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Rees algebras over residual intersections

In memory of Wolmer Vasconcelos, 1937–2021

Abstract: We continue our work on the residual intersections of the ideals I generated by the 2×2 minors of a $2 \times n$ generic matrix with $n \geq 4$, which we initiated in [5]. In this paper we study the Rees algebras over these residual intersections, computing in particular their depths and canonical modules.

Keywords: Rees Algebras, Residual Intersections

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Introduction

We feel that this paper is particularly appropriate for a volume celebrating the life and contributions of our friend and mentor Wolmer Vasconcelos, since it combines several of his core interests: Rees algebras, residual intersections, and (for discovering the results) computational algebra.

Recall that a proper ideal K in a commutative ring S is called an *s-residual intersection* of an ideal I if, for some elements $a_1, \ldots, a_s$ in I, we have $K = (a_1, \ldots, a_s) : I$ and K has codimension $\geq s$. The residual intersection is *geometric* if in addition $I + K$ has codimension $\geq s + 1$.

In [5] we initiated the study of the (sufficiently general) s-residual intersections $R_s = S/K$ of the ideal I generated by the 2×2 minors of a $2 \times n$ generic matrix with $n \geq 4$. These are interesting because for $n-1 < s < 2n-3$ they are among the simplest examples of residual intersections of Cohen–Macaulay ideals that are not Cohen–Macaulay (here $n-1$ and $2n-3$ are the codimension and the analytic spread of I, respectively). However, they are residually S_2: this follows from a general theorem from [3] and a computation of [12] showing the vanishing of certain Ext groups of powers of I.

In [5] we showed that when $s = 2n - 4$, the largest interesting value, the ideal $(IR_s)^j$ is an (up to shift) ω_{R_s}-self-dual Ulrich module whenever $j \geq n - 3$, and we constructed an explicit desingularization of R_s. In this case R_s has depth 1 and dimension 4.

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In [6] we studied the modules $(IR_s)^j$ for $0 \le s \le 2n-4$ and $j \ge -1$, and computed their depths (they are Cohen–Macaulay, for example, when $j = 1$ and $s = n$, which was a surprise to us). We also proved that for generic residual intersections with $s \le 2n-5$, the class of IR_s generates the class group of R_s.

In the current paper we build on this information to study the Rees algebra $\mathcal{R}_s$ of the ideal $IR_s \subset R_s$. In parallel to our discovery of the unexpected properties of the powers $(IR_s)^j$ in [5] and [6], we also study the truncations $(\mathcal{R}_s)_{\ge j}$. We compute the canonical modules of the Rees rings and the depths of all the modules $(\mathcal{R}_s)_{\ge j}$, and we prove a duality among them.

Perhaps most surprising is that the statements about the $\mathcal{R}_s$ are so similar to those about the R_s. In the case $s = n+1$ for example, we show that the truncation $(\mathcal{R}_s)_{\ge 2}$ is a maximal Cohen–Macaulay $\mathcal{R}_s$-module, although depth $\mathcal{R}_s = \dim \mathcal{R}_s - 4$.

In a preliminary section we prove a bigraded version of Theorem 2.4 of [8] computing the canonical modules of the Rees ring and the associated graded ring of an arbitrary homogeneous ideal, under the assumptions that the associated graded ring is Cohen–Macaulay and that its canonical module is generated in one degree. We apply these results to the Rees rings and associated graded rings of an ideal modulo its link. This will be used in our results about determinantal ideals to handle the cases $s \le n-1$.

1 The canonical modules of $\mathrm{gr}_I(S)$ and $\mathcal{R}(I)$

In most of this section we will use the *natural* grading of the Rees algebra: if I is a homogeneous ideal of a graded ring S, then the natural grading on $\mathcal{R}(I) = S[It]$ is the grading induced from the bigrading of the polynomial ring $S[t]$ where t has degree $(0,1)$. This induces the natural grading of the associated graded ring $\mathrm{gr}_I(S) = \mathcal{R}(I)/I\mathcal{R}(I)$.

Theorem 1.1. *Suppose S is a generically Gorenstein positively graded algebra over a field and $I \subsetneq S$ is a homogeneous ideal of codimension $g > 0$ that is generically a complete intersection. If $G := \mathrm{gr}_I(S)$ is Cohen–Macaulay, with graded canonical module generated in degrees $(*, s)$ for fixed s, then in the natural bigrading:*

(a) $\omega_G = \mathrm{gr}_I(\omega_S)(0, -g)$,

(b) *$\mathcal{R}(I)$ is Cohen–Macaulay, with graded canonical module*

$$\omega_{\mathcal{R}(I)} = \begin{cases} \omega_S It\mathcal{R}(I) & \text{if } g = 1, \\ \omega_S (1,t)^{g-2} t\mathcal{R}(I) & \text{if } g \ge 2, \end{cases}$$

considered as a submodule of $\omega_S \otimes_S S[t]$.

Proof. Note that S must be Cohen–Macaulay because G is. We next prove that $\mathcal{R} := \mathcal{R}(I)$ is Cohen–Macaulay. For this, it suffices by [14, Theorem 1.1] to show that $a(G) = -s < 0$. Let P be a minimal prime of I of codimension g. Since ω_G is generated entirely in degree

$(*, s)$, it follows that $(\omega_G)_P = \omega_{G_P}$ is generated in degree s, and G_P is a polynomial ring in g variables. Thus $a(G) = -g$, and $\mathcal{R}(I)$ is Cohen–Macaulay.

By [8, Theorem 2.4 and its proof], $\omega_G = \operatorname{gr}_I(\omega_S)(u, v)$ for some integers u, v, and by the argument above $v = -g$.

To identify u and prove part (b), let K be the total ring of quotients of S. We represent $\omega_{\mathcal{R}}$ as a graded submodule of $K \otimes_S \omega_{\mathcal{R}} = \omega_{K\otimes_S \mathcal{R}}$, which we identify with $tK[t]$. Thus, as a graded module, $\omega_{\mathcal{R}}$ has no components in degree $(*, \leq 0)$, and we may write it as

$$\omega_{\mathcal{R}} = \bigoplus_{i=1}^{\infty} \omega_i t^i,$$

where the ω_i are graded fractional ideals of S.

Next consider the two exact sequences

$$0 \to It\mathcal{R} \to \mathcal{R} \to S \to 0,$$
$$0 \to I\mathcal{R} \to \mathcal{R} \to G \to 0.$$

Since $\mathcal{R}$ is Cohen–Macaulay, dualizing into $\omega_{\mathcal{R}}$ yields exact sequences

$$0 \to \omega_{\mathcal{R}} \to \operatorname{Hom}_{\mathcal{R}}(It\mathcal{R}, \omega_{\mathcal{R}}) \to \omega_S \to 0,$$
$$0 \to \omega_{\mathcal{R}} \to \operatorname{Hom}_{\mathcal{R}}(I\mathcal{R}, \omega_{\mathcal{R}}) \to \omega_G \to 0.$$

We may identify $\operatorname{Hom}_{\mathcal{R}}(I\mathcal{R}, \omega_{\mathcal{R}})$ with $\omega_{\mathcal{R}} :_{K(t)} I$. After tensoring with K, the ideal I becomes the unit ideal, so $\omega_{\mathcal{R}} :_{K(t)} I \subset K \otimes_S \omega_{\mathcal{R}} = tK[t]$ by our previous identification. Thus

$$\omega_{\mathcal{R}} :_{K(t)} I = \omega_{\mathcal{R}} :_{K[t]} I.$$

Also, $\omega_{\mathcal{R}} :_{K[t]} I$ and $\omega_{\mathcal{R}}$ are 0 in all degrees $(*, \leq 0)$. Furthermore,

$$\operatorname{Hom}_{\mathcal{R}}(It\mathcal{R}, \omega_{\mathcal{R}}) = \omega_{\mathcal{R}} :_{K(t)} It = (\omega_{\mathcal{R}} :_{K(t)} I)t^{-1} = (\omega_{\mathcal{R}} :_{K[t]} I)t^{-1}.$$

