\lambda}) = (\mathfrak{m}, \{z_{ij} - \lambda_{ij}\})$

of $T$. We identify the set $\mathcal{A} = \{\mathcal{M}_{\underline{\lambda}} \mid \underline{\lambda} \in k^{gn}\}$ with the set of $k$-rational points of the affine space $\mathbb{A}_k^{gn}$. Since the two ideals $HT$ and $\mathcal{J}^i KT$ coincide locally at $\mathcal{M} = \mathcal{M}_{\underline{\delta}}$ the same holds locally at $\mathcal{M}_{\underline{\lambda}}$ for $\underline{\lambda}$ in a dense open neighborhood of $\underline{\delta}$ in $\mathbb{A}_k^{gn}$. Specializing modulo $\underline{z} - \underline{\lambda}$ we conclude that $H = \mathfrak{a}^i K$. $\square$

**Corollary 4.7.** *With assumptions as in 4.5 one has for every $s \geq r$,*

$$\check{I} = J \colon (J^s \colon I^s)$$

*and*

$$\check{I}_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}} \colon (\mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s).$$

**Proof.** We use Theorems 4.2 and 4.6. $\square$

**Corollary 4.8.** *In addition to the assumptions of 4.5 let $H$ be an ideal integral over $I$. If $J^{t+i} \colon H^t = J^{t+i} \colon I^t$ for some $i \geq 0$ and $t \gg 0$, then $\omega_{R[Ht,t^{-1}]} = \omega_{R[It,t^{-1}]}$.*

**Proof.** Write $A = R[Jt,t^{-1}]$. We have an inclusion of finitely generated graded $A$-modules

$$\omega_{R[Ht,t^{-1}]} \subset \omega_{R[It,t^{-1}]}.$$

By our assumption these modules coincide in degree $g+i-1$ according to Equation (1). By Theorem 4.6 the canonical module $\omega_{R[It,t^{-1}]}$ is generated in degrees $\leq g-1$ as an $A$-module, which forces the two modules to be the same in degrees $\geq g+i-1$. Furthermore the two modules coincide in degrees $\ll 0$. Since they satisfy $S_2$ it then follows that they are equal. $\square$

**Corollary 4.9.** *In addition to the assumptions of 4.5 suppose that $I$ is 0-dimensional.*

(a) *Let $H$ be an ideal integral over $I$ with the same core as $I$. If $H$ and $I$ are generated by forms of the same degree or if $\operatorname{char} k = 0$, then $\omega_{R[Ht,t^{-1}]} = \omega_{R[It,t^{-1}]}$.*
(b) *If $\operatorname{char} k = 0$ then the ideal $\check{I}$ is the largest ideal integral over $I$ with the same core as $I$.*

**Proof.** To prove part (a) notice that $J^{t+1} \colon I^t = \operatorname{core}(I) = \operatorname{core}(H) = J^{t+1} \colon H^t$ for $t \gg 0$ by the first equality in Theorems 2.5 or 2.3. Now apply Corollary 4.8.

Part (b) follows from part (a). Indeed, by (a) if $H$ is an ideal integral over $I$ with the same core as $I$ then $\check{I} = \check{H}$. On the other hand, $\operatorname{core}(I) = \operatorname{core}(\check{I})$ by Theorem 4.3. $\square$

The next corollary shows that in some cases the Rees ring of a monomial ideal is Cohen-Macaulay if it satisfies $S_2$. Monomial algebras in general are Cohen-Macaulay provided they are normal, but the $S_2$ property does not suffice [8, Theorem 1 and Remark 4].

**Corollary 4.10.** *In addition to the assumptions of 4.5 suppose that $d = 2$. One has:*

(a) $r_J(\check{I}) \leq 1$.
(b) $R[\check{I}t]$ *is the $S_2-$ification of $R[It]$ and it is Cohen-Macaulay.*
(c) *If $R[It]$ satisfies $S_2$ then it is Cohen-Macaulay.*

**Proof.** To prove part (a) we may replace $I$ by $\check{I}$ to assume $\check{I} = I$. Observe that by Corollary 4.7, $I_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}} \colon (\mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s)$ for $s \gg 0$. However, $\mathfrak{a}_{\mathfrak{m}} \subset \mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s$ according to the Briançon-Skoda Theorem [20, Theorem 1]. Therefore $\mathfrak{a}_{\mathfrak{m}} \colon I_{\mathfrak{m}} = \mathfrak{a}_{\mathfrak{m}} \colon (\mathfrak{a}_{\mathfrak{m}} \colon (\mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s)) = \mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s$. Since $\mathfrak{a}_{\mathfrak{m}}^s \colon I_{\mathfrak{m}}^s$ is the degree $g - 1$ component of the canonical module of $R_{\mathfrak{m}}[I_{\mathfrak{m}}t, t^{-1}]$, it does not depend on $\mathfrak{a}_{\mathfrak{m}}$. Hence the ideal $I_{\mathfrak{m}}$ is balanced [31, 3.6]. Therefore, $I_{\mathfrak{m}}$ has reduction number at most 1 according to [31, 4.8]. It follows that $r_J(I) \leq 1$.

To prove (b) and (c) observe that part (a), [33, 3.1], and [7, 3.10] imply the Cohen-Macaulayness of the Rees algebra of $\check{I}_{\mathfrak{m}}$ and hence of $\check{I}$. $\qquad\square$

We now turn to the relationship between cores and adjoints as defined in [19, 1.1]. Whenever the core is an adjoint one has a combinatorial description of the former in terms of a Newton polyhedron. In fact Howald has shown that if $I$ is a monomial ideal then its adjoint (or multiplier ideal) $\mathrm{adj}(I)$ is the monomial ideal with exponent set $\{\alpha \in \mathbb{Z}_{\geq 0}^d \mid \alpha + \mathbf{1} \in \mathrm{NP}^\circ(I)\}$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_{\geq 0}^d$ and $\mathrm{NP}^\circ(I)$ denotes the interior of the Newton polyhedron of $I$ [9, Main Theorem] (see also [30, 16.5.3]).

**Theorem 4.11.** *In addition to the assumptions of 4.5 suppose that $I$ is $0$-dimensional. Assume that* $\mathrm{char}\, k = 0$, $\mathrm{char}\, k > r_J(I)$, *or $I$ is generated by monomials of the same degree. If* $\overline{I^{dt}} \subset (I^{dt}, J^{\langle t+1 \rangle})$ *for some $t \geq \max\{r_J(I), d - 1\}$, then* $\mathrm{core}(I) = \mathrm{adj}(I^d)$.

