SYZYGIES OF THE RESIDUE FIELD OVER GOLOD RINGS

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Abstract. Let $(R, \mathfrak{m}, k)$ be a Golod ring. We show a recurrence formula for high syzygies of k in terms of previous ones. In the case of embedding dimension at most 2, we provided complete descriptions of all indecomposable summands of all syzygies of k.

1. Introduction

Since the seminal work of Hilbert significant advances have been made in understanding the structure of finite free resolutions. However, much less is known about infinite free resolutions, which are quite common, as most minimal free resolutions over Noetherian local rings are infinite. Unfortunately, the standard techniques used to study finite free resolutions rarely apply to infinite resolutions.

This paper deals with minimal free resolutions of finitely generated modules over Noetherian local rings, with emphasis on the residue field. While there are numerous results and conjectures about Betti numbers [11, 12, 13, 3, 16, 5, 4, 18], our focus instead is on the structure of syzygy modules and finiteness properties of these in general infinite resolutions.

We will focus on Golod rings. They appear naturally in many contexts. Suppose that $R = S/I$ with $(S, \mathfrak{n})$ regular local (or graded) of dimension e and $I \subset \mathfrak{n}^2$ (so that e is the embedding dimension of R). Then R is Golod, for example, if I has codimension one [6]; or if $e = 2$ and R is not a zero-dimensional complete intersection [20]; or if R is a local Cohen-Macaulay ring of "minimal multiplicity" $e - \dim R + 1$ [3]; or if R is graded and I has linear resolution [7] or is componentwise linear [15]; or if I is a Borel fixed monomial ideal [1, 19]; or if R is graded over a field of characteristic zero and $I = J^s$ is a power (or a symbolic power) of a homogeneous ideal with $s \geq 2$ [14]. The Golod property is stable under factoring out a regular sequence that is part of a regular system of parameters of S [3].

Let $(R, \mathfrak{m}, k)$ be a Noetheran local ring of embedding dimension e. We prove that if R is Golod then every syzygy module of the R-module k is a direct sum of copies of the first $e+1$ syzygy modules, $\operatorname{syz}_i^R(k)$ for $0 \leq i \leq e$, and we give a recursive formula for the number of copies:

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of embedding dimension e. Let $K_\bullet$ be the Koszul complex of a minimal set of generators of $\mathfrak{m}$. If R is Golod then*

$$\operatorname{syz}_{e+1}^R(k) = \bigoplus_{j=0}^{e-1} \operatorname{syz}_j^R(k)^{h_{e-j}}$$

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and, more generally, for every $i > e$,

$$\operatorname{syz}_i^R(k) = \bigoplus_{i-e-1 \le j \le i-2} \operatorname{syz}_j^R(k)^{h_{i-j-1}},$$

where $h_{i-j-1} = \dim_k(H_{i-j-1}(K_\bullet))$.

This structural result provides a new explanation of Golod's well-known formula [9] for the ranks of the free modules in the minimal resolution of k, which is an immediate consequence. It also implies another well-known result that the Poincare series of any finitely generated module M over a Golod ring is rational (see for instance [6]), since the sequence $\{\dim_k \operatorname{Tor}_i^R(k, M)\} = \{\dim_k \operatorname{Tor}_1^R(\operatorname{syz}_{i-1}^R k, M)\}$ satisfies a linear recurrence coming from the decomposition of syzygies described above.

Furthermore, Theorem 1.1 implies that the direct sum decompositions into indecomposables for the first $e+1$ syzygy modules determine such decompositions for all syzygy modules of k. This is in stark contrast to the case of a zero-dimensional Gorenstein ring R with $e \ge 2$, where the (infinitely many) syzygy modules of k are all indecomposable and non-isomorphic.

We will next focus on the case $e = 2$, where the Golod assumption in Theorem 1.1 simply means that R is not a zero-dimensional complete intersection [20]. In this case we will give an explicit description of the direct sum decompositions into indecomposables of the syzygy modules $\operatorname{syz}_i^R(k)$ for all i. By Theorem 1.1 it suffices to do this for $\operatorname{syz}_1^R(k) = \mathfrak{m}$ and $\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m})$.

