A NOTE ON MATHER-JACOBIAN MULTIPLIER IDEALS

Abstract. Using Mather-Jacobian multiplier ideals, we prove a formula comparing the Grauert-Riemenschneider canonical sheaf with the canonical sheaf of a variety over an algebraically closed field of characteristic zero. We also study Mather-Jacobian multiplier ideals on algebraic curves, in which case Mather-Jacobian multiplier ideals can be defined over a ground field of any characteristic. We show that the Mather-Jacobian multiplier ideals on curves are essentially the integrally closed ideals. Finally by comparing the conductor ideal with the Mather-Jacobian multiplier ideal of the structure sheaf, we give a criterion for when an algebraic curve is locally a complete intersection.

1. Introduction

Recently, the theory of Mather-Jacobian multiplier ideals (MJ-multiplier for short) on arbitrary varieties over an algebraically closed field of characteristic zero has been developed by Ein-Ishii-Mustata [EIM16] and Ein-Ishii [EI15] (see also de Fernex-Docampo [dFD14] for a similar theory on normal varieties). This new notion generalizes the classical theory of multiplier ideals on nonsingular varieties (or normal $\mathbb{Q}$-Gorenstein varieties) and has found some interesting applications (for instance [Niu14]). Throughout this paper, the ground field k is always assumed to be *algebraically closed*. By a variety we mean a reduced and irreducible separated scheme of finite type over k.

We first prove the following result that compares the Grauert-Riemenschneider canonical sheaf with the canonical sheaf of a variety by using MJ-multiplier ideals. This result partially generalizes a formula established by de Fernex-Docampo in [dFD14, Theorem C] for normal varieties.

Theorem 1.1. *Let X be a variety over a field of characteristic zero. Let ω_X be the canonical sheaf, ω_X^{GR} the Grauert-Riemenschneider canonical sheaf, and $\widehat{\mathscr{I}}(\mathscr{O}_X)$ the Mather-Jacobian multiplier ideal. Then*

$$\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot \omega_X \subseteq \omega_X^{GR}.$$

The formula of de Fernex-Docampo is stronger than the one above if the variety is normal. However, their formula is not known in general. We also note that when X is locally a complete intersection, it is well-known to experts that $\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot \omega_X = \omega_X^{GR}$ (see Remark 2.6 for details.)

Next we turn to understand MJ-multiplier ideals on algebraic curves, which is the first case one should investigate. Let X be an algebraic curve, i.e., X is a dimension one variety. We first show that the definition of MJ-multiplier ideal can be extended to arbitrary characteristic, see Proposition 2.4. By its definition, any MJ-multiplier ideal is integrally closed and contained in $\widehat{\mathscr{I}}(\mathscr{O}_X)$. We prove that this property actually characterizes MJ-multiplier ideals on curves.

Theorem 1.2. *Let X be an algebraic curve over a field of any characteristic and $\mathfrak{a} \subseteq \mathscr{O}_X$ be an ideal. Then $\mathfrak{a}$ is a Mather-Jacobian multiplier ideal if and only if $\mathfrak{a}$ is integrally closed and $\mathfrak{a} \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X)$.*

2010 *Mathematics Subject Classification.* 13A10, 14Q20.

Key words and phrases. Multiplier ideal, conductor ideal, canonical sheaf.

Note that our theorem is global in nature. The question whether an integrally closed ideal is a multiplier ideal was initially raised by Lipman-Watanabe [LW03]. They proved that on a nonsingular surface, locally a multiplier ideal is the same as an integrally closed ideal. This problem was also studied in work of Favre-Jonsson [FJ05]. Later, it was generalized to log terminal surfaces by Tucker [Tuc09]. Finally the question was completely solved for higher dimensional nonsingular varieties in celebrated work of Lazarsfeld-Lee [LL07] . We wonder whether the result of [LW03] and [FJ05] can be generalized to MJ-multiplier ideals on any surface, see Conjecture 3.6.

On a variety X, there are three intrinsic ideals: the Jacobian ideal Jac_X, the MJ-multiplier ideal $\widehat{\mathscr{I}}(\mathscr{O}_X)$, and the conductor ideal $\mathfrak{C}_X$. They satisfy the inclusion $\overline{\mathrm{Jac}}_X \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X) \subseteq \mathfrak{C}_X$, where $\overline{\mathrm{Jac}_X}$ means the integral closure, and when X is a curve they all capture the singularities of X. Our last theorem gives a criterion for a curve to be a local complete intersection using MJ-multiplier ideals.

Theorem 1.3. *Let X be an algebraic curve over a field of arbitrary characteristic. Let $\widehat{\mathscr{I}}(\mathscr{O}_X)$ be the Mather-Jacobian multiplier ideal and $\mathfrak{C}_X$ be the conductor ideal. Then X is a local complete intersection if and only if $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$.*

This theorem is not true for higher dimensional varieties, see Example 3.12. But it can be established for codimension one points of any variety, see Remark 3.10. It would be interesting to understand algebraic or geometric consequences of the equality between the conductor ideal and the MJ-multiplier ideal of any variety. We give a partial result along these lines in Proposition 3.13.

2. Mather-Jacobian Multiplier ideals

Throughout this section, we assume the ground field k is of *characteristic zero*. We start by recalling the definition of MJ-multiplier ideals defined in [EIM16]. For more details, we refer to the paper [EIM16].

Let X be a variety of dimension d and Ω_X^1 be the sheaf of differentials of X. We write $\Omega_X^d = \wedge^d \Omega_X^1$. The morphism $\mathbb{P}(\Omega_X^d) \to X$ is an isomorphism over the regular locus X_{reg} of X. The closure of X_{reg} in $\mathbb{P}(\Omega_X^d)$ is the Nash blowup of X, and is denoted by $\widehat{X}$ with reduced scheme structure. It comes equipped with the projection $\nu : \widehat{X} \to X$. The line bundle $\mathscr{O}_{\widehat{X}}(1) := \mathscr{O}_{\mathbb{P}(\Omega_X^d)}(1)|_{\widehat{X}}$ is called the *Mather canonical line bundle* of X and sometimes we also write $\widehat{K}_X$ for $\mathscr{O}_{\widehat{X}}(1)$.

Definition 2.1. Let X be a variety over k of dimension d and $f : Y \to X$ be a resolution of singularities factoring through the Nash blow-up of X. Then the image of the canonical homomorphism

$$f^*(\Omega_X^d) \longrightarrow \Omega_Y^d$$

is an invertible sheaf of the form $\mathrm{Jac}_f \cdot \Omega_Y^d$. Here Jac_f is the relative Jacobian ideal, which is invertible and defines an effective divisor, called the *Mather discrepancy divisor* and denoted by $\widehat{K}_{Y/X}$.