**Proof.** One has $\mathrm{adj}(I^d) \subset \mathrm{adj}(I_{\mathfrak{m}}^d) \cap R$ by the definition of the adjoint. On the other hand [19, 1.4.1(ii)] shows that $\mathrm{adj}(I_{\mathfrak{m}}^d) \subset \mathrm{core}(I_{\mathfrak{m}})$. Finally $\mathrm{core}(I_{\mathfrak{m}}) \cap R = \mathrm{core}(I)$ according to Proposition 2.1. Therefore $\mathrm{adj}(I^d) \subset \mathrm{core}(I)$.

To show the reverse inclusion notice that $\mathrm{core}(I) = J^{\langle t+1 \rangle} \colon I^{dt} = J^{\langle t+1 \rangle} \colon \overline{I^{dt}}$, where the first equality holds by Theorems 2.3 and 2.5, and the second equality follows from our assumption on $I$. Thus it suffices to show that $J^{\langle t+1 \rangle} \colon \overline{I^{dt}} \subset \mathrm{adj}(I^d)$.

Write $J = (x_1^{n_1}, \ldots, x_d^{n_d})$ and $L = \mathrm{lcm}(n_1, \ldots, n_d)$. Consider the vectors $\mathbf{n} = (n_1, \ldots, n_d)$, $\omega = (L/n_1, \ldots, L/n_d)$ and $\mathbf{1} = (1, \ldots, 1)$ in $\mathbb{Z}_{\geq 0}^d$. Let $x^\alpha \notin \mathrm{adj}(I^d)$. We need to show that $x^\alpha \notin J^{\langle t+1 \rangle} \colon \overline{I^{dt}}$. As $J^{\langle t+1 \rangle} \subset J^d \subset \mathrm{adj}(I^d)$ we conclude $x^\alpha \notin J^{\langle t+1 \rangle}$. Thus writing $\beta = (t+1)\mathbf{n} - \alpha - \mathbf{1}$, we have $\beta \in \mathbb{Z}_{\geq 0}^d$ and $x^\alpha x^\beta \notin J^{\langle t+1 \rangle}$. It remains to prove that $x^\beta \in \overline{I^{dt}} = \overline{J^{dt}}$ or equivalently that $\omega \cdot \beta \geq dtL$. Indeed, as $x^\alpha \notin \mathrm{adj}(I^d) = \mathrm{adj}(J^d)$, [9, Main Theorem] (see also [30, 16.5.3]) gives $\omega \cdot \alpha \leq dL - \omega \cdot \mathbf{1}$. Hence

$$
\begin{aligned}
\omega \cdot \beta &= (t+1)\omega \cdot \mathbf{n} - \omega \cdot \alpha - \omega \cdot \mathbf{1} \\
&= (t+1)dL - \omega \cdot \alpha - \omega \cdot \mathbf{1} \\
&\geq (t+1)dL - (dL - \omega \cdot \mathbf{1}) - \omega \cdot \mathbf{1} \\
&= dtL.
\end{aligned}
$$

$\qquad\square$

In characteristic 0 one has a characterization for when $\mathrm{core}(I) = \mathrm{adj}(I^d)$ even when the monomial ideal $I$ does not have a reduction generated by a regular sequence of monomial. However, the proof of this fact, which generalizes [16, 5.3.4], is less elementary than the one above.

**Theorem 4.12.** *Let $R = k[x_1,\ldots,x_d]$ be a polynomial ring over a field $k$ of characteristic $0$. Let $I$ be a $0$-dimensional monomial ideal and let $\underline{\alpha}$ be a regular sequence generating a reduction of $I$. Then*

$$\mathrm{adj}(I^d) = (\underline{\alpha})^{t+1} : \overline{I^t} \subset (\underline{\alpha})^{t+1} : I^t = \mathrm{core}(I)$$

*for every $t \geq \max\{r_{(\underline{\alpha})}(I), d-1\}$, and equality holds if and only if $\overline{I^{dt}} \subset (I^{dt}, \underline{\alpha}^{t+1})$ for some $t \geq \max\{r_{(\underline{\alpha})}(I), d-1\}$.*

**Proof.** Let $\overline{B}$ denote the integral closure of $B = R[It, t^{-1}]$ in $R[t, t^{-1}]$. According to [8, Proposition 1] the integral closure $\overline{B}$ is a direct summand of a polynomial ring over $k$, hence [1, Théorème] shows that $\overline{B}$ has only rational singularities. Likewise $\overline{R[It]}$ is Cohen-Macaulay by the same references or [8, Theorem 1]. According to Proposition 2.1 and since $\mathrm{adj}(I^d) = \cap\,\mathrm{adj}(I_{\mathfrak{m}}^d)$, where the intersection is taken over all maximal ideals $\mathfrak{m}$ of $R$, we may replace $R$ by any of its localizations $R_{\mathfrak{m}}$. As $\overline{B}$ has rational singularities, one obtains $\mathrm{adj}(I^d) = [\omega_{\overline{B}}]_d$, which can be deduced from [19, 1.3.1] (see [32] for details). According to [24] the Cohen-Macaulayness of $\overline{R[It]}$ implies that $\overline{I^j} = (\underline{\alpha})^{j-d+1}\overline{I^{d-1}}$ for every $j \geq d-1$. Now a computation as in [23, 2.2.2] yields $[\omega_{\overline{B}}]_d = (\underline{\alpha})^{t+1} : \overline{I^t} = (\underline{\alpha}^{t+1}) : \overline{I^{dt}}$ for every $t \geq d-1$, where the last equality follows as in Lemma 2.2. Therefore $\mathrm{adj}(I^d) = (\underline{\alpha})^{t+1} : \overline{I^t} = (\underline{\alpha}^{t+1}) : \overline{I^{dt}}$. On the other hand $\mathrm{core}(I) = (\underline{\alpha})^{t+1} : I^t = (\underline{\alpha}^{t+1}) : I^{dt}$ for every $t \geq r_{(\underline{\alpha})}(I)$ according to Theorem 2.3, and the assertion follows. $\square$

Notice that if equality holds in the previous theorem then $\mathrm{core}(I) = \mathrm{core}(\overline{I})$. This condition is necessary for the core to be the adjoint of $I^d$ as $\mathrm{adj}(I^d) = \mathrm{adj}(\overline{I}^d) \subset \mathrm{core}(\overline{I}) \subset \mathrm{core}(I)$. On the other hand, the next example shows that the core may not coincide with the adjoint even if the monomial ideal $I$ is integrally closed.