With these identifications, the second pair of exact sequences becomes

$$0 \to \omega_{\mathcal{R}} \to (\omega_{\mathcal{R}} :_{K[t]} I)t^{-1} \to \omega_S \to 0, \tag{9.1}$$
$$0 \to \omega_{\mathcal{R}} \to \omega_{\mathcal{R}} :_{K[t]} I \to \operatorname{gr}_I(\omega_S)(u, -g) \to 0, \tag{9.2}$$

where the left-hand maps are the natural inclusion maps. We may decompose $\omega_{\mathcal{R}} :_{K[t]} I$ into graded components

$$\omega_{\mathcal{R}} :_{K[t]} I = \bigoplus_{i=1}^{\infty} \mathfrak{a}_i t^i,$$

where the $\mathfrak{a}_i$ are graded fractional ideals of S. Note that

$$(\omega_{\mathcal{R}} :_{K[t]} I)t^{-1} = \bigoplus_{i=0}^{\infty} \mathfrak{a}_{i+1}t^i.$$

Using sequence (9.1) and the fact that $(\omega_{\mathcal{R}})_{(*,0)} = 0$, we see that $\mathfrak{a}_1 \cong \omega_S$. On the other hand, for $i > 0$ we have $(\omega_S)_{(*,i)} = 0$, so $\omega_i = \mathfrak{a}_{i+1}$. Using sequence (9.2) and the fact that the component $(\mathrm{gr}_I(\omega_S)(u, -g))_{(*,i)} = 0$ for $i \le g-1$, we see that $\omega_i = \mathfrak{a}_i$ and thus if $g \ge 2$,

$$\omega_S \cong \mathfrak{a}_1 = \omega_1 = \mathfrak{a}_2 = \omega_2 = \cdots = \omega_{g-1} = \mathfrak{a}_g.$$

By sequence (9.2), $\omega_S \cong \mathfrak{a}_g$ maps surjectively to $(\omega_S/I\omega_S)(u)$, so $u = 0$, proving part (a) of the theorem.

For part (b) of the theorem, we will prove by induction that, for $i \ge g$, $\mathfrak{a}_i = I^{i-g}\mathfrak{a}_1$ starting with the case $i = g$, already proven. For $i \ge g$, we use sequence (9.2), giving an exact sequence of graded S-modules

$$0 \to \omega_i \to \mathfrak{a}_i \xrightarrow{\pi} \frac{I^{i-g}\omega_S}{I^{i-g+1}\omega_S} \to 0.$$

By induction, $\mathfrak{a}_i = I^{i-g}\mathfrak{a}_1$. The kernel of the map π obviously contains $I\mathfrak{a}_i$. It factors through an isomorphism $\mathfrak{a}_i/I\mathfrak{a}_i \cong I^{i-g}\omega_S/I^{i-g+1}\omega_S$. Since a surjective endomorphism of a finitely generated module is an isomorphism, the kernel, which is ω_i, must be exactly $I\mathfrak{a}_i = I^{i-g+1}\mathfrak{a}_1$. This completes the induction because $\omega_i = \mathfrak{a}_{i+1}$.

Putting the components together, we see that $\omega_i = \mathfrak{a}_{i+1} = I^{i-g+1}\mathfrak{a}_1$ for $i \ge g$, where $\mathfrak{a}_1 \cong \omega_S$. Thus if $g = 1$ then $\omega_{\mathcal{R}} = \mathfrak{a}_1 It\mathcal{R} \cong \omega_S It\mathcal{R}$, while for $g \ge 2$, $\omega_{\mathcal{R}} = \mathfrak{a}_1(1,t)^{g-2}t\mathcal{R} \cong \omega_S(1,t)^{g-2}t\mathcal{R}$. □

Corollary 1.2. *Let S, I, G be as in Theorem 1.1. Let $a_1, \dots, a_i$ be forms in I that are a superficial sequence for I and generically part of a minimal generating set for I. Write $R := S/(a_1, \dots, a_i)$ and assume R is generically Gorenstein. If $i < g$ then in the natural bigrading:*

(a) *$\mathrm{gr}_{IR}(R)$ is Cohen–Macaulay, with graded canonical module*

$$\omega_{\mathrm{gr}_{IR}(R)} = \mathrm{gr}_{IR}(\omega_R)(0, i-g).$$

(b) *$\mathcal{R}(IR)$ is Cohen–Macaulay, with graded canonical module*

$$\omega_{\mathcal{R}(IR)} = \begin{cases} \omega_R It\mathcal{R}(IR) & \text{if } i = g-1, \\ \omega_R(1,t)^{g-i-2}t\mathcal{R}(IR) & \text{if } i \le g-2. \end{cases}$$

Proof. Since $a_1, \dots, a_i$ form a superficial sequence for I with $i \le g$ and G is Cohen–Macaulay, the initial forms $a_1^*, \dots, a_i^*$ in $G_{(*,1)}$ are a G-regular sequence. So $a_1, \dots, a_i$ are an R-regular sequence and

$$\mathrm{gr}_{IR}(R) \cong G/(a_1^*,\ldots,a_i^*).$$

The assertions now follow by applying Theorem 1.1 to $IR \subset R$. □

Combining the previous result with [8, Theorem 2.6], we get:

Corollary 1.3. *With assumptions as in Corollary 1.2 with $i \le g-2$, the $\mathcal{R}(IR)$-modules $\omega_R(1,t)^j\mathcal{R}(IR)$ and $(1,t)^j\mathcal{R}(IR)$ are Cohen–Macaulay for $-1 \le j \le g-i-1$.*

Theorem 1.4. *Let S, I, G be as in Theorem 1.1. In addition, suppose that S is Gorenstein with $\omega_S = S(\alpha)$, and S/I is Cohen–Macaulay. Let $a_1,\ldots,a_g$ be forms in I of degrees $d_1,\ldots,d_g$ that are a superficial sequence for I and generate I generically. Set $K := (a_1,\ldots,a_g) : I$ and $R := S/K$, and assume R is Gorenstein locally in codimension one. In the natural bigrading:*

(a) *$\mathrm{gr}_{IR}(R)$ is Cohen–Macaulay, with canonical module*

$$\omega_{\mathrm{gr}_{IR}(R)} = \mathrm{gr}_{IR}(R)\Big(\alpha + \sum_i d_i\,,0\Big)_{(*,\ge 1)}.$$

(b) *$\mathcal{R}(IR)$ is Cohen–Macaulay, with canonical module*

$$\omega_{\mathcal{R}(IR)} = (It)^2\mathcal{R}(IR)\Big(\alpha + \sum_i d_i\,,1\Big).$$

Proof. We may harmlessly assume $I \ne (a_1,\ldots,a_g)$ since otherwise $R = 0$. As in the proof of Corollary 1.2, we may factor out the regular sequence $a_1,\ldots,a_g$ and assume $\operatorname{codim} I = 0$ and $\omega_S = S(\alpha+\sum_i d_i)$. By Theorem 1.1, we have $\omega_G = \mathrm{gr}_I(\omega_S) = G(\alpha+\sum_i d_i, 0)$.