All our results are preserved and reflected by completion. For example, the number of summands of a finitely generated R-module M that are isomorphic to the residue field k is $\dim_k(\operatorname{Soc}(M)/(\mathfrak{m}M \cap \operatorname{Soc}(M))$, and this does not change upon completion. Thus we may assume, without loss of generality, that $R = S/I$, where S is a regular local ring and $I \subset \mathfrak{n}^2$. Let $e := \dim S$ be the embedding dimension of R. This will be our notation throughout this paper.

Assume that $e = 2$ and $I \subset \mathfrak{n}^2$ is not an $\mathfrak{n}$-primary complete intersection. We prove that $\operatorname{syz}_1^R(k) = \mathfrak{m}$ is decomposable if and only if $xy \in I$ for some regular system of parameters x, y of S (see Theorem 3.2). In this case $\mathfrak{m} = R/(0:x) \oplus R/(0:y)$ is the direct sum decomposition into indecomposables. For $\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m})$ we obtain (see Proposition 4.1):

Proposition 1.2. *If $\mathfrak{m}$ is decomposable then $\operatorname{syz}_2^R(k) = \mathfrak{m} \oplus k^a$, where*

$$a = \begin{cases} 2 & \text{if } \dim R = 0 \\ 1 & \text{if } \operatorname{depth} R = 0 \text{ and } \dim R = 1 \\ 0 & \text{if } R \text{ is Cohen-Macaulay of dimension 1}. \end{cases}$$

It remains to treat the more general, and more difficult, case where $\mathfrak{m}$ is indecomposable. We only need to give the direct sum decomposition into indecomposables of $\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m})$. The following result combines Theorem 2.1 and Theorem 4.3:

Theorem 1.3. *If $\mathfrak{m}$ is indecomposable, then*

$$\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m}) \cong \operatorname{Hom}(\mathfrak{m}, R) \cong N \oplus k^a,$$

where $a = \dim_k \left(\frac{\mathfrak{n}(I:\mathfrak{n})}{\mathfrak{n}I}\right)$ and N is indecomposable.

We can make the decomposition of $\operatorname{syz}_1^R(\mathfrak{m})$ in Theorem 1.3 very explicit. Let x, y be minimal generators of $\mathfrak{n}$ and $h_1, \ldots, h_n$ be minimal generators of I, write $h_i = f_i x + g_i y$, and let a be as in Theorem 1.3. Choose generators of I so that the images of the last a generators h_i form a k-basis of $\frac{\mathfrak{n}(I:\mathfrak{n})}{\mathfrak{n}I}$ and choose the corresponding f_i and g_i in $I : \mathfrak{n}$. With this notation we will show that $\operatorname{syz}_1^R(\mathfrak{m})$ is the submodule of R^2 generated by the columns of the matrix

$$\begin{pmatrix} \overline{y} & \overline{f_1} & \cdots & \overline{f_n} \\ -\overline{x} & \overline{g_1} & \cdots & \overline{g_n} \end{pmatrix},$$

where $^-$ denotes images in R. Now let N and N' be the submodules of R^2 generated by the first $n+1-a$ columns and by the last a columns, respectively. For these particular submodules N and N' we have:

Corollary 1.4. *If $\mathfrak{m}$ is indecomposable, then*
$$\operatorname{syz}_2^R(k) = N \oplus N',$$
where $N' \cong k^a$ and N is indecomposable.

We generalize the first isomorphism in Theorem 1.3 to second syzygies of some cyclic modules other than k. For instance, if R is Artinian and J is an ideal so that the ring R/J is a complete intersection, then using linkage we show that, if $\operatorname{Fitt}_2(I)R \subset J$, then $\operatorname{syz}_2^R(R/J) \cong J^* := \operatorname{Hom}(J,R)$ (see Theorem 2.5).

A consequence of these results is that at most three non-isomorphic indecomposable modules appear in the direct sum decompositions of all the syzygy modules of k, and that these indecomposable modules are summands of k, $\mathfrak{m}$, $\mathfrak{m}^*$ (see Theorem 5.2). We are also able to characterize when, for any given i, the syzygy module $\operatorname{syz}_i^R(k)$ is indecomposable (see Theorem 5.3).