**Example 4.13.** Let $k[x,y,z]$ be a polynomial ring over an infinite field $k$ with $\mathrm{char}\,k \neq 2$ and let $\mathfrak{m}$ denote the homogeneous maximal ideal. Consider the ideal $I = \overline{(x^3, y^4, z^5)}$ and write $J = (x^3, y^4, z^5)$. One has $r_J(I) = 2$. From the formula of Theorem 2.3 we obtain $\mathrm{core}(I) = \mathfrak{m}I^2$. Notice that $x^2 y^3 z^4 \notin \mathfrak{m}I^2$, whereas $(x^2 y^3 z^4)^2 \in (\mathfrak{m}I^2)^2$. Thus $\mathrm{core}(I)$ is not integrally closed although $I$ is. In particular $\mathrm{core}(I)$ cannot be an adjoint ideal because adjoints are always integrally closed. Also notice that the Rees algebra $R[It]$ is Cohen-Macaulay because $I$ is integrally closed with $r_J(I) \leq 2$, see [11, p. 317], [18, Theorem 1], [33, 3.1], [7, 3.10].

## 5. THE CORE IN WEIGHTED POLYNOMIAL RINGS

For a positively graded ring $S$ and a positive integer $n$ we let $S_{\geq n}$ denote the homogeneous $S$-ideal $\oplus_{\ell \geq n} S_\ell$. Notice that $S_{\geq n}$ is not necessarily generated in degree $n$. In this section we study the core of ideals of the form $S_{\geq n}$, where $S$ is a weighted polynomial ring. The case of section rings of line

bundles has been been considered by Hyry and Smith in connection with a conjecture by Kawamata (see [16, 17]). For us, the ideals $S_{\geq n}$ are mainly interesting because they shed light on the core of monomial ideals in standard graded polynomial rings, as will be explained in Section 7.

**Lemma 5.1.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field $k$, $S = k[x_1^{a_1}, \ldots, x_d^{a_d}]$, $n$ a multiple of $\mathrm{lcm}(a_1, \ldots, a_d)$, and $J$ the $S$-ideal generated by $x_1^n, \ldots, x_d^n$. The following hold:*

(a) *$J^i$ is a reduction of $S_{\geq in}$ for every $i \geq 1$.*

(b) *If the $S$-ideal $S_{\geq n}$ is normal then*

$$J^{\langle t+1 \rangle} :_S (S_{\geq n})^{dt} = J^{\langle t+1 \rangle} :_S S_{\geq dnt} = S_{\geq dn - \sum a_i + 1} \quad \text{for } t \geq d - 1.$$

**Proof.** For every monomial $f \in S_{\geq in}$ we have $f^n \in J^{in}$. This gives part (a).

To prove part (b) notice that $(S_{\geq n})^{dt} = S_{\geq dnt}$ by part (a) as $(S_{\geq n})^{dt}$ is integrally closed. Thus it suffices to show the second equality. Since $t \geq d - 1$ we have $J^{\langle t+1 \rangle} \subset S_{\geq n(t+1)} \subset S_{\geq dn - \sum a_i + 1}$, and we may pass to the ring $A = S/J^{\langle t+1 \rangle}$. Notice that $A$ is an Artinian graded Gorenstein ring with socle degree $dn(t+1) - \sum a_i$. Therefore $0 :_A (A_{\geq dnt}) = A_{\geq dn - \sum a_i + 1}$. Indeed, to see that the left hand side is contained in the right hand side, let $f \neq 0$ be a homogeneous element in $0 :_A (A_{\geq dnt})$. There exists a homogeneous element $\lambda \in A$ such that $0 \neq \lambda f \in \mathrm{soc}(A)$. In particular $\deg(\lambda) < dnt$ and $\deg(\lambda f) = dn(t+1) - \sum a_i$. This implies $\deg(f) \geq dn - \sum a_i + 1$, hence $f \in A_{\geq dn - \sum a_i + 1}$. $\square$

**Proposition 5.2.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field $k$, $S = k[x_1^{a_1}, \ldots, x_d^{a_d}]$, and $n$ a multiple of $\mathrm{lcm}(a_1, \ldots, a_d)$. Assume that $\mathrm{char}\, k = 0$ or the $S$-ideal $S_{\geq n}$ is generated by monomials of degree $n$. If $S_{\geq n}$ is a normal $S$-ideal then $\mathrm{core}(S_{\geq n}) = S_{\geq dn - \sum a_i + 1}$.*

**Proof.** The assertion follows from Theorems 2.3 and 2.5, and Lemma 5.1. $\square$

**Corollary 5.3.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field $k$, $S = k[x_1^{a_1}, \ldots, x_d^{a_d}]$, $a = \mathrm{lcm}(a_1, \ldots, a_d)$, and $n = sa$. Assume that $\mathrm{char}\, k = 0$ or the $S$-ideal $S_{\geq n}$ is generated by monomials of degree $n$. If $s \geq d - 1$ then $\mathrm{core}(S_{\geq n}) = S_{\geq dn - \sum a_i + 1}$.*

**Proof.** By [26, 3.5] the $S$-ideal $S_{\geq n}$ is normal. Now the assertion follows from Proposition 5.2. $\square$

**Corollary 5.4.** *Let $k[x, y, z]$ be a polynomial ring over an infinite field $k$, $S = k[x^a, y^b, z^c]$ with $a, b, c$ pairwise relatively prime, and $n$ a multiple of $abc$. Assume that $\mathrm{char}\, k = 0$ or the $S$-ideal $S_{\geq n}$ is generated by monomials of degree $n$. Then $\mathrm{core}(S_{\geq n}) = S_{\geq 3n - a - b - c + 1}$.*

**Proof.** The $S$-ideal $S_{\geq n}$ is normal according to [35, 3.13] and [26, 3.5]. Again the assertion follows from Proposition 5.2. $\square$

The next example shows that Proposition 5.2 does not hold without the normality assumption.

**Example 5.5.** Let $k[x, y, z]$ be a polynomial ring over a field $k$ with $\mathrm{char}\, k = 0$ and consider the subring $S = k[x^{30}, y^{35}, z^{42}]$. We take $n = \mathrm{lcm}(30, 35, 42) = 210$, in which case $3n - a - b - c + 1 = 524$. It turns out that $S_{\geq 524} \subsetneq \mathrm{core}(S_{\geq 210}) \subsetneq \overline{\mathrm{core}(S_{\geq 210})} = S_{\geq 520}$.