In this situation, I is generically 0 and $K = 0 : I$ is a geometric link, so $I \cap K = 0$. Since $I^j \cap K = 0$ for all $j > 0$, there is an exact sequence of graded G-modules

$$0 \to \frac{I+K}{I} \to G \to \mathrm{gr}_{IR}(R) \to 0.$$

Note that $(I+K)/I \cong \omega_{S/I}(-\alpha - \sum_i d_i, 0)$ is a maximal Cohen–Macaulay G-module since S/I is a maximal Cohen–Macaulay G-module. As G is also Cohen–Macaulay, it follows that $\operatorname{depth} \mathrm{gr}_{IR}(R) \ge \dim G - 1$. We dualize into ω_G obtaining

$$0 \to \omega_{\mathrm{gr}_{IR}(R)} \to \omega_G \xrightarrow{\ \pi\ } \mathrm{Hom}_G\Big(\frac{I+K}{I}, \omega_G\Big) \to \mathrm{Ext}^1_G(\mathrm{gr}_{IR}(R), \omega_G) \to 0.$$

To prove that $\mathrm{gr}_{IR}(R)$ is Cohen–Macaulay, it suffices to show that π is surjective.

The map π is the dual of a nonzero map of maximal Cohen–Macaulay G-modules, so it is nonzero. Moreover,

$$\mathrm{Hom}_G\Big(\frac{I+K}{I}, \omega_G\Big) \cong \mathrm{Hom}_G(\omega_{S/I}, \omega_G)\Big(\alpha + \sum_i d_i, 0\Big) = (S/I)\Big(\alpha + \sum_i d_i, 0\Big),$$

where the last equality uses the identification $\mathrm{Hom}_G(\omega_{S/I}, \omega_G) = \mathrm{Hom}_G(G, S/I)$. Since $\omega_G = G(a + \sum_i d_i, 0)$, the source and target of π are both generated in degree $(-a - \sum_i d_i, 0)$ and the components of that degree are $k := S_0$, a field. So π is the projection of G onto the $(*, 0)$ component, up to multiplication by a nonzero element of k. Thus $\mathrm{gr}_I(R)$ is Cohen–Macaulay, and

$$\omega_{\mathrm{gr}_{IR}(R)} = (\omega_G)_{(*,\geq 1)} = G\Big(a + \sum_i d_i\,, 0\Big)_{(*,\geq 1)} = \mathrm{gr}_{IR}(R)\Big(a + \sum_i d_i\,, 0\Big)_{(*,\geq 1)},$$

proving part (a).

By linkage the ring R is Cohen–Macaulay, $\omega_R = IR(a + \sum_i d_i)$, and IR is unmixed of codimension 1. Recall that R is Gorenstein locally in codimension 1, hence IR is generically a complete intersection. The assumptions of Theorem 1.1 are now satisfied for the ideal $IR \subset R$, so $\mathcal{R}(IR)$ is Cohen–Macaulay and

$$\begin{aligned}\omega_{\mathcal{R}(IR)} &= \omega_R \cdot IRt \cdot \mathcal{R}(IR) = IR \cdot IRt \cdot \mathcal{R}(IR)\Big(a + \sum_i d_i\,, 0\Big)\\ &\cong (It)^2 \mathcal{R}(IR)\Big(a + \sum_i d_i\,, 1\Big),\end{aligned}$$

proving part (b). □

In the case where S is a standard graded algebra over a field k and $I \subset S$ is generated by forms of a single degree d, we may regard $\mathcal{R}(I)$ as a bigraded subalgebra of $S[t]$, giving the linear forms of S degree $(1, 0)$ and giving t degree $(-d, 1)$. This also induces a bigrading on $\mathrm{gr}_I(S)$. We call these the *standard* bigradings; with these gradings the k-algebras $\mathcal{R}(I)$ and $\mathrm{gr}_I(S)$ are generated in degrees $(1, 0)$ and $(0, 1)$.

Corollary 1.5. *Let $m \leq n$ be positive integers, and let*

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix}$$

be a generic $m \times n$ matrix over the standard graded polynomial ring $S = k[x_{1,1}, \dots, x_{m,n}]$ over a field. Let $I \subset S$ be the ideal of $m \times m$ minors of X. The rings $G := \mathrm{gr}_S(I)$ and $\mathcal{R} := \mathcal{R}(I)$ are Cohen–Macaulay domains and in the standard bigrading:

(a) $\omega_G = G(m - m^2, -n + m - 1)$, *so G is Gorenstein.*

(b) $\omega_{\mathcal{R}} = (1, t)^{n-m-1} \mathcal{R}(m - mn, -1)$.

Proof. The Cohen–Macaulay property of the Rees ring is proven in [4, Proposition 2.6], and the fact that G is a Cohen–Macaulay domain follows from [10, Proposition 1.1 and Corollary 2.1]. Since G is a Cohen–Macaulay domain, it is Gorenstein by [8, Proposition 1.1] following [9, Proposition, p. 55].

Thus $I \subset S$ satisfies the assumptions of Theorem 1.1, where I is generated by forms of degree m, $g = n - m + 1$, and $\omega_S = S(-mn)$. We apply the theorem, noting that now t has degree $(-m, 1)$ instead of $(0, 1)$. An ungraded version of the computation of $\omega_{\mathcal{R}}$ also follows from [8, Theorem 2.4 and Corollary 2.9]. □

2 Depths of truncated Rees algebras

We use the following notation throughout the rest of this paper: Let

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \end{pmatrix}$$

be a generic $2 \times n$ matrix over the ring $S = k[x_{1,1}, \ldots, x_{2,n}]$, where k is a field of characteristic 0 and $n \geq 4$ (the case $n = 3$ being easy and special). Write $\mathfrak{m}$ for the maximal homogeneous ideal of S, and let $I = I_2(X)$ be the ideal of 2×2 minors of X. The codimension and analytic spread of I are $g := n - 1$ and $\ell := 2n - 3$, respectively (for the notions analytic spread, reduction, and superficial sequence, we refer the reader to [13]). Let J be a homogeneous minimal reduction of I and $a_1, \ldots, a_\ell$ be general quadrics in J that generate J. Set $J_i = (a_1, \ldots, a_i) \subset I$, $K_i = J_i : I$, and $R_i = S/K_i$. Since $\dim S/(J : I) = 0$ and $a_1, \ldots, a_\ell$ are chosen generally in J, it follows that K_i is a geometric i-residual intersection (see, for instance, [2, Lemma 2.3] or [15, Corollary 1.6(a)]), and $a_1, \ldots, a_\ell$ form a superficial sequence for I. We henceforth assume that $0 \leq i \leq \ell - 1 = 2n - 4$ because the ℓ-residual intersection would be primary to the maximal homogeneous ideal. By [5, Corollary 3.2], R_i is unmixed of dimension $2n - i$. We write $\mathcal{R}_i = \mathcal{R}(IR_i) \subset R_i[t]$ for the Rees ring of $IR_i \subset R_i$, and we give t bidegree $(-2, 1)$, making $\mathcal{R}(IR_i)$ a standard bigraded algebra over k.

Write $\mathfrak{n} := \mathfrak{m}R_i + It\mathcal{R}_i$ for the maximal ideal of $\mathcal{R}_i$ that is homogeneous in the total grading, where i will be clear from context.

If $\mathfrak{a} \subset T$ is an ideal in any ring we write $\mathfrak{a}^0 = T$ and, for $j < 0$, $\mathfrak{a}^j = T :_{\mathrm{Quot}(T)} \mathfrak{a}^{-j}$.