Definition 2.2. Let X be a variety over k and $\mathfrak{a} \subseteq \mathscr{O}_X$ a nonzero ideal on X. Given a log resolution $f : Y \to X$ of $\mathrm{Jac}_X \cdot \mathfrak{a}$, we denote by Z and $J_{Y/X}$ the effective divisors on Y such that $\mathfrak{a} \cdot \mathscr{O}_Y = \mathscr{O}_Y(-Z)$ and $\mathrm{Jac}_X \cdot \mathscr{O}_Y = \mathscr{O}_Y(-J_{Y/X})$ (such a resolution automatically factors through the Nash blow-up, see Remark 2.3 of [EIM16]). The *Mather-Jacobian multiplier ideal* of $\mathfrak{a}$ of exponent $t \in \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{\mathscr{I}}(X, \mathfrak{a}^t) := f_* \mathscr{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ \rfloor),$$

where $\lfloor \quad \rfloor$ means the round down of an $\mathbb{R}$-divisor. Sometimes we simply write it as $\widehat{\mathscr{I}}(\mathfrak{a}^t)$ and call it *MJ-multiplier ideal*.

Remark 2.3. (1) It has been shown in [EIM16, Corrolary 2.14] that the sheaf $\widehat{\mathscr{I}}(X, \mathfrak{a}^t)$ defined above is an ideal of $\mathscr{O}_X$.

(2) If X is a curve, then we choose its normalization $f : X' \to X$ as a resolution of singularities. In this case, there is no need to refer to the Nash blow-up.

(3) It is clear that any MJ-multiplier ideal is contained in the ideal $\widehat{\mathscr{I}}(\mathscr{O}_X)$. It would be interesting to understand more about this intrinsic ideal $\widehat{\mathscr{I}}(\mathscr{O}_X)$.

For a variety X, let $f : X' \to X$ be a resolution of singularities (or simply take X' to be the normalization of X). Then the conductor ideal $\mathfrak{C}_X$ is defined to be

$$\mathfrak{C}_X := \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(f_* \mathscr{O}_{X'}, \mathscr{O}_X) = \mathrm{ann}(f_* \mathscr{O}_{X'}/\mathscr{O}_X).$$

We refer the reader to [ZS75, V.5] and [Kun13] for details of conductor ideals. The following easy proposition shows that MJ-multiplier ideals are contained in the conductor ideal. See also [EIM16, Corrolary 2.14].

Proposition 2.4. *Let X be a variety over k and $\mu : X' \to X$ be the normalization. Then $\widehat{\mathscr{I}}(\mathscr{O}_X)$ is an ideal of $\mu_* \mathscr{O}_{X'}$. In particular, $\widehat{\mathscr{I}}(\mathscr{O}_X) \subseteq \mathfrak{C}_X$.*

Proof. Let $f : Y \to X$ be a log resolution of Jac_X. There exists a morphism $f' : Y \to X'$ such that $f = \mu \circ f'$. Thus $\widehat{\mathscr{I}}(\mathscr{O}_X) = f_*(\mathscr{O}_Y(\widehat{K}_{Y/X} - J_{Y/X})) = \mu_*(f'_*(\widehat{K}_{Y/X} - J_{Y/X}))$. Note that $f'_*(\widehat{K}_{Y/X} - J_{Y/X})$ is a module over $f'_* \mathscr{O}_Y = \mathscr{O}_{X'}$. Hence $\mu_*(f'_*(\widehat{K}_{Y/X} - J_{Y/X}))$ is a module over $\mu_* \mathscr{O}_{X'}$. This means that $\widehat{\mathscr{I}}(\mathscr{O}_X)$ is a $\mu_* \mathscr{O}_{X'}$ module. But we know $\widehat{\mathscr{I}}(\mathscr{O}_X)$ is contained in $\mathscr{O}_X \subseteq \mu_* \mathscr{O}_{X'}$ thus it is an ideal of $\mu_* \mathscr{O}_{X'}$. The rest of the result follows from the fact that $\mathfrak{C}_X$ is maximal with respect to the property of being an ideal of $\mu_* \mathscr{O}_{X'}$ contained in $\mathscr{O}_X$ (cf. [Kun13, Excersice 6, p.103]). □

Recall that for a variety X, if $f : X' \to X$ is a proper birational morphism with X' nonsingular, then the *Grauert-Riemenschneider canonical sheaf* ω_X^{GR} of X is defined as $f_* \omega_{X'}$. It is easy to check that ω_X^{GR} does not depend on the choice of the birational morphism f. Notice that $\omega_X^{GR} \subseteq \omega_X$. It has been discussed in [dFD14] that if X is normal, then one can use a special ideal to compare ω_X with ω^{GR} (see Remark 2.6(2)). Here we show that MJ-multiplier turns out to be a more natural option for such comparison on arbitrary varieties in the following theorem.

Theorem 2.5. *Let X be a variety over k. Let ω_X be its canonical sheaf, ω_X^{GR} its Grauert-Riemenschneider canonical sheaf, and $\widehat{\mathscr{I}}(\mathscr{O}_X)$ its Mather-Jacobian multiplier ideal. Then one has*

$$\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot \omega_X \subseteq \omega_X^{GR}.$$

Proof. The question is local, so we can assume that X is affine and embedded in an affine space $\mathbb{A}^N$ such that $d := \dim X$ and $c := \mathrm{codim}_{\mathbb{A}^N} X$. Assume that $I_X := (F_1, F_2, \cdots, F_r)$. We can take the generators $F_1, \cdots, F_r$ general such that any c of them provide a general link of X (for instance, if $I_X = (f_1, \cdots, f_d)$ is generated by polynomials f_i, then we can take for $1 \leq i \leq d$, $F_i = \sum_{j=1}^{d} a_{i,j} f_j$ where the $a_{i,j}$ are general elements in k. See [EM09, Section 9.2]). Specifically, let $J \subset \{1, 2, \cdots, r\}$ such that $|J| = c$. Then let I_{V_J} be the ideal generated by $\{F_i \mid i \in J\}$. Then the subscheme V_J defined by I_{V_J} is a general link of X, which is a complete intersection in $\mathbb{A}^N$. Denote $q_J := (I_{V_J} : I_X) \cdot \mathscr{O}_X$ and $\mathrm{Jac}_J := \mathrm{Jac}_{V_J} \cdot \mathscr{O}_X$. Consider the following morphisms

$$\Omega^d_X \xrightarrow{c_X} \omega_X \xrightarrow{u_J} \omega_{V_J}|_X \xrightarrow{w_J} \mathscr{O}_X,$$

where c_X is the fundamental class, u_J is a natural inclusion, and w_J is induced by the localization at the generic points of X so that it maps $\omega_{V_j}|_X$ isomorphic to $\mathscr{O}_X$ (for details see [EM09, Proposition 9.1]). It has been proved in *loc. cit.* that

(1) the image of u_J is $q_J \otimes \omega_{V_J}|_X$ and therefore if we set $\alpha_J := w_J \circ u_J$, we get an isomorphism

$$\alpha_J : \omega_X \longrightarrow q_J;$$

(2) the image of $u_J \circ c_X$ is $\mathrm{Jac}_J \otimes \omega_{V_J}|_X$ and under the isomorphism α_J above the image of c_X is Jac_J.