In experiments with rings of embedding dimension > 2 we have seen an analogous phenomenon:

Conjecture 1.5. If $(R, \mathfrak{m}, k)$ is a local Golod ring of embedding dimension e, then there is a set of at most $e+1$ indecomposable modules from which every R-syzygy of k may be built as a direct sum.

Our results were suggested by Macaulay 2 computations [10], performed at an AIM meeting in September 2023 with the help of Mahrud Sayrafi and Devlin Mallory, using their *DirectSummands* package [17]. Without this support we might never have guessed that the results of this paper could be true.

2. $\operatorname{syz}_1^R(\mathfrak{m})$ IS $\mathfrak{m}^*$

Theorem 2.1. *If $(R, \mathfrak{m})$ is a Noetherian local ring of embedding dimension 2 that is not a zero-dimensional complete intersection, then $\operatorname{syz}_1^R(\mathfrak{m}) \cong \mathfrak{m}^*$.*

We postpone the proof until after Theorem 2.5.

Theorem 2.2. *Let S be a ring and let $I \subset J$ be ideals of S. Set $R = S/I$ and write $(-)^* = \operatorname{Hom}_S(-, R)$ for the R-dual. The following conditions are equivalent:*

- *(1) The dual $(JR)^* \to J^*$ of the natural surjection $J \to JR$ is an isomorphism.*
- *(2) The restriction map $J^* \to I^*$ is 0.*
- *(3) The natural map $\operatorname{Ext}^1_R(S/J, R) \to \operatorname{Ext}^1_S(S/J, R)$ is an isomorphism.*

If these conditions are satisfied for J then they are satisfied for any ideal containing J.

Example 2.3. The conditions (1)–(3) of Theorem 2.2 are satisfied if the R-ideal $J/I = JR$ contains a nonzerodivisor. The natural exact sequence of R-modules
$$0 \to I/IJ \to J \otimes_S R \to JR \to 0$$
gives an exact sequence
$$0 \to (JR)^* \to J^* \to \operatorname{Hom}(I/IJ, R).$$
But $\operatorname{Hom}(I/IJ, R) = 0$ since the R-module I/IJ is annihilated by JR. Thus the map $(JR)^* \to J^*$ is an isomorphism as in condition (1) of Theorem 2.2.

Proof of Theorem 2.2. (1)$\iff$ (2): Dualizing the exact sequence
$$0 \to I \to J \to (J/I = JR) \to 0$$
yields the result.

(1) $\iff$ (3): We have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & JR & \longrightarrow & R & \longrightarrow & R/JR & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow\cong & & \\
0 & \longrightarrow & J & \longrightarrow & S & \longrightarrow & S/J & \longrightarrow & 0
\end{array}.$$

Dualizing into R, we get the diagram

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathrm{Ext}^1_R(R/JR, R) & \longleftarrow & (JR)^* & \longleftarrow & R & \longleftarrow & (R/JR)^* & \longleftarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \downarrow\cong & & \\
0 & \longleftarrow & \mathrm{Ext}^1_S(S/J, R) & \longleftarrow & J^* & \longleftarrow & R & \longleftarrow & (S/J)^* & \longleftarrow & 0
\end{array}.$$

The equivalence now follows from the "five lemma".

The last statement follows at once from condition (2). ■

Proposition 2.4. *With notation as in Theorem 2.2, if J is generated by an S-regular sequence x, y, then $J^* \cong \mathrm{syz}_1^R(JR)$ via the map $f \mapsto (f(y), -f(x))$. In particular, if $J^* = (JR)^*$ then f indices an isomorphism*

$$\mathrm{syz}_1^R(JR) \cong (JR)^*.$$

Proof. We may write $\mathrm{syz}_1^R(JR) = \{(\overline{a}, \overline{b}) \mid ax + by \in I\}$ where $^-$ denotes images in R. Consider the maps

$$S \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} -b & a \end{pmatrix}} R.$$