Theorem 2.1. *Assume that* $0 \leq i \leq \ell - 1$. *With hypotheses as above:*
(a) *The rings* $\mathcal{R}_i$ *are locally Cohen–Macaulay away from* $\mathfrak{n}$. *They are Cohen–Macaulay if* $i \leq g = n - 1$.
(b) *The canonical module of* $\mathcal{R}_i$ *is*

$$\omega_{\mathcal{R}_i} \cong (It\mathcal{R}_i)^{i-g+2}(-4, i - n + 2) \tag{9.3}$$

$$\cong ((1, t)\mathcal{R}_i)^{g-i-2}(2i - 2n + 2, -1). \tag{9.4}$$

(c) *If* $i \geq g + 1 = n$ *then* $\operatorname{depth} \mathcal{R}_i = \dim \mathcal{R}_i - 4 = 2n - i - 3$ *and the local cohomology of* $\mathcal{R}_i$ *with support in* $\mathfrak{n}$ *is nonzero only in cohomological degrees* $\operatorname{depth} \mathcal{R}_i$ *and*

$\dim \mathcal{R}_i$. *Using part* (a) *it follows that if* $i \le \ell - 2 = 2n - 5$ *then* $\mathcal{R}_i$ *satisfies Serre's condition* S_2.

Proposition 2.2. *Assume that* $0 \le i \le \ell - 1$. *With hypotheses as above, if* $j \ge 0$ *or* $i \le g - 2$, *then* $(It\mathcal{R}_i)^j = ((t^{-1}, 1)\mathcal{R}_i)^{-j}$.

Theorem 2.3. *Assume that* $0 \le i \le \ell - 1$ *and* $j \ge 0$. *With hypotheses as above:*

(a) *The* $\mathcal{R}_i$*-module* $(It\mathcal{R}_i)^j$ *is Cohen–Macaulay away from* $\mathfrak{n}$ *for* $j \le i - g + 3$.

(b) *If* $j \le 2$ *then the local cohomology of* $(It\mathcal{R}_i)^j$ *with support in* $\mathfrak{n}$ *vanishes except at the depth, given below, and the dimension,* $2n - i + 1$.

(c) *If* $j \le i - g + 3$ *and one of the conditions*

(i) $i \le \ell - 2$; *or*

(ii) $i = \ell - 1$ *and* $j \ge i - g$,

is satisfied then the multiplication maps

$$(It\mathcal{R}_i)^j \otimes_{\mathcal{R}_i} (It\mathcal{R}_i)^{(i-g+2)-j} \to (It\mathcal{R}_i)^{i-g+2}$$

induce duality isomorphisms

$$\operatorname{Hom}_{\mathcal{R}_i}((It\mathcal{R}_i)^{(i-g+2)-j}, \omega_{\mathcal{R}_i}) \cong (It\mathcal{R}_i)^j(-4, i - n + 2), \tag{1}$$

and thus $(It\mathcal{R}_i)^j$ *satisfies Serre's condition* S_2.

Furthermore, if $2 \le p \le \dim \mathcal{R}_i - 1$, *then*

$$\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_i + 1 - p}((It\mathcal{R}_i)^{(i-g+2)-j})^\vee \cong \mathrm{H}_{\mathfrak{n}}^p((It\mathcal{R}_i)^j)(-4, i - n + 2), \tag{2}$$

where $-^\vee$ *denotes the graded k-dual.*

(d) *See Figure* 9.1 *for a graphic representation of the following:*

$$\operatorname{depth}(It\mathcal{R}_i)^j = \begin{cases} \dim \mathcal{R}_i & \text{if } i - g \le j \le 1, \\ \dim \mathcal{R}_i - 4 & \text{if } j \le \min\{2, i - g - 1\}, \\ n + 3 & \text{if } j = 2 \text{ and } i \le g - 1, \\ \dim \mathcal{R}_i & \text{if } j = 2 \text{ and } g \le i \le g + 2, \\ \min\{\dim \mathcal{R}_i - 4,\ 5\} & \text{if } 3 \le j \le i - g - 1, \\ 5 & \text{otherwise, that is, if } \max\{3, i - g\} \le j. \end{cases}$$

Remark 2.4. As a consequence of Theorem 2.3, we see that $(It\mathcal{R}_i)^j$ is a maximal Cohen–Macaulay $\mathcal{R}_i$-module if and only if:

– $i - g \le j \le 1$; or

– $j = 2$ and $g - 1 \le i \le g + 2$; or

– $i - g \le j$ and $i = \ell - 1$.

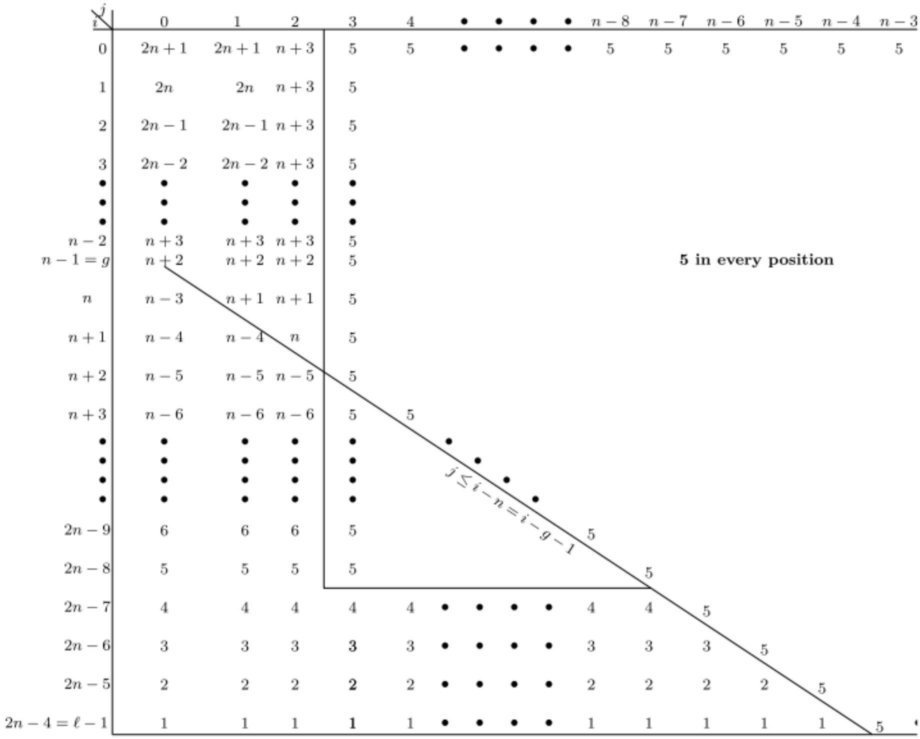

Figure 9.1: Depth of $(It\mathcal{R}_i)^j$.

Proof of Proposition 2.2. Suppose first $j \geq 0$. Because $It(t^{-1},1) \subset \mathcal{R}_i$, the product $(It\mathcal{R}_i)((t^{-1},1)\mathcal{R}_i)$ is in $\mathcal{R}_i$, and thus $(It\mathcal{R}_i)^j \subset ((t^{-1},1)\mathcal{R}_i)^{-j}$. On the other hand, the fractional ideal $((t^{-1},1)\mathcal{R}_i)^j$ contains $\mathcal{R}_i$, so $(((t^{-1},1)\mathcal{R}_i)^j)^{-1}$ is a homogeneous ideal in $\mathcal{R}_i$. Since $t^{-j} \in ((t^{-1},1)\mathcal{R}_i)^j$, the inverse ideal must have initial t-degree $\geq j$, and thus is contained in $(It\mathcal{R}_i)^j$.