## 6. MONOMIALS OF THE SAME DEGREE: DIMENSION TWO

In this section we prove a formula for the core of ideals generated by monomials of the same degree in a polynomial ring in two variables. We start with a number theoretic lemma.

**Lemma 6.1.** *Let $k_1, \ldots, k_s$ be non negative integers, $n$ a positive integer, and write $\delta = \gcd(k_1, \ldots, k_s, n)$. Every integer $t$ divisible by $\delta$ can be written in the form*

$$t = \alpha n + \sum_{i=1}^{s} \beta_i k_i \,,$$

*where $\beta_i \geq 0$ for all $i$ and $\sum_{i=1}^{s} \beta_i < n/\delta$. Furthermore, if $t \gg 0$ we can take $\alpha \geq 0$.*

**Proof.** The second assertion follows trivially from the first, since $\sum \beta_i < n/\delta$ and $n$ and the $k_i$ are fixed.

Replacing $t, k_i, n$ by $t/\delta, k_i/\delta$, and $n/\delta$, respectively, we may assume that $\delta = 1$. For any $t \in \mathbb{Z}$, we can write $t = \alpha n + \sum_{i=1}^{s} \beta_i k_i$ where $\alpha, \beta_i \in \mathbb{Z}$ since $\gcd(k_1, \ldots, k_s, n) = 1$. We proceed by induction on $s$. Let $s = 1$. Write $\beta_1 = qn + r$ with $0 \leq r \leq n - 1$. Then $t = \alpha n + \beta_1 k_1 = (\alpha + qk_1)n + rk_1$. So the assertion holds for $s = 1$.

Now assume $s > 1$ and the first assertion holds for $s - 1$. Let $\delta_j = \gcd(k_1, \ldots, \widehat{k_j}, \ldots, k_s, n)$ for $1 \leq j \leq s$. If $\delta_j = 1$ for some $j$ then the conclusion follows from the induction hypothesis. So assume that $\delta_j > 1$ for all $j$. For each $1 \leq j \leq s$ choose a prime $p_j$ that divides $\delta_j$; notice that $p_j \nmid k_j$. Hence $p_1, \ldots, p_s$ are distinct primes, $\prod p_j \mid n$ and $\prod_{j \neq i} p_j \mid k_i$. Thus $\prod_{j \neq i} p_j \mid \gcd(n, k_i)$ and $\prod_{j \neq i} p_j \geq 2^{s-1} \geq s$, hence $\gcd(n, k_i) \geq s$. Changing $\beta_i$ modulo $n/\gcd(n, k_i)$ using the division algorithm, we can assume that $0 \leq \beta_i \leq \frac{n}{\gcd(n, k_i)} - 1 \leq \frac{n}{s} - 1$ and hence $\sum \beta_i \leq n - 1$. $\square$

**Assumptions 6.2.** Let $R = k[x, y]$ be a polynomial ring over a field $k$ and write $\mathfrak{m}$ for the homogeneous maximal ideal of $R$. Let $I$ be an $R$-ideal generated by monomials of the same degree. Write $I = \mu(x^n, y^n, x^{n-k_1}y^{k_1}, \ldots, x^{n-k_s}y^{k_s})$ with $\mu$ a monomial and $0 < k_1 < \cdots < k_s < n$, and set $\delta = \gcd(k_1, \ldots, k_s, n)$.

**Lemma 6.3.** *In addition to the assumptions of 6.2 suppose that $\mu = 1$ and $\delta = 1$. Then for $t \gg 0$,*

$$\mathfrak{m}^{2nt} \subset I^{2t} + (x^{n(t+1)}, y^{n(t+1)}).$$

**Proof.** Consider a monomial generator $x^u y^v$ of $\mathfrak{m}^{2nt}$. Thus $u + v = 2nt$ and we may assume $u < n(t+1)$ and $v < n(t+1)$. Since $u + v = 2nt = n(t+1) + n(t-1)$, we must have $v > n(t-1)$. By Lemma 6.1 we can write

$$v = \alpha n + \sum_{i=1}^{s} \beta_i k_i,$$

where $\beta_i \geq 0$ and $\sum_{i=1}^{s} \beta_i \leq n - 1$. As $v > n(t-1)$ and $t \gg 0$, we can take $\alpha \geq 0$; we also have $\alpha \leq t$ since $v < n(t+1)$.

Now

$$
\begin{aligned}
u &= 2nt - \alpha n - \sum \beta_i k_i \\
&= 2nt - \alpha n - \sum \beta_i n + \sum \beta_i (n - k_i) \\
&= (2t - \alpha - \sum \beta_i) n + \sum \beta_i (n - k_i).
\end{aligned}
$$

Notice that $2t - \alpha - \sum \beta_i \geq 0$, because $t \gg 0$ and $\alpha + \sum \beta_i \leq t + n - 1 \leq 2t$. Thus

$$(u, v) = (2t - \alpha - \sum \beta_i)(n, 0) + \sum \beta_i (n - k_i, k_i) + \alpha(0, n)$$

is the exponent of a monomial in $I^{2t}$.  □

We are now ready to prove the main theorem of the section.

**Theorem 6.4.** *In addition to the assumptions of 6.2 suppose that $k$ is an infinite field. Then*

$$\mathrm{core}(I) = \mu (x^\delta, y^\delta)^{2\frac{n}{\delta} - 1}.$$

**Proof.** First, we may assume $\mu = 1$, since $\mathrm{core}(\mu I) = \mu \, \mathrm{core}(I)$ for any non zero divisor $\mu$. Passing to the subring $k[x^\delta, y^\delta]$ over which $k[x, y]$ is flat, we may further suppose that $\delta = 1$. Indeed the core of 0-dimensional ideals is preserved by flat base change according to Proposition 2.1 and [4, 4.8]. Now we are left to prove that $\mathrm{core}(I) = \mathfrak{m}^{2n-1}$. But

$$
\begin{aligned}
\mathrm{core}(I) &= (x^{n(t+1)}, y^{n(t+1)}) : (I^{2t}, x^{n(t+1)}, y^{n(t+1)}) &&\text{by Theorem 2.5} \\
&= (x^{n(t+1)}, y^{n(t+1)}) : \mathfrak{m}^{2nt} &&\text{by Lemma 6.3} \\
&= \mathfrak{m}^{2n-1}.
\end{aligned}
$$

□

**Corollary 6.5.** *In addition to the assumptions of 6.2 suppose that $\mu = 1$ and $\delta = 1$. Then $\check{I} = \mathfrak{m}^n$.*

**Proof.** We may assume that $k$ is infinite. By Theorem 6.4 we have $\mathrm{core}(I) = \mathrm{core}(\mathfrak{m}^n)$. Now the assertion follows from Corollary 4.9(a).  □

For any integrally closed ideal $I$ in a two-dimensional regular local ring it is known that $\mathrm{core}(I) = \mathrm{adj}(I^2)$, by work of Huneke and Swanson and of Lipman [12, 19]. The next corollary shows that this equality may hold even for ideals that are far from being integrally closed.