The composition is 0 if and only if $ax + by \in I$, and since J is the cokernel of $\begin{pmatrix} y \\ -x \end{pmatrix}$, this is the condition that $(-b, a)$ induces a homomorphism $J \to R$. ■

Theorem 2.5. *Let S be a Noetherian local ring and let x, y be an S-regular sequence. Let $I \subset S$ be an ideal of projective dimension one, so that we may write $I = aI'$, where I' is perfect of grade 2 and a is a nonzerodivisor.*

If $I' \subset J := (x, y)$ then conditions $(1) - (3)$ of Theorem 2.2 are equivalent to the condition that $a \cdot \mathrm{Fitt}_2(I) \subset J$.

Proof. Write $R = S/I$ and $(-)^* = \mathrm{Hom}_S(-, R)$. We denote images in R by $^-$.

We will show that the restriction map $\rho : J^* \to I^*$ is 0 if and only if $a \cdot \mathrm{Fitt}_2(I) \subset J$.

Let $h'_1, \dots, h'_n$ be generators of I' so that the elements $h_i = ah'_i$ generate I. Extending the ground field if necessary, we may assume that h'_i, h'_j form a regular sequence for every $i \neq j$ and that h'_i, a form a regular sequence for every i. Let

$$\varphi' = \begin{pmatrix} f'_1 & \dots & f'_n \\ g'_1 & \dots & g'_n \end{pmatrix}$$

be a matrix with entries in S satisfying $\begin{pmatrix} h'_1 & \dots & h'_n \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \cdot \varphi'$, and let $\varphi = a\varphi'$.

We first prove that $\rho = 0$ if and only if $aI_2(\varphi') \subset I'$. Consider presentations of J and I with respect to the generators x, y and $h_1, \dots, h_n$, respectively, and a morphism between them,

$$\begin{array}{ccccccc}
S & \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} & S^2 & \longrightarrow & J & \longrightarrow & 0 \\
\uparrow & & \uparrow\varphi & & \uparrow & & \\
S^{n-1} & \longrightarrow & S^n & \longrightarrow & I & \longrightarrow & 0\,.
\end{array}$$

Dualizing into R we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & & R^{n+1} & & & & \\
 & & & \searrow^{\overline{\psi}} & & & \\
0 & \longrightarrow & J^* & \longrightarrow & S^{2*} = R^{2*} & \xrightarrow{(\overline{y}, -\overline{x})} & S^* = R^* \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \varphi^* = \overline{\varphi}^*} & & \downarrow \\
0 & \longrightarrow & I^* & \longrightarrow & S^{n*} = R^{n*} & \longrightarrow & S^{n-1*} = R^{n-1*} .
\end{array}$$

Thus $\rho = 0$ if and only if $\overline{\varphi}^*$ is zero when restricted to the image of J^*. This image is the syzygy module of $\overline{y}, -\overline{x}$ in R^{2*}, which in turn is generated by the columns of the matrix $\overline{\psi}$, where

$$\psi = \begin{pmatrix} x & g_1 & \dots & g_n \\ y & -f_1 & \dots & -f_n \end{pmatrix} .$$

Therefore $\rho = 0$ if and only if $\overline{\varphi}^* \overline{\psi} = 0$.

Since

$$\varphi^t \psi = \begin{pmatrix} h_1 & 0 & \Delta_{1,2} & \dots & \Delta_{1,n} \\ h_2 & -\Delta_{1,2} & 0 & \dots & \Delta_{2,n} \\ \vdots & \vdots & & \ddots & \vdots \\ h_n & -\Delta_{1,n} & -\Delta_{2,n} & \dots & 0 \end{pmatrix}$$

where $\Delta_{i,j}$ is the determinant of the submatrix of φ involving columns i, j, we see that $\rho = 0$ if and only if $I_2(\varphi) \subset I$, and this is the case if and only if $aI_2(\varphi') \subset I'$, as claimed.