Next if $i \leq g-2$ and $j = -k$ is negative, then the product of $(It\mathcal{R}_i)^k$ and $((t^{-1},1)\mathcal{R}_i)^k$ is contained in $\mathcal{R}_i$, so $(It\mathcal{R}_i)^{-k} \supset ((t^{-1},1)\mathcal{R}_i)^k$. For the other inclusion, we note that $i \leq g-2 \leq g-1$, so the ideal IR_i has positive codimension and by Corollary 1.2(a), which applies due to Corollary 1.5(a), the associated graded ring $\mathrm{gr}_{IR_i}(R_i)$ is Cohen–Macaulay. Hence the irrelevant ideal of $\mathrm{gr}_{IR_i}(R_i)$ has positive grade. It follows that $(IR_i)^{k+\ell} :_{R_i} (IR_i)^k = (IR_i)^\ell$ for any $\ell \geq 0$. Since $i \leq g-2$ the grade of IR_i is ≥ 2, and thus $(It\mathcal{R}_i)^{-k} \subset R_i[t,t^{-1}]$ and is homogeneous with initial t-degree $\geq -k$, as $(It\mathcal{R}_i)^k$ has initial t-degree k.

The ideal $((t^{-1},1)\mathcal{R}_i)^k$ is 0 in degrees $< -k$, coincides with $R_i[t,t^{-1}]$ in degrees $-k,\dots,0$, and coincides with $\mathcal{R}_i$ in positive degrees. Thus to prove that $(It\mathcal{R}_i)^{-k} \subset ((t^{-1},1)\mathcal{R}_i)^k$, it is enough to show that, in strictly positive degrees, $(It\mathcal{R}_i)^{-k}$ is contained in $\mathcal{R}_i$.

If $ut^h \in (It\mathcal{R}_i)^{-k}$ with $u \in R_i$ and $h \geq 1$, then

$$ut^h (IR_i)^k t^k \subset (IR_i)^{h+k} t^{h+k},$$

so $u \in (IR_i)^{k+h} :_{R_i} (IR_i)^k = (IR_i)^h$, completing the proof of Proposition 2.2. □

Proof of Theorems 2.1 *and* 2.3. If $j \geq 0$, we set

$$N_{i,j} := (It\mathcal{R}_i)^j = (\mathcal{R}_i)_{\geq j} = \bigoplus_{\nu \geq j} (IR_i)^\nu t^\nu.$$

We regard $(IR_i)^j t^j$ and $a_{i+1}(IR_i)^{-1}$ as bigraded $\mathcal{R}$-modules via the natural map $\mathcal{R} \to \mathcal{R}/It\mathcal{R} = R$.

For the proofs, we will use the three families of short exact sequences of bigraded $\mathcal{R}$-modules:

$$\begin{array}{lll}
(a_{i,j}) & 0 \longrightarrow N_{i,j-1}(0,-1) \xrightarrow{a_{i+1}t} N_{i,j} \longrightarrow N_{i+1,j} \longrightarrow 0 & \text{for } j \geq 1,\\
(b_{i,j}) & 0 \longrightarrow N_{i,j} \xrightarrow{\iota} N_{i,j-1} \longrightarrow (IR_i)^{j-1}t^{j-1} \longrightarrow 0 & \text{for } j \geq 1,\\
(c_{i,j}) & 0 \longrightarrow a_{i+1}(IR_i)^{j-1}t^j \longrightarrow \dfrac{N_{i,j}}{a_{i+1}tN_{i,j}} \longrightarrow N_{i+1,j} \longrightarrow 0 & \text{for } j \geq 1,\\
(c_{i,0}) & 0 \longrightarrow K_{i+1}R_i \longrightarrow \dfrac{N_{i,0}}{a_{i+1}tN_{i,0}} \longrightarrow N_{i+1,0} \longrightarrow 0. &
\end{array}$$

We begin by proving that these sequences are exact. The exactness of $(a_{i,j})$ for $j \geq 1$ and of $(c_{i,j})$ for $j \geq 0$ follows directly from the exactness of the sequences $(*_{i,\nu-1})$ in item (IV) of [6, Section 3], which give

$$\frac{(IR_i)^\nu}{a_{i+1}(IR_i)^{\nu-1}} = (IR_{i+1})^\nu$$

for $\nu \geq 0$ and $a_{i+1}(IR_i)^{-1} = K_{i+1}R_i$. In the sequence $(b_{i,j})$, the map ι is the natural inclusion, and the exactness follows directly from the definitions.

We are now going to prove parts (a) and (d) of Theorem 2.3, except for the upper bounds in (d), together with

(*) Let U be the polynomial ring $S[\{y_{p,q}\}_{1 \leq p < q \leq n}]$, which is the ambient polynomial ring of the Rees algebra $\mathcal{R}(I)$, graded with both the x_i and the $y_{p,q}$ of degree 1. Let $F_\bullet^{i,j}$ be the minimal U-free resolution of $N_{i,j}(j)$. If $j \geq i - g$ then $F_k^{i,j}$ is generated in degree k for $k \geq i + \binom{n}{2}$.

Suppose first that $i = 0$. We do induction on j. If $j = 0$ then $(It\mathcal{R}_i)^j = \mathcal{R}(I) = N_{0,j}$ is Cohen–Macaulay by [4, Proposition 2.6] and items (a) and (d) follow immediately. It also follows that the projective dimension of $N_{0,j} = (It\mathcal{R})^j$ as a U-module is $< \binom{n}{2}$, proving (*).

If $j = 1$ we use the exact sequence $(b_{0,1})$. From the Cohen–Macaulayness of $N_{0,0}$ and the depth of S, we see that $N_{0,1}$ is also a maximal Cohen–Macaulay module, and again this implies all the assertions.

Item (a) is vacuous for $j \neq 0$. Using the sequence $(b_{0,j})$, the depth assertion of item (d) follows from the induction hypothesis because I^{j-1} has depth $n+2$ when $j = 2$ and depth 4 when $j \geq 3$; the last assertion follows from [1, Theorem 5.4 and the beginning of its proof]. The same reference also shows that $I^{j-1}(2j-2)$ has linear resolution for $j \geq 2$. Thus using the long exact sequence in Tor associated to $(b_{0,j})$ and applying the inductive hypothesis, we obtain the assertion of (*). This finishes the case $i = 0$, so from now on we suppose that $i \geq 1$.

Item (a)

We must show that if P is a prime of $\mathcal{R}_i$ not containing $\mathfrak{n} = \mathfrak{m}R_i + It\mathcal{R}_i$ then $(N_{i,j})_P$ is Cohen–Macaulay whenever $j \leq i-g+3$. If P does not contain It, we first consider the case $j = 0$, and use the exact sequence $(c_{i-1,0})$. The $\mathcal{R}_{i-1}$-module K_iR_{i-1} is concentrated in one degree in the $y_{p,q}$, and is thus annihilated by It, so $(\mathcal{R}_{i-1}/(a_it))_P \cong (\mathcal{R}_i)_P$ by the same exact sequence. On the other hand, a_i is a nonzerodivisor on R_{i-1} by [6, Proposition 2.4(d)]. Thus a_it is a nonzerodivisor on $\mathcal{R}_{i-1}$. By induction $(\mathcal{R}_{i-1})_P$ is Cohen–Macaulay, completing the argument in this case. Also, $(N_{i,j})_P = (\mathcal{R}_i)_P$.

Next suppose that $\mathfrak{m}R_i \not\subset P$, and let $p := S \cap P$. We do induction on j, starting with the case $j = 0$. The ideal I_p is either the unit ideal or a complete intersection equal to $(J_\ell)_p$ and, as above, IR_i contains a nonzerodivisor on R_i. Thus by [11, Theorem 3.1 (i) and (iii)] $(R_i)_p$ is Cohen–Macaulay and $(IR_i)_p$ is a strongly Cohen–Macaulay ideal.