Now consider $J_0 := \{1, 2, \cdots, c\}$ and write $V_0 := V_{J_0}$, $\mathrm{Jac}_0 := \mathrm{Jac}_{J_0}$, and $q_0 = q_{J_0}$. From the surjective morphism

$$\Omega^d_X \longrightarrow \mathrm{Jac}_0 \otimes \omega_{V_0}|_X,$$

we deduce that the Nash blowup satisfies $\widehat{X} = \mathrm{Bl}_{\mathrm{Jac}_0} X$ and $\widehat{K}_X = \mathrm{Jac}_0 \cdot \mathscr{O}_{\widehat{X}} \otimes \nu^* \omega_{V_0}|_X$, where $\nu : \widehat{X} \to X$ is the projection (indeed, if one has a surjection $\Omega^d_X \to \mathfrak{a} \otimes \mathscr{L}$ where $\mathfrak{a} \subseteq \mathscr{O}_X$ is an ideal and $\mathscr{L}$ is an line bundle, then $\widehat{X} = \mathrm{Bl}_{\mathfrak{a}} X$ by the definition of $\widehat{X}$, and $\widehat{K}_X = \mathfrak{a} \cdot \mathscr{O}_{\widehat{X}} \otimes \nu^* \mathscr{L}$). Consider a log resolution $f : X' \to X$ of Jac_X, Jac_0 and q_0 such that $\mathrm{Jac}_X \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-J_{X/X'})$, $\mathrm{Jac}_0 \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-Z_0)$, and $q_0 \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-Q_0)$. This f factors through $\widehat{X}$ and therefore we have

$$\widehat{K}_{X'/X} = \omega_{X'} \otimes \mathscr{O}_{X'}(Z_0) \otimes f^* \omega^{-1}_{V_0}|_X,$$

which implies that

$$\omega^{GR}_X = f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - Z_0) \otimes \omega_{V_0}|_X. \tag{2.5.1}$$

Note that $f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - Z_0)$ is an ideal sheaf because it is contained in $\widehat{\mathscr{I}}(\mathscr{O}_X)$ since $\mathrm{Jac}_0 \subseteq \mathrm{Jac}_X$. On the other hand, we have the equality $\omega_X = q_0 \otimes \omega_{V_0}|_X$. Hence we deduce that

$$\omega^{GR}_X : \omega_X = (f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - Z_0) \otimes \omega_{V_0}|_X) : (q_0 \otimes \omega_{V_0}|_X) = f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - Z_0) : q_0,$$

since $\omega_{V_0}|_X$ is invertible.

In order to prove the theorem we need to show $\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot q_0 \subseteq f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - Z_0)$. But note that $\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot q_0 \subseteq \widehat{\mathscr{I}}(q_0) = f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - J_{X/X'} - Q_0)$. Thus it suffices to show

$$\mathrm{Jac}_X \cdot q_0 \subseteq \mathrm{Jac}_0\,. \tag{2.5.2}$$

To this end, consider for any $J \subset \{1, 2, \cdots, r\}$ with $|J| = c$ the isomorphism

$$\alpha_J \circ \alpha_0^{-1} : q_0 \longrightarrow q_J.$$

Since the ideals q_0 and q_J contain some non zero-divisors of $\mathscr{O}_X$, the isomorphism $\alpha_J \circ \alpha_0^{-1}$ is given by $\alpha_J \circ \alpha_0^{-1}(r) = \frac{b_J}{a_J} \cdot r$ for any $r \in q_0$, where b_J and a_J are some non zero-divisors in $\mathscr{O}_X$. Thus we have $q_J = \frac{b_J}{a_J} \cdot q_0$ and $\mathrm{Jac}_J = \frac{b_J}{a_J} \cdot \mathrm{Jac}_0$, as indicated in the following commutative diagram

$$\begin{array}{ccccccc}
 & & & \overset{\alpha_J}{\nearrow} & q_J & \hookleftarrow & \alpha_J(\mathrm{Im}\, c_X) = \mathrm{Jac}_J \\
\wedge^d \Omega^1_X & \xrightarrow{c_X} & \omega_X & & \uparrow{\scriptstyle \alpha_J \circ \alpha_0^{-1}} & & \\
 & & & \underset{\alpha_0}{\searrow} & q_0 & \hookleftarrow & \alpha_0(\mathrm{Im}\, c_X) = \mathrm{Jac}_0\,.
\end{array}$$

But notice that

$$\operatorname{Jac}_X = \sum_J \operatorname{Jac}_J,$$

and therefore

$$\operatorname{Jac}_X \cdot q_0 = \sum_J \operatorname{Jac}_J \cdot q_0 = \sum_J \frac{b_J}{a_J} \cdot \operatorname{Jac}_0 \cdot q_0 = \sum_J \frac{b_J}{a_J} \cdot q_0 \cdot \operatorname{Jac}_0 = \sum_J q_J \cdot \operatorname{Jac}_0 \subseteq \operatorname{Jac}_0 .$$

This proves the inclusion (2.5.2) and the theorem then follows. □

Remark 2.6. (1) When X is locally a complete intersection, the image of the canonical morphism $\Omega^d_X \to \omega_X$ is $\operatorname{Jac}_X \otimes \omega_X$. So $\widehat{X} = \operatorname{Bl}_{\operatorname{Jac}_X} X$ and using a log resolution $f : X' \to X$ of Jac_X, one obtains as in (2.5.1) that

$$\omega^{GR}_X = f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - J_{X/X'}) \otimes \omega_X = \widehat{\mathscr{I}}(\mathscr{O}_X) \cdot \omega_X.$$

This equality was also mentioned explicitly or implicitly in [EIM16] and [dFD14].