Let $\Delta'_{i,j}$ be the minor of φ' involving columns i, j. Since $\begin{pmatrix} h'_i & ah'_j \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} f'_i & af'_j \\ g'_i & ag'_j \end{pmatrix}$ and h'_i, ah'_j form a regular sequence, a theorem of Gaeta (see for instance [2, Example 3.2(b)]) gives

$$(h'_i, ah'_j) : J = (h'_i, ah'_j, a\Delta'_{i,j}) .$$

As I' is perfect of grade 2, it follows that $a \cdot \Delta'_{i,j} \in I'$ if and only if $(h'_i, ah'_j) : I' \subset J$ by the symmetry of linkage.

By the same theorem of Gaeta, the link $(h'_i, ah'_j) : I'$ of I' is generated by h'_i and a times the $n-2$ minors of the presentation matrix of $I' \cong I$ with rows i and j deleted. This proves that $a \cdot I_2(\varphi') \subset I'$ if and only if $a \cdot \mathrm{Fitt}_2(I) \subset J$. ■

Proof of Theorem 2.1. We apply the previous results with $J = \mathfrak{n}$. If I is principal, we use Example 2.3 and Proposition 2.4. If I is not principal, we may write $I = aI'$ with I' perfect of grade 2. We see from Proposition 2.4 and Theorem 2.5 that the result holds unless both (a) and $\mathrm{Fitt}_2(I)$ are unit ideals. If a is a unit, then $I = I'$ has grade 2. If in addition $\mathrm{Fitt}_2(I) = S$, then I is a zero dimensional complete intersection. ■

3. The decomposition of $\mathfrak{m}$

In this section $(S, \mathfrak{n}, k)$ denotes a regular local ring of dimension 2 and $I \subset S$ is an ideal. Write $R = S/I$ and $\mathfrak{m} = \mathfrak{n}R$.

Lemma 3.1. *Suppose that $\mathfrak{n} = (x, y)$. If $xy \in I$ then I can either be written as*

(a) $I = (xy, ux^\alpha + vy^\beta)$ where u, v are each either 0 or units and α, β are non-negative integers; or as

(b) (xy, x^α, y^β), where α and β are positive.

Proof. If I has codimension 1 then I has a proper common divisor, which we may take to be x. Writing $I = x(J + (y))$, we see that either $I = (xy)$ or $I = (xy, x^\alpha)$ for some $\alpha \geq 1$ because $S/(y)$ is a discrete valuation ring with parameter x.

Any element of an Artinian local ring can be written as a polynomial in the generators of the maximal ideal with unit coefficients. In particular, if I has codimension 2, then any element of $S/(xy, \mathfrak{n}I)$ is the image of an element of the form $f = ux^\alpha + vy^\beta$, where each of u and v is either 0 or a unit of S and α, β are non-negative. Note that if $u \neq 0$ then, modulo xy, every x^μ with $\mu > \alpha$ is a multiple of f and similarly for v and y.

If $I/(xy)$ is principal, then we may write $I = (xy, ux^\alpha + vy^\beta)$, and we are done. Otherwise, modulo xy, we may write two of the generators of I as $ux^\alpha + vy^\beta,\ px^\gamma + qy^\delta$, where u and q are units and α and δ are minimal. Thus we may assume that $p = 0$, and $v = 0$, so $I = (xy, x^\alpha, y^\delta)$. Notice that α, β have to be positive. ∎

Theorem 3.2. *The following are equivalent:*

(*1*) *The module* $\mathfrak{m}$ *is decomposable.*

(*2*) *We may write* $\mathfrak{n} = (x, y)$ *with* $xy \in I$ *and* R *is neither a discrete valuation ring nor a zero-dimensional complete intersection.*

(*3*) *We may write* $\mathfrak{n} = (x, y)$ *in such a way that* $I = (xy, ux^\alpha, vy^\beta)$, *where each of* u, v *is a unit of* S *or 0 and* α, β *are* ≥ 2.

In this case $\mathfrak{m} \cong R/(0 : x) \oplus R/(0 : y)$.