For q any prime ideal of S with $I \subset q \subset p \subsetneq \mathfrak{m}$, let $t = \dim S_q$. Notice that $I_q = J_q$ is generated by ℓ elements. We claim that in fact I_q is generated by t elements. If $t < \ell$ then K_t is a geometric t-residual intersection of I, and it follows that $I_q = (a_1, \dots, a_t)S_q$. So $(IR_i)_q = (a_{i+1}, \dots, a_t)(R_i)_q$ can be generated by $t - i = \dim(R_i)_q$ elements. By [7, Theorem 2.6] (in the case where the ideal has positive height), the Rees algebra $(\mathcal{R}_i)_p$ is Cohen–Macaulay, and thus the further localization $(\mathcal{R}_i)_P$ is also Cohen–Macaulay, completing the proof of item (a) in the case $j = 0$.

Now suppose $j > 0$. We use the exact sequence $(b_{i,j})$ in the localized form

$$0 \longrightarrow (N_{i,j})_P \xrightarrow{\iota} (N_{i,j-1})_P \longrightarrow (IR_it)_p^{j-1} \longrightarrow 0.$$

By [6, Theorem 1.3(b)], the module $(IR_it)_p^{j-1}$ is a maximal Cohen–Macaulay $(R_i)_p$-module because $j-1 \leq i-g+2$. On the other hand,

$$1 + \dim(R_i)_p = \dim(\mathcal{R}_i \otimes_{R_i} (R_i)_p) \geq \dim(N_{i,j-1})_P = \dim(N_{i,j})_P.$$

Thus $(N_{i,j})_P$ is a maximal Cohen–Macaulay $(\mathcal{R}_i)_P$-module. This completes the proof of item (a).

Statement (*) and the lower bounds in (d)

We first note that for $i \le g$ and $j = 0$ the module $N_{i,j}$ is Cohen–Macaulay. This follows from Corollary 1.2(b) and Theorem 1.4(b), which apply by Corollary 1.5(a) (and because K_{g+1} is a geometric $(g+1)$-residual intersection, so R_g is Gorenstein locally in codimension one).

The assertion of (*) for $i = 0$ is proven above. For $j = 0$, the modules $F_k^{i,j}$ are zero because $N_{i,j}$ is Cohen–Macaulay of dimension $2n - i + 1$. Now using induction on i and the sequences $(a_{i-1,j})$, the assertion of (*) follows in general.

As for item (d), we begin by establishing the given depth of $N_{i,j}$ as a lower bound in the cases $j \ge i - g$. The case $j = 0$ has just been treated. For $1 \le j \le 3$, the sequences $(a_{i-1,j})$, together with induction on i and j, suffice, except when $j = 3$ and $n \le 5$. However, $i \le 2n - 4 = \ell - 1$, so in the cases $j = 3$ and either $n = 4$ or $n = 5$ we have $i \le g + 1$ or $i \le g + 2$, respectively. Therefore the sequences $(a_{i-1,j})$ suffice in these cases as well.

If now $j \ge 4$ we must show that depth $N_{i,j} \ge 5$. By induction on i and j, the depths of $N_{i-1,j-1}$ and $N_{i-1,j}$ are at least 5. By (*), $F_k^{i-1,j-1}$ is generated in degree k for $k \ge i - 1 + \binom{n}{2}$. In particular, this holds for $k \ge 2n + \binom{n}{2} - 5$ since $i \le 2n - 4$. From sequence $(a_{i-1,j})$ and [6, Proposition 2.2(a)], the inequality depth $N_{i,j} \ge 5$ follows.

Finally, we come to the cases $j \le i - g - 1$. First, if $j = 0$, we do induction on i using the sequence $(c_{i-1,0})$. From the definition of R_i, we have an exact sequence

$$0 \to K_i R_{i-1} \to R_{i-1} \to R_i \to 0.$$

From [6, Theorem 1.2(a)], we know that depth $R_{i-1} \ge 2n - i - 2$ and depth $R_i \ge 2n - i - 3$, so depth $K_i R_{i-1} \ge 2n - i - 2$. By induction, $\mathcal{R}_{i-1}$ has depth $\ge 2n - i - 2$. Also, as above, $a_i t$ is a nonzerodivisor on $\mathcal{R}_{i-1}$, so the sequence $(c_{i-1,0})$ shows that depth $\mathcal{R}_i \ge 2n - i - 3$, as required for the case $j = 0$.

Next, if $j \ge 1$, we use the sequence $(b_{i,j})$ and induction on j. If $j \le 2$, we have depth $N_{i,j-1} \ge 2n - i - 3$ by induction, while depth$(IR_i)^{j-1} = 2n - i - 3$ by [6, Theorem 1.3(e)]. Thus depth $N_{i,j} \ge 2n - i - 3$. If, on the other hand, $j \ge 3$, we have depth $N_{i,j-1} \ge \min\{2n - i - 3, 5\}$ by induction, while depth$(IR_i)^{j-1} = \min\{2n - i - 3, 4\}$ by [6, Theorem 1.3(e)], so depth $N_{i,j} \ge \min\{2n - i - 3, 5\}$, completing the proof of the lower bounds in item (d). □

The canonical modules

We next prove Theorem 2.1(b). By Proposition 2.2, the expressions (9.3) and (9.4) are equal.

Corollary 1.2(b) applies because of Corollary 1.5(a), and proves this result for $i \le g-2$.

We now do induction on i, and suppose that $i \ge g-1$. We claim that

$$\omega_{\mathcal{R}_i} \cong \left[\left(\frac{\omega_{\mathcal{R}_{i-1}}}{a_i t \omega_{\mathcal{R}_{i-1}}}\right)(0,1)\right]_{(*,\ge 1)}. \tag{9.5}$$

Dualizing the exact sequence $(c_{i-1,0})$ into ω_U, we get an exact sequence

$$0 \to \omega_{\mathcal{R}_i} \to \omega_{(\frac{\mathcal{R}_{i-1}}{a_i t \mathcal{R}_{i-1}})} \to \operatorname{Ext}_U^{\dim U - \dim R_{i-1}}(K_i R_{i-1}, \omega_U).$$

Local duality shows that the right-hand module is concentrated in degree $(*, 0)$.

We claim that $\omega_{\mathcal{R}_i}$ is concentrated in degrees $(*, \ge 1)$. To see this, let L be the total ring of quotients of R_i, and observe that

$$\omega_{\mathcal{R}_i} \subset L \otimes \omega_{\mathcal{R}_i} = \omega_{L \otimes \mathcal{R}_i}.$$

However, $L \otimes \mathcal{R}_i = L[t]$ because I contains a nonzerodivisor on R_i (see [6, Proposition 2.4(d)]), so $\omega_{L\otimes\mathcal{R}_i} = t\omega_L \otimes_L L[t]$ is concentrated in positive t-degrees, proving the claim.

This shows that

$$\omega_{\mathcal{R}_i} \cong [\omega_{(\frac{\mathcal{R}_{i-1}}{a_i t \mathcal{R}_{i-1}})}]_{(*,\ge 1)}. \tag{9.6}$$

To understand the right-hand side of the last expression, we consider the sequence

$$0 \to \mathcal{R}_{i-1}(0,-1) \xrightarrow{a_i t} \mathcal{R}_{i-1} \to \mathcal{R}_{i-1}/(a_i t \mathcal{R}_{i-1}) \to 0.$$

Dualizing into ω_U, we get

$$\omega_{\mathcal{R}_{i-1}} \xrightarrow{a_i t} \omega_{\mathcal{R}_{i-1}}(0,1) \to \omega_{(\frac{\mathcal{R}_{i-1}}{a_i t \mathcal{R}_{i-1}})} \to \operatorname{Ext}_U^{\dim U - \dim \mathcal{R}_{i-1}+1}(\mathcal{R}_{i-1}, \omega_U).$$

We are now going to show that the last module in the sequence vanishes. By local duality,

$$\operatorname{Ext}_U^{\dim U - \dim \mathcal{R}_{i-1}+1}(\mathcal{R}_{i-1}, \omega_U) \cong (\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_{i-1}-1}(\mathcal{R}_{i-1}))^\vee,$$

where $(-)^\vee$ indicates the graded k-dual. Furthermore, if $i \le g+1$, the ring $\mathcal{R}_{i-1}$ is Cohen–Macaulay by the lower bound (d), so this local cohomology vanishes.