(2) In their work [dFD14], when X is normal, de Fernex-Docampo proved that

$$\mathscr{I}^{\diamond}(\mathfrak{d}_X^{-1}) \cdot \omega_X \subseteq \omega_X^{GR},$$

where the ideal $\mathscr{I}^{\diamond}(\mathfrak{d}_X^{-1})$ is a special multiplier ideal defined on normal varieties. It is not clear to us right now that this notion can be generalized to any variety. It would be interesting if a similar result can be proved for arbitrary varieties.

Example 2.7. It is easy to see that in general we do not have the equality $\widehat{\mathscr{I}}(\mathscr{O}_X) \cdot \omega_X = \omega_X^{GR}$ for any variety. Indeed, we can take X to be a variety with rational singularities that are not all MJ-canonical. Then $\omega_X = \omega_X^{GR}$ but $\widehat{\mathscr{I}}(\mathscr{O}_X)$ is not trivial.

3. Mather-Jacobian multiplier ideals on curves

In this section, we study MJ-multiplier ideals on algebraic curves. We allow the ground field k to have *any characteristic*. We shall first prove that the definition of MJ-multiplier ideals still works for such a ground field k.

We start by briefly recalling some facts about general Noether normalizations. Assume that $X = \operatorname{Spec} R$ is an affine variety of dimension d and $R = k[x_1, \ldots, x_n]$ is a finitely generated k-algebra generated by $x_1, \ldots, x_n$. We can choose $x_1, \ldots, x_n$ general (for example, we can take the forms $a_{i1}x_1 + \ldots + a_{in}x_n$, where the a_{ij}'s are general elements in the ground field k) such that any d of them, say $x_{j_1}, \ldots, x_{j_d}$, give a Noether normalization $k[x_{j_1}, \cdots, x_{j_d}] \subseteq k[x_1, \cdots, x_n]$ and the quotient field $Q(R)$ is separable over $k(x_{j_1}, \cdots, x_{j_d})$. The following proposition is well-known to experts and can be easily proved by a calculation.

Proposition 3.1. *Let $X = \operatorname{Spec} R$ be an affine variety of dimension d over k, where $R = k[x_1, \cdots, x_n]$, and let X' be the normalization of X. For any $J \subset \{1, \cdots, n\}$ with $|J| = d$, write $\boldsymbol{x}_J = \{x_j \mid j \in J\}$ and $A_J := \operatorname{Spec} k[\boldsymbol{x}_J]$ and consider the following diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & A_J & \end{array}$$

Then we can take the generators $x_1, \dots, x_n$ to be general such that X is generically étale and finite over each A_J and we have

$$\mathrm{Jac}_X = \sum_J \mathrm{Jac}(X/A_J) \quad \textit{and} \quad \mathrm{Jac}(X'/X) = \sum_J \mathrm{Jac}(X'/A_J), \tag{3.1.1}$$

where $\mathrm{Jac}(X'/X)$*,* $\mathrm{Jac}(X/A_J)$*, and* $\mathrm{Jac}(X'/A_J)$ *are relative Jacobian ideals in the diagram.*

Now let X be an algebraic curve over k and let $f : X' \to X$ be the normalization of X. Let Jac_X be the Jacobian ideal of X and write $\mathrm{Jac}_X \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-J_{X/X'})$ for an effective divisor $J_{X/X'}$ on X'. Let $\mathrm{Jac}(X'/X)$ be the relative Jacobian ideal of f and $\widehat{K}_{X'/X}$ be the effective divisor on X' with $\mathrm{Jac}(X'/X) = \mathscr{O}_{X'}(-\widehat{K}_{X'/X})$. For an ideal $\mathfrak{a}$ of $\mathscr{O}_X$ we write $\mathfrak{a} \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-Z)$ where Z is an effective divisor on X'. Then formally we can form an Mather-Jacobian multiplier ideal of $\mathfrak{a}$ of exponent $t \in \mathbb{R}_{\geq 0}$ by

$$\widehat{\mathscr{I}}(X, \mathfrak{a}^t) := f_* \mathscr{O}_{X'}(\widehat{K}_{X'/X} - J_{X/X'} - \lfloor tZ \rfloor).$$

The crucial point is that we need to show this fractional ideal sheaf $\widehat{\mathscr{I}}(X, \mathfrak{a}^t)$ is indeed inside $\mathscr{O}_X$ if k has any characteristic. As any such fraction ideal is contained in $\widehat{\mathscr{I}}(\mathscr{O}_X)$, it suffices to show that $\widehat{\mathscr{I}}(\mathscr{O}_X)$ is an ideal of $\mathscr{O}_X$. When k has characteristic zero, this is proved in [EIM16]. For arbitrary characteristic, we prove it in the following proposition by using a result from Lipman and Sathaye.

Proposition 3.2. *In the setting above,* $\widehat{\mathscr{I}}(\mathscr{O}_X)$ *is contained in the conductor ideal* $\mathfrak{C}_X$ *of* X*.*

Proof. The question is local, so we assume that $X = \operatorname{Spec} R$ is affine, where R is a k-algebra generated by $x_1, \dots, x_n$. Write S for the normalization of R in L and $X' = \operatorname{Spec} S$ is the normalization of X. We choose $x_1, \dots, x_n$ as in Proposition 3.1. It suffices to show that $(S : \mathrm{Jac}(X'/X)) \cdot \mathrm{Jac}_X$ is an R-ideal. To this end, for a Noether normalization A_J as in Proposition 3.1, consider ring extensions $A_J \subseteq R \subseteq L$, where L is the quotient field of R. Then [LS81, Theorem 2] says that

$$(S :_L \mathrm{Jac}(X'/A_J)) \cdot \mathrm{Jac}(X/A_J) \subseteq \mathfrak{C}_X.$$

Note that the notations $\{A_J, R, S, L, \mathrm{Jac}(X/A_J), \mathrm{Jac}(X'/A_J)\}$ used here correspond to the notations $\{R, S, \bar{S}, L, J, \bar{J}\}$ used in *loc. cit.* Now by (3.1.1) and by the fact that $S :_L \mathrm{Jac}(X'/X) \subseteq S :_L \mathrm{Jac}(X'/A_J)$, we see that

$$(S :_L \mathrm{Jac}(X'/X)) \cdot \mathrm{Jac}_X \subseteq \mathfrak{C}_X.$$

Finally since the conductor ideal $\mathfrak{C}_X$ is inside R, the result follows. □

Remark 3.3. When $\operatorname{char} k = 0$, it was shown in [EIM16] that $\widehat{\mathscr{I}}(\mathscr{O}_X) \subseteq \mathscr{O}_X$ by reducing to the usual multiplier ideal on a nonsingular variety. Using this we proved Proposition 2.4. The method requires the existence of resolution of singularities.