Proof. (1) $\Rightarrow$ (2) If $\mathfrak{m}$ is decomposable, then it has to decompose as $\mathfrak{m} = xR \oplus yR$ where $\mathfrak{m} = (x, y)$. This implies $xy \in I$. Every ideal of a domain is indecomposable, and every non-zero ideal of a zero-dimensional local Gorenstein ring contains the socle, and thus is indecomposable.
(2) $\Rightarrow$ (3) This follows from Lemma 3.1 because R is not a discrete valuation ring or a zero-dimensional complete intersection.
(3) $\Rightarrow$ (1) One easily check that $(I, x) \cap (I, y) = I$. ∎

4. The decomposition of $\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m})$

Again in this section $(S, \mathfrak{n}, k)$ is a regular local ring of dimension 2 and I is an ideal. Set $R = S/I$ and $\mathfrak{m} = \mathfrak{n}R$. We denote images in R by $^-$.

Proposition 4.1. *If* $\mathfrak{m}$ *is decomposable then* $\operatorname{syz}_2^R(k) = \operatorname{syz}_1^R(\mathfrak{m}) \cong \mathfrak{m} \oplus k^a$, *where*

$$a = \begin{cases} 2 & \textit{if } \dim R = 0 \\ 1 & \textit{if } \operatorname{depth} R = 0 \textit{ and } \dim R = 1 \\ 0 & \textit{if } R \textit{ is Cohen-Macaulay of dimension 1}. \end{cases}$$

Proof. We apply the analysis of Theorem 3.2. This shows that $I \subset \mathfrak{n}^2$ and we may assume $\mathfrak{m} = \overline{x}R \oplus \overline{y}R$. Furthermore, if $\dim R = 0$ then we can write $I = (x^\alpha, xy, y^\beta)$, where α, β are ≥ 2. In this case $\overline{x}R \cong R/(\overline{x}^{\alpha-1}, \overline{y})$ and $\overline{y}R \cong R/(\overline{x}, \overline{y}^{\beta-1})$. Thus $\operatorname{syz}_1^R(\mathfrak{m}) = (\overline{x}^{\alpha-1}, \overline{y}) \oplus (\overline{x}, \overline{y}^{\beta-1}) \cong k \oplus \overline{y}R \oplus \overline{x}R \oplus k \cong \mathfrak{m} \oplus k^2$ since $\overline{x}^{\alpha-1}, \overline{y}^{\beta-1}$ are in the socle of R.

If $\operatorname{depth} R = 0$ and $\dim R = 1$, then Theorem 3.2 shows that we may write $R = S/(xy, x^\alpha)$ with $\alpha \geq 2$. Now $\overline{x}R \cong R/(\overline{y}, \overline{x}^{\alpha-1})$ and $\overline{y}R \cong R/\overline{x}R$, so $\operatorname{syz}_1^R(\mathfrak{m}) = (\overline{y}, \overline{x}^{\alpha-1}) \oplus \overline{x}R \cong \overline{y}R \oplus k \oplus \overline{x}R \cong \mathfrak{m} \oplus k$ because $\overline{x}^{\alpha-1}$ is in the socle of R.

Finally, if R is Cohen-Macaulay of dimension 1, then $I = (xy)$ by Theorem 3.2 and all the modules $\operatorname{syz}_i^R(k)$ for $i \geq 1$ are isomorphic. ∎

Given generators x, y of $\mathfrak{n}$ and $h_1, \ldots, h_n$ of I we consider, as in the proof of Theorem 2.5, a $2 \times (n+1)$ matrix with entries in S

$$L = \begin{pmatrix} L_0 & L_1 & \ldots & L_n \end{pmatrix} = \begin{pmatrix} y & f_1 & \cdots & f_n \\ -x & g_1 & \cdots & g_n \end{pmatrix}$$

such that $\begin{pmatrix} h_1 & \dots & h_n \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} f_1 & \dots & f_n \\ g_1 & \dots & g_n \end{pmatrix}$.

Lemma 4.2. *Suppose that $\mathfrak{m}$ is indecomposable, and let L be a matrix as above. If R is not a complete intersection, then:*

- (*a*) *$I + \mathfrak{n}^3$ does not contain any element xy such that $\mathfrak{n} = (x, y)$.*
- (*b*) $\dim_k(I + \mathfrak{n}^3)/\mathfrak{n}^3 \leq 1$.
- (*c*) *There exists a choice of generators x, y of $\mathfrak{n}$ and a choice of f_i, g_i such that the entries of every column of the form $L_0 + \sum_{i>0} \lambda_i L_i$ generate $\mathfrak{n}$ for all $\lambda_i \in S$.*

Proof. Since R is not a complete intersection, we must have $I \subset \mathfrak{n}^2$.