On the other hand,

$$\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_{i-1}-1}(\mathcal{R}_{i-1}) = \mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_{i-1}-1}(\operatorname{Hom}_{\mathcal{R}_{i-1}}(\omega_{\mathcal{R}_{i-1}}, \omega_{\mathcal{R}_{i-1}})) = \mathrm{H}_{\mathfrak{n}}^2(\omega_{\mathcal{R}_{i-1}})^\vee,$$

where the first equality holds because $\mathcal{R}_{i-1}$ satisfies Serre's condition S_2 by item (a) and the lower bound in (d), and the second follows from [6, Proposition 2.3], which applies because $\omega_{\mathcal{R}_{i-1}}$ is Cohen–Macaulay on the punctured spectrum, by item (a). By induction on i, $\omega_{\mathcal{R}_{i-1}}$ is up to shift isomorphic to $N_{i-1,i-g+1}$. Hence, by the lower bound in (d), the depth of $\omega_{\mathcal{R}_{i-1}}$ is at least 3 because $i-g+1 \geq (i-1)-g$, proving the vanishing.

This shows that

$$\omega_{(\frac{\mathcal{R}_{i-1}}{a_i t \mathcal{R}_{i-1}})} \cong \left(\frac{\omega_{\mathcal{R}_{i-1}}}{a_i t \omega_{\mathcal{R}_{i-1}}}\right)(0,1). \tag{9.7}$$

Combining equations (9.6) and (9.7), claim (9.5) follows.

By induction, $\omega_{\mathcal{R}_{i-1}} = (It\mathcal{R}_{i-1})^{i-g+1}(-4, i-n+1)$. Now (9.5) gives

$$\begin{aligned}\omega_{\mathcal{R}_i} &= \left[\left(\frac{(It\mathcal{R}_{i-1})^{i-g+1}}{a_i t (It\mathcal{R}_{i-1})^{i-g+1}}\right)(-4, i-n+2)\right]_{(*,\geq 1)} \\ &= \left(\frac{(It\mathcal{R}_{i-1})^{i-g+2}}{a_i t (It\mathcal{R}_{i-1})^{i-g+1}}\right)(-4, i-n+2).\end{aligned}$$

This is isomorphic to

$$(It\mathcal{R}_i)^{i-g+2}(-4, i-n+2)$$

by the exactness of the sequence $(a_{i-1,i-g+2})$ as $i-g+2 \geq 1$. □

Duality

We now turn to Theorem 2.3(c). Notice that for every integer k, the fractional ideal $(It\mathcal{R}_i)^k$ contains a homogeneous nonzerodivisor of $\mathcal{R}_i$ of degree $(*, k)$; if $k \leq 0$ this holds because $t^k \in (It\mathcal{R}_i)^k$, and if $k \geq 1$ one uses the fact that IR_i contains a nonzerodivisor on R_i by [6, Proposition 2.4(d)]. Now it follows by degree reasons that the inclusion

$$(It\mathcal{R}_i)^j \subset (It\mathcal{R}_i)^{i-g+2} :_{\mathcal{R}_i} (It\mathcal{R}_i)^{(i-g+2)-j}$$

is an equality.

Writing $\mathcal{K}_i$ for the total ring of quotients of $\mathcal{R}_i$, we claim that the inclusion

$$(It\mathcal{R}_i)^{i-g+2} :_{\mathcal{R}_i} (It\mathcal{R}_i)^{(i-g+2)-j} \subset (It\mathcal{R}_i)^{i-g+2} :_{\mathcal{K}_i} (It\mathcal{R}_i)^{(i-g+2)-j}$$

is also an equality. Because

$$(It\mathcal{R}_i)^{i-g+2} :_{\mathcal{R}_i} (It\mathcal{R}_i)^{(i-g+2)-j} = (It\mathcal{R}_i)^j$$

has depth ≥ 2 by the lower bound in Theorem 2.3(d), it suffices to check equality locally on the punctured spectrum, and there we must prove that

$$(It\mathcal{R}_i)^{i-g+2} :_{\mathcal{K}_i} (It\mathcal{R}_i)^{(i-g+2)-j} \subset \mathcal{R}_i.$$

One has

$$(It\mathcal{R}_i)^{i-g+2} :_{\mathcal{K}_i} (It\mathcal{R}_i)^{(i-g+2)-j} \subset (It\mathcal{R}_i)^{i-g+2} :_{\mathcal{K}_i} (It\mathcal{R}_i)^{i-g+2}$$
$$= \operatorname{End}_{\mathcal{R}_i}(\omega_{\mathcal{R}_i}) \supset \mathcal{R}_i.$$

Here the equality holds because up to a shift $\omega_{\mathcal{R}_i}$ is isomorphic to $(It\mathcal{R}_i)^{i-g+2}$ by Theorem 2.1(b), and the last containment is an equality locally on the punctured spectrum because there $\mathcal{R}_i$ is Cohen–Macaulay by Theorem 2.1(a).

Finally,

$$(It\mathcal{R}_i)^{i-g+2} :_{\mathcal{K}_i} (It\mathcal{R}_i)^{(i-g+2)-j} \cong \operatorname{Hom}_{\mathcal{R}_i}\big((It\mathcal{R}_i)^{(i-g+2)-j}, (It\mathcal{R}_i)^{i-g+2}\big)$$

and $(It\mathcal{R}_i)^{i-g+2} \cong \omega_{\mathcal{R}_i}(4, n-i-2)$ by Theorem 2.1(b), completing the proof of equation (1) in Theorem 2.3(c).

Equation (2) in Theorem 2.3(c) is a consequence of (1) in Theorem 2.3(c) and [6, Proposition 2.3] once we have shown that $(It\mathcal{R}_i)^{(i-g+2)-j}$ is Cohen–Macaulay locally on the punctured spectrum. If $(i-g+2)-j \geq 0$, this holds by Theorem 2.3(a). Otherwise $(i-g+2)-j = -1$ and $(It\mathcal{R}_i)^{-1} \cong \operatorname{Hom}_{\mathcal{R}_i}(It\mathcal{R}_i, \mathcal{R}_i)$ satisfies S_2 locally on the punctured spectrum because $\mathcal{R}_i$ does according to Theorem 2.3(a). So the $\omega_{\mathcal{R}_i}$-dual of the isomorphism (1) in Theorem 2.3(c) implies that for every prime ideal P of $\mathcal{R}_i$ with $P \neq \mathfrak{n}$, $(It\mathcal{R}_i)_P^{-1}$ is isomorphic to $\operatorname{Hom}_{\mathcal{R}_i}((It\mathcal{R}_i)^{i-g+3}, \omega_{\mathcal{R}_i})_P$, which is Cohen–Macaulay by Theorem 2.3(a). □

One intermediate local cohomology

We next prove Theorem 2.3(b), and assume $j \leq 2$. If $i \leq g$ then $N_{i,0}$ and $N_{i,1}$ have depth equal to $\dim \mathcal{R}_i$ by the lower bound in Theorem 2.3(d). Moreover, the $\mathcal{R}_i$-module IR_i has dimension $\dim \mathcal{R}_i - 1$ and at most one more nonvanishing local cohomology module according to [6, Theorem 1.3(c)]. So the sequence $(b_{i,2})$ shows that also $N_{i,2}$ has at most two nonvanishing local cohomology modules. Hence we may assume that $i \geq g+1$.