If $\operatorname{char} k > 0$, although we have proved the above proposition for curves, it is not clear how to extend the theory of MJ-multiplier ideals to varieties of any dimension. The above proposition may provide some evidence in this direction.

Theorem 3.4. *Let* X *be an algebraic curve over* k *and* $\mathfrak{a} \subseteq \mathscr{O}_X$ *be an ideal. Then* $\mathfrak{a}$ *is a MJ-multiplier ideal if and only if* $\mathfrak{a}$ *is integrally closed and* $\mathfrak{a} \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X)$*.*

Proof. The necessity of the conditions is clear by the definition of Mather-Jacobian multiplier ideals. So we prove the sufficiency by assuming that $\mathfrak{a}$ is integrally closed and $\mathfrak{a} \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X)$.

Let $f : X' \to X$ be the normalization of X. We write $\mathfrak{a} \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-E)$ and decompose E as

$$E = \sum_{i=1}^{t} a_i E_i,$$

where E_i are distinct prime divisors and $a_i > 0$. Let $\widehat{K}_{X'/X}$ be the Mather discrepancy divisor. Let Jac_X be the Jacobian ideal of X, and write $\mathrm{Jac}_X \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-J_{X/X'})$. Since $\mathfrak{a} \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X)$ we have an inequality

$$-E \leq \widehat{K}_{X'/X} - J_{X/X'}. \tag{3.4.1}$$

Thus there exists an effective divisor B such that

$$-E = \widehat{K}_{X'/X} - J_{X/X'} - B.$$

We construct an effective divisor

$$B' = nB + E$$

by fixing a number $n \gg 0$ so that $\lfloor \frac{1}{n} B' \rfloor = B$. As $\mathscr{O}_{X'}(-B') \subseteq \mathscr{O}_{X'}(-E)$, we see that

$$f_* \mathscr{O}_{X'}(-B') \subseteq f_* \mathscr{O}_{X'}(-E) = \mathfrak{a} \subseteq \mathscr{O}_X,$$

where $\mathfrak{a} = f_* \mathscr{O}_{X'}(-E)$ because $\mathfrak{a}$ is integrally closed. Thus we set $\mathfrak{b} := f_* \mathscr{O}_{X'}(-B')$, which is an ideal. Now since f is a finite morphism, we then get the surjection

$$f^* f_* \mathscr{O}_{X'}(-B') \longrightarrow \mathscr{O}_{X'}(-B') \longrightarrow 0. \tag{3.4.2}$$

(Indeed, the question is local so we can assume $X' = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and $\mathscr{O}_{X'}(-B') = \widetilde{M}$ for a B-module M. Then the surjectivity of the map in (3.4.2) follows from the fact that the natural map $M \otimes_A B \to M$ is surjective.) Thus we see that $\mathfrak{b} \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-B')$. Finally, we check using the definition that $\mathfrak{a} = \widehat{\mathscr{I}}(\mathfrak{b}^{\frac{1}{n}})$. □

Remark 3.5. (1) Note that the theorem above is global, i.e., X need not to be affine.

(2) Theorem 3.4 is certainly not true for higher dimensional varieties by the result of Lazarsfeld-Lee [LL07]. However, for surfaces, in light of the work of Lipman-Watanabe [LW03], Favre-Jonsson [FJ05], and Tucker [Tuc09], we may expect the following conjecture.

Conjecture 3.6. Let X be an algebraic surface over k and $\mathfrak{a} \subseteq \mathscr{O}_X$ be an ideal. Let $\mathfrak{p} \in X$ be a closed point. Then $\mathfrak{a}_\mathfrak{p}$ is a MJ-multiplier ideal at $\mathfrak{p}$ if and only if $\mathfrak{a}_\mathfrak{p}$ is integrally closed and $\mathfrak{a}_\mathfrak{p} \subseteq \widehat{\mathscr{I}}(\mathscr{O}_X)_\mathfrak{p}$.

Next we prove Theorem 1.3. We first state two easy lemmas used in the proof.

Lemma 3.7. *Let X be a variety over k and $f : X' \to X$ be its normalization. Let $\mathfrak{C}_X$ be the conductor ideal of X. Then one has $\mathfrak{C}_X \cdot \omega_X \subseteq f_* \omega_{X'}$. If furthermore X is Gorenstein then one has $\mathfrak{C}_X = f_* \omega_{X'} \otimes \omega_X^{-1}$.*

Proof. Recall that $\mathfrak{C}_X = \mathscr{H}om(f_* \mathscr{O}_{X'}, \mathscr{O}_X)$. Then we have the following diagram

$$\begin{array}{ccc} & & \mathscr{H}om(f_* \mathscr{O}_{X'}, \omega_X) = f_* \omega_{X'} \\ & \nearrow & \downarrow \\ \mathfrak{C}_X \otimes \omega_X & \longrightarrow & \omega_X. \end{array}$$

This implies that $\mathfrak{C}_X \cdot \omega_X \subseteq f_* \omega_{X'}$. If X is Gorenstein, then by definition one has that $\mathfrak{C}_X = f_* \omega_{X'} \otimes \omega_X^{-1}$. □

Lemma 3.8. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one with a canonical module ω_R. Let M be a finite torsion-free R-module and N be a submodule of M such that the length $\lambda(M/N)$ is finite. Then one has*

$$\lambda(M/N) = \lambda(\operatorname{Hom}_R(N,\omega_R)/\operatorname{Hom}_R(M,\omega_R)).$$

Proof. Apply $\operatorname{Hom}_R(\,\_\,,\omega_R)$ to the exact sequence $0 \to N \to M \to M/N \to 0$. From the assumption that M/N has finite length and M is torsion-free, we deduce an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M,\omega_R) \longrightarrow \operatorname{Hom}_R(N,\omega_R) \longrightarrow \operatorname{Ext}^1_R(M/N,\omega_R) \longrightarrow 0.$$

Now by local duality and Matlis duality, we have

$$\lambda(M/N) = \lambda(\operatorname{Ext}^1_R(M/N,\omega_R)),$$

and the result follows. □

Theorem 3.9. *Let X be an algebraic curve over k. Let $\widehat{\mathscr{I}}(\mathscr{O}_X)$ be the MJ-multiplier ideal and $\mathfrak{C}_X$ be the conductor ideal. Then X is a local complete intersection if and only if $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$.*

Proof. Let $f : X' \to X$ be the normalization of X. If X is a local complete intersection, the fundamental class $\Omega^1_X \to \omega_X$ has image $\operatorname{Jac}_X \otimes \omega_X$, where Jac_X is the Jacobian ideal of X. We deduce that $\omega_{X'} = \mathscr{O}_{X'}(\widehat{K}_{X'/X} - J_{X/X'}) \otimes f^*\omega_X$, where $\operatorname{Jac}_X \cdot \mathscr{O}_{X'} = \mathscr{O}_{X'}(-J_{X/X'})$ for an effective divisor $J_{X/X'}$ on X'. The projection formula gives that $f_*\omega_{X'} = \widehat{\mathscr{I}}(\mathscr{O}_X) \otimes \omega_X$. From Lemma 3.7, we conclude $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$.