**Corollary 6.6.** *In addition to the assumptions of 6.2 suppose that $k$ is an infinite field, $\mu = 1$, and $\delta = 1$. Then $\mathrm{core}(I) = \mathrm{adj}(I^2)$.*

**Proof.** The assertion follows from Theorem 4.11 via Lemma 6.3.  □

**Alternative Proof of Theorem 6.4.** Again assuming $\mu = 1$ and $\delta = 1$ we wish to prove that $\mathrm{core}(I) = \mathfrak{m}^{2n-1}$. But $\mathfrak{m}^n$ is integral over $I$ and $\mathrm{core}(\mathfrak{m}^n) = \mathfrak{m}^{2n-1}$ by Corollary 5.3 for instance. Hence $\mathrm{core}(I) \supset \mathrm{core}(\mathfrak{m}^n) = \mathfrak{m}^{2n-1}$. Thus we only need to establish the inclusion $\mathrm{core}(I) \subset \mathfrak{m}^{2n-1}$. Since $\mathrm{core}(I)$ is a monomial ideal it suffices to prove that $\mathfrak{m}^{2n-1}$ is the maximal monomial ideal

contained in some reduction $J$ of $I$, i.e. $\mathfrak{m}^{2n-1} = \mathrm{mono}(J)$. We take $J = (y^n - x^n, f)$ for $f = b_0 y^n - b_1 x^{n-k_1} y^{k_1} - \cdots - b_s x^{n-k_s} y^{k_s}$ with $(b_0, \ldots, b_s) \in k^{s+1}$ general. Notice $\underline{\beta} = x^{2n}, y^{2n}$ is a regular sequence of monomials contained in $J$ and $(\underline{\beta}) \colon \mathfrak{m}^{2n} = \mathfrak{m}^{2n-1}$. Thus according to Lemma 3.2 the equality $\mathrm{mono}(J) = \mathfrak{m}^{2n-1}$ follows once we have shown that $\mathrm{Mono}((\underline{\beta}) \colon J) = \mathfrak{m}^{2n}$. To compute $(\underline{\beta}) \colon J = (x^{2n}, y^{2n}) \colon (y^n - x^n, f)$ we write $x^{2n} = h(y^n - x^n) + gf$ where $h, g$ are forms of degree $n$ and $\deg_y g \leq n-1$. We have

$$
\begin{aligned}
x^{2n} &= h(y^n - x^n) + gf \\
y^{2n} &= (h + y^n + x^n)(y^n - x^n) + gf.
\end{aligned}
$$

Hence $(x^{2n}, y^{2n}) \colon (y^n - x^n, f) = (x^{2n}, y^{2n}, \Delta)$, where

$$
\Delta = \begin{vmatrix} h & g \\ h + y^n + x^n & g \end{vmatrix} = -(y^n + x^n)g.
$$

To prove that $\mathrm{Mono}(x^{2n}, y^{2n}, \Delta) = \mathfrak{m}^{2n}$ it suffices to show that the monomial support of $\Delta = -(y^n + x^n)g$ is the set of all monomials of degree $2n$ except for $y^{2n}$. To this end we establish that the monomial support of $g$ is the set of all monomials of degree $n$ except for $y^n$. After dehomogenizing the latter claim follows from a general fact about polynomials in $k[y]$:

**Lemma 6.7.** *Let $k[y]$ be a polynomial ring over an infinite field $k$, and $f = b_0 y^n - b_1 y^{k_1} - \ldots - b_s y^{k_s} \in k[y]$, where $0 < k_1 < \ldots < k_s < n$ are integers with $\gcd(k_1, \ldots, k_s, n) = 1$ and $(b_0, \ldots, b_s) \in k^{s+1}$ is general. If $1 = h(y^n - 1) + gf$ with $h \in k[y]$ and $g = c_0 + c_1 y + \ldots + c_{n-1} y^{n-1} \in k[y]$, then $c_i \neq 0$ for every $i$.*

To prove Lemma 6.7 we are led to study Hankel matrices with strings of zeros and variables. We need to determine under which conditions on the distance between the strings of variables the ideal generated by the maximal minors of the matrix has generic grade. We solve this problem, which is interesting in its own right, by using techniques from Gröbner basis theory. On the other hand, Lemma 6.7 is actually equivalent to Theorem 6.4. Therefore the first proof of Theorem 6.4 also provides a less involved proof of Lemma 6.7.

## 7. MONOMIALS OF THE SAME DEGREE: DIMENSION THREE

In this section we study the core of ideals generated by monomials of the same degree in three variables. However, our results are less complete than in the two dimensional case.

**Notation and Discussion 7.1.** Let $R = k[x, y, z]$ be a polynomial ring over an infinite field $k$ and consider the $R$-ideal $I = (x^n, y^n, z^n, \{x^{n-k_i}y^{k_i}\}, \{x^{n-\ell_i}z^{\ell_i}\}, \{y^{n-m_i}z^{m_i}\}) \neq R$. Write

$$
\begin{aligned}
a &= \gcd(n, k_i\text{'s}, \ell_i\text{'s}) \\
b &= \gcd(n, k_i\text{'s}, m_i\text{'s}) \\
c &= \gcd(n, \ell_i\text{'s}, m_i\text{'s}) \\
S &= k[x^a, y^b, z^c].
\end{aligned}
$$

Notice that $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = \gcd(a, b, c)$. For the purpose of computing the core of $I$ we may assume that $\delta = \gcd(a, b, c) = 1$, since we may first compute the core of the corresponding ideal in the polynomial ring $k[x^\delta, y^\delta, z^\delta]$ and then use the fact that the core is preserved under flat base change according to Proposition 2.1 and [4, 4.8]. Thus throughout this section we will assume that $gcd(a, b, c) = 1$, and hence that, $a$, $b$, $c$ are pairwise relatively prime. Furthermore by relabeling the variables we can assume that $a \leq b \leq c$.

Let $J$ be the $R$-ideal generated by $x^n$, $y^n$, $z^n$, let $K$ be the $R$-ideal generated by the monomials in $S$ of degree $n$, and $L$ the $R$-ideal generated by $S_{\geq n}$. Clearly $J \subset I \subset K \subset L$.