The duality statement (2) of Theorem 2.3(c) implies that, up to twist, $H^p_{\mathfrak{n}}(N_{i,j})$ is the graded dual of $H^{\dim \mathcal{R}_i+1-p}_{\mathfrak{n}}(N_{i,i-g+2-j})$ for $2 \leq p \leq \dim \mathcal{R}_i - 1$; notice that this statement applies because $i-g \leq i-g+2-j \leq i-g+3$. Again since $i-g+2-j \geq i-g$, the lower bound in Theorem 2.3(d) shows that $N_{i,i-g+2-j}$ has depth ≥ 5, except possibly when $n=4$ and $i = n+1$. But this possibility is ruled out by our blanket assumption $i \leq 2n-4$.

Thus $H^{\dim \mathcal{R}_i+1-p}_{\mathfrak{n}}(N_{i,i-g+2-j}) = 0$ for $p \geq \dim \mathcal{R}_i - 3$, so $H^p_{\mathfrak{n}}(N_{i,j}) = 0$ for $\dim \mathcal{R}_i - 3 = \max\{2, \dim \mathcal{R}_i - 3\} \leq p \leq \dim \mathcal{R}_i - 1$. This suffices because $\operatorname{depth} N_{i,j} \geq \dim \mathcal{R}_i - 4$ by the lower bound of Theorem 2.3(d). □

Upper bound on depth

Finally, we prove equality in Theorem 2.3(d). The lower bound for the depth of $N_{i,j}$ that we have already proven is equal to the dimension of $N_{i,j}$ if $j \geq i - g = i - n + 1$ and one of the following holds:

$$j \leq 1,$$
$$j = 2 \quad \text{and} \quad i \geq n-2,$$
$$i = 2n-4.$$

Next, if $j = 2$ and $i \leq n-3$, then depth $N_{i,1} \geq n+4$. By [6, Theorem 1.3(e)], depth $IR_i = n+2$, so using the sequence $(b_{i,2})$ we see that depth $N_{i,2} = n+3$ as claimed.

In the case $i = 2n-4$, we must show that if $j \leq n-4$ then depth $N_{i,j} \leq 1$. We have an inclusion $a_{2n-3}t(It\mathcal{R}_{2n-4})^j \subset (It\mathcal{R}_{2n-4})^{j+1}$. Let Q be the quotient.

We claim that this is an equality away from $\mathfrak{n} = \mathfrak{m}R_{2n-4} + It\mathcal{R}_{2n-4}$ and thus Q has finite length. Let P be a prime ideal of the polynomial ring $U = S[\{y_{p,q}\}_{1\leq p<q\leq n}]$. Recall that the elements $a_1, \dots, a_{2n-3}$ generate a reduction of I. If P does not contain $\mathfrak{m}$, then the ideal I_P is a complete intersection, so $a_1, \dots, a_{2n-3}$ generate I_P. As $a_1, \dots, a_{2n-4}$ map to 0 in R_{2n-4}, the claim follows in this case.

On the other hand, if the image of P in $\mathcal{R}_{2n-4}$ does not contain $It\mathcal{R}_{2n-4}$, then it does not contain $a_{2n-3}t\mathcal{R}_{2n-4}$, because $\sqrt{I} = \sqrt{(a_1, \dots, a_{2n-3})S}$ and thus $\sqrt{It\mathcal{R}_{2n-4}} = \sqrt{a_{2n-3}t\mathcal{R}_{2n-4}}$, proving the claim in this case as well.

By [6, Proposition 2.4(d)], the element $a_{2n-3}t$ is a nonzerodivisor on R_{2n-4}. Proceeding by contradiction, suppose that $\operatorname{depth}(It\mathcal{R}_{2n-4})^j \geq 2$ for some $j \leq n-4$. It follows that depth $a_{2n-3}t(It\mathcal{R}_{2n-4})^j \geq 2$ as well. Since Q has finite length, we see that $Q = 0$. Thus

$$a_{2n-3}t(It\mathcal{R}_{2n-4})^j = (It\mathcal{R}_{2n-4})^{j+1}.$$

It follows that $(a_{2n-3}t)^{j'-j}(It\mathcal{R}_{2n-4})^j = (It\mathcal{R}_{2n-4})^{j'}$ for all $j' \geq j$, so $(IR_{2n-4})^j \cong (IR_{2n-4})^{j'}$. But, by [6, Theorem 1.3(e)], the former has depth 1 and the latter, when $j' \gg 0$, has depth 4, a contradiction, showing that depth $N_{i,j} \leq 1$ in this case.

Next we consider the case where $i \geq \ell - 4 = 2n-7$ and $j \leq i - g - 1 = i - n$, and we do decreasing induction on i to show that depth $N_{i,j} \leq \dim \mathcal{R}_i - 4$. The previous case establishes this for $i = 2n-4$, so we assume that $i \leq 2n-5$. Since $N_{i,j+1}$ has depth $\geq \dim \mathcal{R}_i - 4$, the sequence $(a_{i,j+1})$ and the induction hypothesis show that depth $N_{i,j} \leq \dim \mathcal{R}_i - 4$ as required.

By the duality isomorphism (2) in Theorem 2.3(c), if

$$i \leq 2n-5, \quad j \leq i - g + 3, \quad 2 \leq p \leq \dim \mathcal{R}_i - 1,$$

then

$$\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_i + 1 - p}(N_{i,i-g+2-j})^{\vee} \cong \mathrm{H}_{\mathfrak{n}}^{p}(N_{i,j})(-4, i-n+2),$$

where $-^\vee$ denotes graded k-dual.

Now assume that $2n-7 \le i \le 2n-5$ and $j = i-g$. From the depth computation above, we know that depth $N_{i,2} = \dim \mathcal{R}_i - 4$, which gives $\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_i+1-5}(N_{i,2}) \ne 0$. By the duality isomorphism, $\mathrm{H}_{\mathfrak{n}}^5(N_{i,j}) \ne 0$, and therefore depth $N_{i,j} \le 5$ as asserted.

Next suppose that we are in one of the cases

$$j \ge 3 \quad \text{and} \quad i \le 2n-8; \quad \text{or}$$
$$j \ge i-g+1 \quad \text{and} \quad i \ge 2n-7.$$

In these cases $\operatorname{depth}(IR_i)^{j-1} = 4$ by [6, Theorem 1.3(e)] and depth $N_{i,j-1} \ge 5$, so using the sequence $(b_{i,j})$ we see that depth $N_{i,j} \le 5$ as claimed.

Finally, if $j \le \min\{2, i-g-1\}$ and $i \le 2n-8$, then by the duality isomorphism above $\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_i-4}(N_{i,j})$ is, up to a shift, the graded dual of $\mathrm{H}_{\mathfrak{n}}^5(N_{i,i-g+2-j})$. Since $j \le i-g-1$, we have $i-g+2-j \ge 3$. By the previous case, depth $N_{i,i-g+2-j} = 5$, so $\mathrm{H}_{\mathfrak{n}}^5(N_{i,i-g+2-j}) \ne 0$. Thus $\mathrm{H}_{\mathfrak{n}}^{\dim \mathcal{R}_i-4}(N_{i,j}) \ne 0$ and therefore depth $N_{i,j} \le \dim \mathcal{R}_i - 4$, completing the argument for Theorem 2.3(d). □

This also completes the proof of Theorem 2.1(a) and (c): part (a) follows from Theorem 2.3(d), Theorem 2.3(a), and Theorem 2.3(c); while part (c) follows from Theorem 2.3(d) and Theorem 2.3(b). □

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