Next we prove the sufficient part of the theorem assuming that $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$. The question is local, so we can assume that X is an affine curve. Since f is finite the condition $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$ is the same as

$$\operatorname{Jac}_X \cdot \mathscr{O}_{X'} = \mathfrak{C}_X \cdot \operatorname{Jac}(X'/X), \tag{3.9.1}$$

where $\operatorname{Jac}(X'/X)$ is the relative Jacobian of f. Let $\mathfrak{p} \in X$ be a closed point and $R := \mathscr{O}_\mathfrak{p}$ be the local ring at $\mathfrak{p}$. Consider the following fiber product

$$\begin{array}{ccc} X'_\mathfrak{p} & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow f \\ \operatorname{Spec} R & \longrightarrow & X \end{array}$$

where $X'_\mathfrak{p} := \operatorname{Spec} R \times_X X'$, and write $X'_\mathfrak{p} = \operatorname{Spec} S$. Note that S is a regular semilocal noetherian ring and is a finitely generated R-module. Furthermore since S is locally a principal ideal domain, it is then a principal ideal domain, i.e., any ideal of S is generated by an element of S.

Now we shall take a Noether normalization $\mu : X \to \mathbb{A} = \operatorname{Spec} k[y]$ for some y in $\mathscr{O}_X$. Write $\operatorname{Jac}(X/\mathbb{A})$ and $\operatorname{Jac}(X'/\mathbb{A})$ for the relative Jacobian ideals. We make the following crucial claim in our proof.

Claim 3.9.2. We can choose y general so that

(1) $\operatorname{Jac}(X'/X) \cdot S = \operatorname{Jac}(X'/\mathbb{A}) \cdot S$;
(2) $(\operatorname{Jac}_X) \cdot S = \operatorname{Jac}(X/\mathbb{A}) \cdot S$.

Proof of claim. We can assume $R = k[x_1, \dots, x_n]$. Consider the natural exact sequence

$$S \otimes_R \Omega^1_{R/k} \xrightarrow{u} \Omega^1_{S/k} \longrightarrow \Omega^1_{S/R} \longrightarrow 0.$$

The image of u is generated by $1 \otimes dx_1, \ldots, 1 \otimes dx_n$ as S-module. Because S is regular and semilocal, we see that $\Omega^1_{S/k}$ is isomorphic to S. For the same reason the image of u, as an S-submodule, can be generated by one element of the form $1 \otimes dy$, where $y = a_1x_1 + \ldots + a_nx_n$ and $a_1, \ldots, a_n$ are general elements of the infinite field k. Thus we take this y to produce the Noether normalization $\mathbb{A} = \operatorname{Spec} k[y]$ with the morphism $\mu : X \to \mathbb{A}$. Write $\mathfrak{q} := \mu(\mathfrak{p})$ and $A := k[y]_{\mathfrak{q}}$. Then in the sequence

$$S \otimes_A \Omega^1_{A/k} \xrightarrow{u_A} \Omega^1_{S/k} \longrightarrow \Omega^1_{S/A} \longrightarrow 0,$$

the image of u_A is $S \cdot (1 \otimes dy)$. This shows that $\Omega^1_{S/R} = \Omega^1_{S/A}$ and then statement (1) follows.

Since the image of u is a free S-module generated by $1 \otimes dy$, we obtain

$$S \otimes_R \Omega^1_{R/k} = S \cdot (1 \otimes dy) \oplus T,$$

where T is the torsion part. We can check that $T \cong S \otimes_R \Omega^1_{R/A}$. Also from the splitting above we see that $\operatorname{Fitt}^1(S \otimes_R \Omega^1_{R/k}) = \operatorname{Fitt}^0(T)$ and then statement (2) follows. Thus Claim 3.9.2 is proved.

Having Claim 3.9.2 in hand, we see that on $X'_{\mathfrak{p}}$, the condition (3.9.1) becomes

$$\operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}} \cdot S = \mathfrak{C}_{X,\mathfrak{p}} \cdot (\operatorname{Jac}(X'/\mathbb{A}) \cdot S). \tag{3.9.3}$$

Here we shall use the notion of *Dedekind complementary modules*. We briefly recall the definition here and the details can be found in [Kun08, Definition 8.4]. Write $L = Q(R)$ for the the quotient field of X. The quotient field of $\mathbb{A}$ is $k(y)$ and L is a vector space over $k(y)$. In addition, one has

$$\operatorname{Hom}_{k(y)}(L, k(y)) \cong L \cdot \operatorname{tr}_{L/k(y)}$$

is generated by the trace map $\operatorname{tr}_{L/k(y)}$ from L to $k(y)$. So the image of the natural map

$$\operatorname{Hom}_{k[y]}(R, k[y]) \longrightarrow \operatorname{Hom}_{k(y)}(L, k(y))$$

can be written as $\mathcal{C}(X/\mathbb{A}) \cdot \operatorname{tr}_{L/k(y)}$ for a R-ideal $\mathcal{C}(X/\mathbb{A})$ in L, which is called the Dedekind complementary modules of $X/\mathbb{A}$.

Let $\mathcal{C}(X/\mathbb{A})$ and $\mathcal{C}(X'/\mathbb{A})$ be the Dedekind complementary modules of $X/\mathbb{A}$ and $X'/\mathbb{A}$. They are the canonical modules of $X/\mathbb{A}$ and $X'/\mathbb{A}$ respectively. We have inclusions $R \subseteq S \subseteq \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}} \subseteq \mathcal{C}(X/\mathbb{A})_{\mathfrak{p}}$. On the other hand, we have inclusions

$$\operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}} \subseteq R :_L \mathcal{C}(X/\mathbb{A})_{\mathfrak{p}} \subseteq R :_L \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}}. \tag{3.9.4}$$