We will show that the core of $I$ is always equal to the core of $K$; in particular $K$ is contained in the first coefficient ideal of $I$ according to Corollary 4.9(a). If $a = 1$, we will actually show that $\mathrm{core}(I) = \mathrm{core}(K) = \mathrm{core}(L)$ and that $L$ is the first coefficient ideal of $I$. We first need some technical lemmas. For their proofs set $k = \gcd(n, k_i\text{'s})$, $\ell = \gcd(n, \ell_i\text{'s})$, and $m = \gcd(n, m_i\text{'s})$.

**Lemma 7.2.** *With assumptions as in 7.1 one has $K^{3t} \subset S_{3nt}R \subset I^{3t} + J^{\langle t+1 \rangle}$ for $t \gg 0$.*

**Proof.** It suffices to show that for a monomial $x^{au}y^{bv}z^{cw}$ of $S_{3nt}$ that is not in $J^{\langle t+1 \rangle}$, we have $x^{au}y^{bv}z^{cw} \in I^{3t}$. Thus $au + bv + cw = 3nt$ and $au, bv, cw < n(t+1)$. Since the sum of any two of $au$, $bv$, $cw$ is strictly less than $2n(t+1)$ we have $au, bv, cw > (t-2)n$. In particular, when $t \gg 0$ each summand $au, bv, cw \gg 0$. Applying Lemma 6.1 to the integers $n$, $\ell_i$, $m_i$ we can write

$$
(3) \qquad\qquad cw = \alpha n + \sum \beta_i \ell_i + \sum \gamma_i m_i,
$$

where $\sum \beta_i + \sum \gamma_i < n/c$ and $\alpha, \beta_i, \gamma_i \geq 0$. In particular

$$
(4) \qquad\qquad \alpha n = cw - \left( \sum \beta_i \ell_i + \sum \gamma_i m_i \right) > (t - 2 - n/c)n.
$$

Next we wish to apply Lemma 6.1 to the integers $n$, $k_i$, $\ell m$. Since $\sum \gamma_i(n - m_i) < n^2/c$ we have $bv - \sum \gamma_i(n - m_i) \gg 0$. We first observe that $\gcd(n, k_i\text{'s}, \ell m) = \gcd(k, \ell m) = ab$. This follows since $a = \gcd(k, \ell)$, $b = \gcd(k, m)$, and $\gcd(a, b) = 1$. Now we want to prove that $bv - \sum \gamma_i(n - m_i)$ is divisible by $ab$. Clearly $b$ divides $bv - \sum \gamma_i(n - m_i)$. Since $au = 3nt - bv - \alpha n - \sum \beta_i \ell_i - \sum \gamma_i m_i$ by (3), we see that $a$ divides $bv + \sum \gamma_i m_i$ and hence divides $bv - \sum \gamma_i(n - m_i)$. As $\gcd(a, b) = 1$, $bv - \sum \gamma_i(n - m_i)$ is a multiple of $ab$. Hence according to Lemma 6.1 we can write

$$
(5) \qquad\qquad bv - \sum \gamma_i(n - m_i) = \mu n + \sum \nu_i k_i + \eta \ell m,
$$

where $\sum \nu_i + \eta < n/ab$ and $\mu, \nu_i, \eta \geq 0$. Therefore

$$(6) \qquad bv = \mu n + \sum \gamma_i(n - m_i) + \sum \nu_i k_i + \eta \ell m.$$

Now we apply Lemma 6.1 to the integers $n$, $n - m_i$. By (5) we have $\mu n + \eta \ell m \gg 0$ as $\sum \nu_i k_i < n^2/ab$. Hence we may write

$$\mu n + \eta \ell m = \rho n + \sum \gamma_i'(n - m_i),$$

where $\sum \gamma_i' < n/m$ and $\rho, \gamma_i' \geq 0$. Substituting the last equality into (6) we obtain

$$(7) \qquad bv = \rho n + \sum \gamma_i'(n - m_i) + \sum \gamma_i(n - m_i) + \sum \nu_i k_i.$$

Next consider $au - \sum \beta_i(n - \ell_i) - \sum \nu_i(n - k_i)$, which is $\gg 0$ when $t \gg 0$. We wish to see that $au - \sum \beta_i(n - \ell_i) - \sum \nu_i(n - k_i)$ is divisible by $\ell$. Indeed

$$
\begin{aligned}
au - \sum \beta_i(n - \ell_i) - \sum \nu_i(n - k_i) &\equiv au + \sum \nu_i k_i \mod \ell \\
&\equiv au + cw - \sum \gamma_i m_i + \sum \nu_i k_i \mod \ell && \text{by (3)} \\
&\equiv au + cw + bv \mod \ell && \text{by (6)} \\
&\equiv 3nt \mod \ell \\
&\equiv 0 \mod \ell.
\end{aligned}
$$

Therefore $au - \sum \beta_i(n - \ell_i) - \sum \nu_i(n - k_i)$ is a multiple of $\ell$. Thus we may apply Lemma 6.1 to the integers $n$, $n - \ell_i$ to write

$$au - \sum \beta_i(n - \ell_i) - \sum \nu_i(n - k_i) = \zeta n + \sum \beta_i'(n - \ell_i),$$

where $\sum \beta_i' < n/\ell$ and $\zeta, \beta_i' \geq 0$. Hence

$$(8) \qquad au = \zeta n + \sum \beta_i(n - \ell_i) + \sum \nu_i(n - k_i) + \sum \beta_i'(n - \ell_i).$$

Combining equations (8), (7), and ( 3) we obtain

$$
\begin{aligned}
(au, bv, cw) = \ & \zeta(n, 0, 0) + \rho(0, n, 0) + \alpha(0, 0, n) + \sum (\beta_i + \beta_i')(n - \ell_i, 0, \ell_i) \\
& + \ \sum \nu_i(n - k_i, k_i, 0) + \sum (\gamma_i + \gamma_i')(0, n - m_i, m_i) \\
& - \ (0, 0, \sum \beta_i' \ell_i + \sum \gamma_i' m_i).
\end{aligned}
$$

Taking the sum of the components on each side we see that $\sum \beta_i' \ell_i + \sum \gamma_i' m_i = \lambda n$ for some $\lambda \geq 0$. Thus

$$
\begin{aligned}
(au, bv, cw) = \ & \zeta(n, 0, 0) + \rho(0, n, 0) + (\alpha - \lambda)(0, 0, n) + \sum (\beta_i + \beta_i')(n - \ell_i, 0, \ell_i) \\
& + \ \sum \nu_i(n - k_i, k_i, 0) + \sum (\gamma_i + \gamma_i')(0, n - m_i, m_i).
\end{aligned}
$$