Since S is a principal ideal domain, $\operatorname{Jac}(X'/\mathbb{A}) \cdot S = S \cdot y$ for a non zero-divisor $y \in S$. Because X' is a nonsingular curve, we have $\mathcal{C}(X'/\mathbb{A}) = \mathscr{O}_{X'} :_L \operatorname{Jac}(X'/\mathbb{A})$ [Ber64, Satz 2] and therefore we have $\mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}} = S :_L (\operatorname{Jac}(X'/\mathbb{A}) \cdot S) = S \cdot y^{-1}$. Thus $R :_L \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}} = R :_L (S \cdot y^{-1}) = (R :_L S)y = \mathcal{C}_{X,\mathfrak{p}} \cdot y$. But $\mathfrak{C}_{X,\mathfrak{p}}$ is also an S-ideal so $\mathfrak{C}_{X,\mathfrak{p}} \cdot y = \mathfrak{C}_{X,\mathfrak{p}} \cdot (S \cdot y) = \mathfrak{C}_{X,\mathfrak{p}} \cdot (\operatorname{Jac}(X'/\mathbb{A}) \cdot S)$. Thus we deduce that

$$R :_L \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}} = \mathfrak{C}_{X,\mathfrak{p}} \cdot (\operatorname{Jac}(X'/\mathbb{A}) \cdot S),$$

and therefore (3.9.3) is equivalent to

$$R :_L \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}} = \operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}} \cdot S. \tag{3.9.5}$$

Again, since S is a principal ideal domain, we have $\operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}} \cdot S = S \cdot \delta$, where $\delta \in \operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}}$. We show that this δ generates $\operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}}$ as an R-ideal. To this end, consider the inclusions of (3.9.4),

$$R \cdot \delta \subseteq \operatorname{Jac}(X/\mathbb{A})_{\mathfrak{p}} \subseteq R :_L \mathcal{C}(X/\mathbb{A})_{\mathfrak{p}} \subseteq R :_L \mathcal{C}(X'/\mathbb{A})_{\mathfrak{p}}.$$

By equality (3.9.5), we have $R :_L \mathcal{C}(X'/\mathbb{A})_\mathfrak{p} = \operatorname{Jac}(X/\mathbb{A})_\mathfrak{p} \cdot S = S \cdot \delta$. Thus we obtain

$$R \cdot \delta \subseteq \operatorname{Jac}(X/\mathbb{A})_\mathfrak{p} \subseteq R :_L \mathcal{C}(X/\mathbb{A})_\mathfrak{p} \subseteq R :_L \mathcal{C}(X'/\mathbb{A})_\mathfrak{p} = S \cdot \delta.$$

Granting the inequality in Claim 3.9.6 for the time being, we immediately get $R \cdot \delta = \operatorname{Jac}(X/\mathbb{A})_\mathfrak{p}$. This means that $\operatorname{Jac}(X/\mathbb{A})$ is principal at $\mathfrak{p}$.

Now by using [Lip69, Lemma 1], we see that the projective dimension $\operatorname{pd}_R(\Omega^1_{X/\mathbb{A}})_\mathfrak{p} \leq 1$ since $\operatorname{Jac}(X/\mathbb{A})$ is principal at $\mathfrak{p}$. Also notice that we have a short exact sequence

$$0 \longrightarrow \mu^*\Omega^1_{\mathbb{A}/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/\mathbb{A}} \longrightarrow 0,$$

where $\Omega^1_{\mathbb{A}/k}$ is locally free. Then we obtain that $\operatorname{pd}_R(\Omega^1_{X/k})_\mathfrak{p} \leq 1$ and therefore $\operatorname{pd}_{\mathscr{O}_X} \Omega^1_{X/k} \leq 1$, which implies that X is locally a complete intersection by the well-known result of [Vas68] or [Fer67].

Finally, we need to prove the following claim to finish our proof.

Claim 3.9.6. Use λ to denote the length of modules. Then one has

$$\lambda\left(\frac{R : \mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}{R : \mathcal{C}(X/\mathbb{A})_\mathfrak{p}}\right) \geq \lambda\left(\frac{\mathcal{C}(X/\mathbb{A})_\mathfrak{p}}{\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}\right) = \lambda\left(\frac{S}{R}\right)$$

Proof. First note that

$$\frac{\operatorname{Hom}_R(R, \mathcal{C}(X/\mathbb{A})_\mathfrak{p})}{\operatorname{Hom}_R(S, \mathcal{C}(X/\mathbb{A})_\mathfrak{p})} \cong \frac{\mathcal{C}(X/\mathbb{A})_\mathfrak{p} :_L R}{\mathcal{C}(X/\mathbb{A})_\mathfrak{p} :_L S} = \frac{\mathcal{C}(X/\mathbb{A})_\mathfrak{p}}{\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}.$$

Now the asserted equality of lengths in the claim follows from Lemma 3.8 since $\mathcal{C}(X/\mathbb{A})_\mathfrak{p} \cong \omega_R$.

To show the inequality, we note that there exits a canonical module K of R such that $R \subseteq K \subseteq S$. Indeed, we can first choose a canonical ideal ω of R, then there is an element $t \in \omega$ such that $St = \omega \cdot S$ since S is a principal ideal domain and k is infinite. Then we simply take $K := t^{-1}\omega$. Also note that such K satisfies the condition $KS = S$.

Now by the choice of K, we have $\mathcal{C}(X/\mathbb{A})_\mathfrak{p} \subseteq K\mathcal{C}(X/\mathbb{A})_\mathfrak{p}$ and $\mathcal{C}(X'/\mathbb{A})_\mathfrak{p} = K\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}$, which implies that

$$\lambda\left(\frac{\mathcal{C}(X/\mathbb{A})_\mathfrak{p}}{\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}\right) \leq \lambda\left(\frac{K\mathcal{C}(X/\mathbb{A})_\mathfrak{p}}{K\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}\right).$$

On the other hand, we have

$$K :_L K\mathcal{C}(X'/\mathbb{A})_\mathfrak{p} = (K :_L K) :_L \mathcal{C}(X'/\mathbb{A})_\mathfrak{p} = R :_L \mathcal{C}(X'/\mathbb{A})_\mathfrak{p}$$

since $K :_L K = R$. Similarly, $K :_L K\mathcal{C}(X/\mathbb{A})_\mathfrak{p} = R :_L \mathcal{C}(X/\mathbb{A})_\mathfrak{p}$. Thus by Lemma 3.8 we have

$$\lambda\left(\frac{K\mathcal{C}(X/\mathbb{A})_\mathfrak{p}}{K\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}\right) = \lambda\left(\frac{\operatorname{Hom}_R(K\mathcal{C}(X'/\mathbb{A})_\mathfrak{p}, K)}{\operatorname{Hom}_R(K\mathcal{C}(X/\mathbb{A})_\mathfrak{p}, K)}\right) = \lambda\left(\frac{R :_L \mathcal{C}(X'/\mathbb{A})_\mathfrak{p}}{R :_L \mathcal{C}(X/\mathbb{A})_\mathfrak{p}}\right),$$

which proves the asserted inequality.

□

Remark 3.10. The same proof shows that Theorem 3.9 holds for codimension one points on any variety. Let X be a variety of any dimension and let $\mathfrak{p} \in X$ be a codimension one point, then X is a local complete intersection at $\mathfrak{p}$ if and only if $\widehat{\mathscr{I}}(\mathscr{O}_X)_\mathfrak{p} = \mathfrak{C}_{X,\mathfrak{p}}$.

Example 3.11. We adopt this example from [EIM16] and [Eis95, Excercise 11.16]. Consider $X = \operatorname{Spec} k[x^2, x^3]$. It is a curve with a cusp at the origin. One can calculate that the MJ-multiplier ideal $\widehat{\mathscr{I}}(\mathscr{O}_X) = (x^2) \subseteq k[x^2, x^3]$ and the conductor ideal $\mathfrak{C}_X = (x^2)$. Thus $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$.

Example 3.12. (1) Consider X to be the cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ over a field of characteristic zero. Then X is MJ-canonical so that $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathscr{O}_X = \mathfrak{C}_X$. But X is not a local complete intersection. For more details, we refer to [EI15, Example 3.13].

(2) Consider X to be the cone over a nonsingular hypersurface of degree $d \geq 3$ in $\mathbb{P}^{d-1}$ over $\mathbb{C}$. Then X is a normal locally complete intersection. Moreover, X is MJ-log canonical but not MJ-canonical. Thus $\widehat{\mathscr{I}}(\mathscr{O}_X) \neq \mathscr{O}_X$ while $\mathfrak{C}_X = \mathscr{O}_X$. See [EI15, Example 3.12] for more details.

To finish, we discuss some cases of higher dimensional varieties where the MJ-multiplier ideal and the conductor ideal coincide.

Proposition 3.13. *Let X be a Gorenstein variety over a field of characteristic zero, $\widehat{\mathscr{I}}(\mathscr{O}_X)$ be the MJ-multiplier ideal, and $\mathfrak{C}_X$ be the conductor ideal. Let X' be the normalization of X. Assume that $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$. Then one has $\omega_{X'} = \omega_{X'}^{GR}$, where $\omega_{X'}^{GR}$ is the Grauert-Riemenschneider canonical sheaf of X'.*

Proof. Let $f : X' \to X$ be the normalization morphism from X' to X. By Lemma 3.7, one has

$$\mathfrak{C}_X = f_*\omega_{X'} \otimes \omega_X^{-1}.$$

On the other hand, by Theorem 2.5, we have $\widehat{\mathscr{I}}(\mathscr{O}_X) \subseteq \omega_X^{GR} \otimes \omega_X^{-1}$. Thus the assumption $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$ implies that $f_*\omega_{X'} \subseteq \omega_X^{GR}$. On the other hand, $\omega_X^{GR} = f_*\omega_{X'}^{GR} \subseteq f_*\omega_{X'}$. So we have $f_*\omega_{X'}^{GR} = f_*\omega_{X'}$. Since f is affine and finite, we deduce that $\omega_{X'}^{GR} = \omega_{X'}$. □

Corollary 3.14. *Assume that X is a Gorenstein surface over a field of characteristic zero such that $\widehat{\mathscr{I}}(\mathscr{O}_X) = \mathfrak{C}_X$. Then the normalization of X has rational singularities.*

Proof. Let X' be the normalization of X. Then by the proposition above, $\omega_{X'} = \omega_{X'}^{GR}$. But X' is Cohen-Macaulay, thus X' has rational singularities. □

References

[Ber64] Robert Berger. Differenten regulärer Ringe. *J. Reine Angew. Math.*, 214:441–442, 1964.

[dFD14] Tommaso de Fernex and Roi Docampo. Jacobian discrepancies and rational singularities. *J. Eur. Math. Soc. (JEMS)*, 16:165–199, 2014.

[EI15] Lawrence Ein and Shihoko Ishii. Singularities with respect to Mather-Jacobian discrepancies. In *Com. Algebra and Noncom. Algebraic Geometry I, MSRI Publ.*, 67:125–168, 2015.

[EIM16] Lawrence Ein, Shihoko Ishii, and Mircea Mustaţă. Multiplier ideals via Mather discrepancy. In *Minimal models and extremal rays (Kyoto, 2011)*, volume 70 of *Adv. Stud. Pure Math.*, pages 9–28. Math. Soc. Japan, Tokyo, 2016.

[EM09] Lawrence Ein and Mircea Mustaţă. Jet schemes and singularities. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 505–546. Amer. Math. Soc., Providence, RI, 2009.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[FJ05] Charles Favre and Mattias Jonsson. Valuations and multiplier ideals. *J. Amer. Math. Soc.*, 18:655–684, 2005.

[Fer67] Daniel Ferrand. Suite régulière et intersection complète. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A427–A428, 1967.

[Kun08] Ernst Kunz. *Residues and duality for projective algebraic varieties*, volume 47 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. With the assistance of and contributions by David A. Cox and Alicia Dickenstein.

[Kun13] Ernst Kunz. *Introduction to commutative algebra and algebraic geometry*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. Translated from the 1980 German original [MR0562105] by Michael Ackerman, With a preface by David Mumford, Reprint of the 1985 edition [MR0789602].

[LL07] Robert Lazarsfeld and Kyungyong Lee. Local syzygies of multiplier ideals. *Invent. Math.*, 167:409–418, 2007.

[Lip69] Joseph Lipman. On the Jacobian ideal of the module of differentials. *Proc. Amer. Math. Soc.*, 21:422–426, 1969.

[LS81] Joseph Lipman and Avinash Sathaye. Jacobian ideals and a theorem of Briançon-Skoda. *Michigan Math. J.*, 28:199–222, 1981.

[LW03] Joseph Lipman and Kei-ichi Watanabe. Integrally closed ideals in two-dimensional regular local rings are multiplier ideals. *Math. Res. Lett.*, 10:423–434, 2003.

[Niu14] Wenbo Niu. Geometric Nullstellensatz and symbolic powers on arbitrary varieties. *Math. Ann.*, 359:745–758, 2014.

[Tuc09] Kevin Tucker. Integrally closed ideals on log terminal surfaces are multiplier ideals. *Math. Res. Lett.*, 16:903–908, 2009.

[Vas68] Wolmer V. Vasconcelos. A note on normality and module of differentials. *Math. Z.*, 105:291–293, 1968.

[ZS75] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. 1*, volume No. 28 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg-Berlin, 1975. With the cooperation of I. S. Cohen, Corrected reprinting of the 1958 edition.