Since $\sum \beta_i' < n/\ell$ and $\sum \gamma_i' < n/m$ we must have $\lambda n < (n/\ell + n/m)n$, and consequently $\lambda < n/\ell + n/m$. As $\alpha > t - 2 - n/c$ by (4), we have $\alpha - \lambda \geq 0$ for $t \gg 0$. Finally, since the sum of the components on the left hand side is $3nt$ we deduce that the right hand side is the exponent vector of a monomial in $I^{3t}$, as desired. $\qquad \square$

**Lemma 7.3.** *With assumptions as in 7.1 the S-ideal $S_{\geq j}$ is generated by monomials of degrees at most $j + b - 1$ for every integer multiple $j$ of $c$.*

**Proof.** Let $x^{au}y^{bv}z^{cw}$ be a minimal monomial generator of $S_{\geq j}$. Suppose that $au + bv + cw \geq j + b$. Since $a \leq b$ it follows that $u = v = 0$ because the monomial $x^{au}y^{bv}z^{cw}$ is a minimal generator of $S_{\geq j}$. Hence $cw \geq j + b > j$ which implies $z^{cw} = z^j z^{c(w-j/c)}$, a contradiction.                     $\square$

**Lemma 7.4.** *With assumptions as in 7.1 and $a = b = 1$ the S-ideal $S_{\geq j}$ is generated by monomials of degree $j$ for every integer multiple $j$ of $c$; in particular $L = K$.*

**Proof.** This follows immediately from Lemma 7.3.                     $\square$

**Lemma 7.5.** *With assumptions as in 7.1 and $a = 1$ one has $L^{3t} \subset S_{\geq 3nt}R \subset I^{3t} + J^{\langle t+1 \rangle}$ for $t \gg 0$.*

**Proof.** It suffices to show that every minimal monomial generator $x^u y^{bv} z^{cw}$ of the $S$-ideal $S_{\geq 3nt}$ that is not in $J^{\langle t+1 \rangle}$ is in $I^{3t}$. Lemma 7.3 gives $u + bv + cw = 3nt + \varepsilon$ with $0 \leq \varepsilon \leq b - 1$. Since $x^u y^{bv} z^{cw} \notin J^{\langle t+1 \rangle}$ we have $bv, cw < n(t+1)$, hence $bv + cw < 2n(t+1)$. As $u + bv + cw \geq 3nt$ we obtain $u > (t-2)n$. In particular $u \geq \varepsilon$ for $t \geq 3$. Now $x^u y^{bv} z^{cw} = x^\varepsilon x^{u-\varepsilon} y^{bv} z^{cw}$ with $x^{u-\varepsilon} y^{bv} z^{cw} \in S_{3nt}R$, and the assertion follows from Lemma 7.2.                     $\square$

From now on we will assume that the field $k$ is infinite.

**Theorem 7.6.** *With assumptions as in 7.1 one has $\mathrm{core}(I) = \mathrm{core}(K)$. In particular $K \subset \check{I}$, the first coefficient ideal of $I$.*

**Proof.** Lemma 7.2 gives $K^{3t} + J^{\langle t+1 \rangle} = I^{3t} + J^{\langle t+1 \rangle}$ for $t \gg 0$. Thus $\mathrm{core}(K) = \mathrm{core}(I)$ by Theorem 2.5. Corollary 4.9(a) then implies that $\check{K} = \check{I}$.                     $\square$

We are now ready to give an explicit formula for the core of $I$.

**Theorem 7.7.** *With assumptions as in 7.1 and $a = 1$ one has*

$$\mathrm{core}(I) = \mathrm{core}(K) = \mathrm{core}(L) = (S_{\geq 3n-b-c})R.$$

**Proof.** The $R$-ideal $J = (x^n, y^n, z^n)$ is a reduction of $L$ according to Lemma 5.1(a) and the $S$-ideal $S_{\geq n}$ is normal by [35, 3.13] and [26, 3.5]. Now we obtain for $t \gg 0$,

$$
\begin{aligned}
J^{\langle t+1 \rangle} :_R L^{3t} &= J^{t+1} :_R L^t & &\text{by Lemma 2.2} \\
&\subset \mathrm{core}(L) & &\text{by Proposition 2.1 and [23, 4.8]} \\
&\subset \mathrm{core}(K) & &\text{since } K \text{ is a reduction of } L \\
&\subset \mathrm{core}(I) & &\text{since } I \text{ is a reduction of } K \\
&= J^{\langle t+1 \rangle} :_R I^{3t} & &\text{by Theorem 2.5} \\
&= J^{\langle t+1 \rangle} :_R L^{3t} & &\text{by Lemma 7.5} \\
&= (S_{\geq 3n-b-c})R & &\text{by Lemma 5.1(b).}
\end{aligned}
$$

$\square$

The next example shows that Theorem 7.7 does not hold when $a = 2$.

**Example 7.8.** Let $R = k[x,y,z]$ be a polynomial ring over a field $k$ with $\text{char}\, k = 0$ and consider the ideal $I = (x^{30}, y^{30}, z^{30}, x^6y^{24}, x^{10}z^{20}, y^{15}z^{15})$. In this case $a = 2$, $b = 3$, $c = 5$ and $S = k[x^2, y^3, z^5]$. One has $L = K + (x^{26}z^5, x^{20}y^6z^5, x^{16}z^{15}, x^{14}y^{12}z^5, x^{10}y^6z^{15}, x^8y^{18}z^5, x^4y^{12}z^{15}, x^2y^{24}z^5) + (y^{27}z^5, y^{12}z^{20})$. It turns out that $\text{core}(L) = S_{\geq 81}R \subsetneq \text{core}(I) = \text{core}(K)$.

**Theorem 7.9.** *With assumptions as in 7.1 and $a = 1$ one has*

   (a) $\check{I} = L$.
   (b) $R[\check{I}t] = R[Lt]$ *is the $S_2$-ification of $R[It]$.*
   (c) $R[\check{I}t] = R[Lt]$ *is a Cohen-Macaulay ring.*

**Proof.** The ideal $L$ is integral over $I$ by Lemma 5.1(a). Furthermore $J^{t+1} : L^t = J^{t+1} : I^t$ for $t \gg 0$ according to Lemmas 7.5 and 2.2. Now Corollary 4.8 implies that $\check{L} = \check{I}$. Thus the theorem follows once we have shown that $R[Lt]$ is Cohen-Macaulay. The Rees algebra $S[S_{\geq n}t]$ is normal by [35, 3.13] and [26, 3.5], and hence Cohen-Macaulay according to [8, Theorem 1]. But $R[Lt]$ is a finite free module over $S[S_{\geq n}t]$ and thus a Cohen-Macaulay ring as well.          $\square$

The next two corollaries show that for $a = b = 1$ our formula for the core becomes more explicit, akin to the case of two variables.

**Corollary 7.10.** *In addition to the assumptions of 7.1 suppose that $a = b = 1$ and write $q = \frac{3n}{c} - 1$. One has*

   (a) $\check{I} = K = L = ((x,y)^c, z^c)^{n/c}$.
   (b) $\text{core}(I) = (z^{qc}) + \sum_{i=0}^{q-1} z^{ic}(x,y)^{(q-i)c-1}$.

**Proof.** The first two equalities in part (a) follow from Lemma 7.4 and Theorem 7.9(a), whereas the last equation is immediate from the definition of $K$. To prove part (b) one uses Theorem 7.7.          $\square$

**Corollary 7.11.** *With assumptions as in 7.1 and $a = b = c = 1$ one has*

   (a) $\check{I} = K = L = \mathfrak{m}^n$.
   (b) $\text{core}(I) = \mathfrak{m}^{3n-2} = \text{adj}(I^3)$.

**Proof.** In light of Corollary 7.10 it suffices to prove that $\text{core}(I) = \text{adj}(I^3)$ in part (b). Indeed, part (a) and Lemma 7.2 show that the assumptions of Theorem 4.11 are satisfied. Now apply that theorem.          $\square$

## References

[1] J.-F. Boutot, Singularités rationnelles et quotients par les groupes réductifs, Invent. Math. **88** (1987), 65-68.

[2] C. Ciupercă, First coefficient ideals and the $S_2$-ification of a Rees algebra, J. Algebra **242** (2001), 782-794.

[3] C. Ciupercă, A numerical characterization of the $S_2$-ification of a Rees algebra, J. Pure and Appl. Algebra **178** (2003), 25-48.

[4] A. Corso, C. Polini and B. Ulrich, The structure of the core of ideals, Math. Ann. **321** (2001), 89-105.

[5] A. Corso, C. Polini and B. Ulrich, Core and residual intersections of ideals, Trans. Amer. Math. Soc. **354** (2002), 2579-2594.

[6] L. Fouli, C. Polini and B. Ulrich, The core of ideals in arbitrary characteristic, preprint.

[7] S. Goto and Y. Shimoda, On the Rees algebras of Cohen-Macaulay local rings, in *Commutative algebra*, Lecture Notes in Pure and Appl. Math. **68**, Dekker, New York, 1982, 201-231.

[8] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. **96** (1972), 318-337.

[9] J. Howald, Multiplier ideals of monomial ideals, Trans. Amer. Math Soc. **353** (2001), 2665-2671.

[10] R. Hübl and I. Swanson, Adjoints of ideals, in preparation.

[11] C. Huneke, Hilbert functions and symbolic powers, Michigan Math. J. **34** (1987), 293-318.

[12] C. Huneke and I. Swanson, Cores of ideals in 2-dimensional regular local rings, Michigan Math. J. **42** (1995), 193-208.

[13] C. Huneke and N. V. Trung, On the core of ideals, Compositio Math. **141** (2005), 1-18.

[14] C. Huneke and B. Ulrich, Divisor class groups and deformations, Amer. J. Math. **107** (1985), 1265-1303.

[15] E. Hyry, Coefficient ideals and the Cohen–Macaulay property of Rees algebras, Proc. Amer. Math. Soc. **129** (2001), 1299–1308.

[16] E. Hyry and K. Smith, On a non-vanishing conjecture of Kawamata and the core of an ideal, Amer. J. Math. **125** (2003), 1349–1410.

[17] E. Hyry and K. Smith, Core versus graded core, and global sections of line bundles, Trans. Amer. Math. Soc. **356** (2004), 3143–3166.

[18] S. Itoh, Integral closures of ideals generated by regular sequences, J. Algebra **117** (1988), 390-401.

[19] J. Lipman, Adjoints of ideals in regular local rings, Math. Res. Lett. **1** (1994), 739-755.

[20] J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon–Skoda, Michigan Math. J. **28** (1981), 199-222.

[21] G. Lyubeznik, A property of ideals in polynomial rings, Proc. Amer. Math. Soc. **98** (1986), 399-400.

[22] D. G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. **50** (1954), 145-158.

[23] C. Polini and B. Ulrich, A formula for the core of an ideal, Math. Ann. **331** (2005), 487-503.

[24] C. Polini, B. Ulrich, R. Villarreal and W. V. Vasconcelos, Indices of normalization of ideals, preprint.

[25] D. Rees and J. D. Sally, General elements and joint reductions, Michigan Math. J. **35** (1988), 241-254.

[26] L. Reid, L. G. Roberts and M. A. Vitulli, Some results on normal homogeneous ideals, Comm. Algebra **31** (2003), 4485-4506.

[27] M. Saito, B. Sturmfels and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, Springer, Berlin, 2000.

[28] K. Shah, Coefficient ideals, Trans. Amer. Math. Soc. **327** (1991), 373-384.

[29] A. Simis, B. Ulrich and W. V. Vasconcelos, Cohen-Macaulay Rees algebras and degrees of polynomial relations, Math. Ann. **301** (1995), 421-444.

[30] I. Swanson and C. Huneke, *Integral closure of ideals, rings and modules*, Cambridge Univ. Press, Cambridge, 2006.

[31] B. Ulrich, Ideals having the expected reduction number, Amer. J. Math. **118** (1996), 17-38.

[32] B. Ulrich, Briançon-Skoda theorems, adjoints, and canonical modules of extended Rees algebras, preprint.

[33] P. Valabrega and G. Valla, Form rings and regular sequences, Nagoya Math. J. **72** (1978), 93–101.

[34] W. V. Vasconcelos, Reduction numbers of ideals, J. Algebra **216** (1999), 652-664.

[35] M. A. Vitulli, Serre's condition $R_\ell$ for affine semigroups rings, preprint.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556
*E-mail address*: cpolini@nd.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907
*E-mail address*: ulrich@math.purdue.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403
*E-mail address*: vitulli@math.oregon.edu