(a): Suppose first that $\dim R = 1$. Since S is factorial and I is not a complete intersection, we may write $I = aI'$ where I' is an ideal of codimension 2. If I contains an element of order 2, then a must have order 1. By condition (2) of Theorem 3.2, any element of I' that has order 1 must be a multiple of a, completing the proof in the 1-dimensional case.

Now assume that $\dim R = 0$. Suppose that I contains an element $xy + f$ with $\operatorname{ord} f \geq 3$ such that $\mathfrak{n} = (x, y)$. If $\mathfrak{n}^p \subset I$ and $\operatorname{ord} f \geq p$ then $xy \in I$, which is impossible by Theorem 3.2(2). Otherwise, suppose there is an expression $xy + f \in I$ such that $\mathfrak{n} = (x, y)$ with order $\operatorname{ord} f$ maximal and $< p$.

We may write $f = xf_1 + yf_2 + g$ with $\min\{\operatorname{ord} f_1, \operatorname{ord} f_2\} \geq \operatorname{ord} f - 1$ and $\operatorname{ord} g > \operatorname{ord} f$. Thus $xy + f = (x + f_2)(y + f_1) + (g - f_1 f_2)$. Note that $\operatorname{ord}(g - f_1 f_2) > \operatorname{ord} f$. We may replace x, y by $x + f_2, y + f_1$, thus increasing the order of f, a contradiction.

(b): Suppose on the contrary that $ax^2 + bxy + cy^2, a'x^2 + b'xy + c'y^2$ are linearly independent elements of $I + \mathfrak{n}^3/\mathfrak{n}^3$, where x, y are generators of $\mathfrak{n}/\mathfrak{n}^3$ and the coefficients $a, \dots, c'$ are in k. By taking a linear combination, we may assume that $a = 0$, in which case we are done by part (a) unless also $b = 0$. If on the other hand $b = 0$, then $c \neq 0$, so we may assume that $c' = 0$. Now we are done unless $b' = 0$. If $b' = 0$, then x^2 and y^2 are in $I + \mathfrak{n}^3/\mathfrak{n}^3$, but xy is not by part (a). Thus the associated graded ring of R, and with it R itself, is a zero-dimensional complete intersection, a contradiction. This shows that $\dim_k(I + \mathfrak{n}^3)/\mathfrak{n}^3 \leq 1$, completing the proof.

(c): By part (b) the quotient $(I + \mathfrak{n}^3)/\mathfrak{n}^3$ is cyclic. If $I + \mathfrak{n}^3 = (\ell^2) + \mathfrak{n}^3$ for some element ℓ of order 1, we choose generators $x = \ell, y$ for $\mathfrak{n}$. Otherwise we make an arbitrary choice. Furthermore, we may choose f_i and g_i to be in $\mathfrak{n}^2$ for $i > 1$. If $I \subset \mathfrak{n}^3$ we also choose f_1 and g_1 to be in $\mathfrak{n}^2$. If $I + \mathfrak{n}^3 = (x^2) + \mathfrak{n}^3$, we choose $f_1 \equiv x \bmod \mathfrak{n}^2$ and $g_1 \in \mathfrak{n}^2$.

Now consider the 2×2 matrix

$$L' := \begin{pmatrix} L_0 & L_1' \end{pmatrix} := \begin{pmatrix} y & \sum_{i>0} \lambda_i f_i \\ -x & \sum_{i>0} \lambda_i g_i \end{pmatrix}.$$

If $\lambda_1 \in \mathfrak{n}$ or $I \subset \mathfrak{n}^3$, then $L_0 + L_1' \equiv L_0 \bmod (\mathfrak{n}^2 \oplus \mathfrak{n}^2)$ and the claim follows. Thus we can assume that λ_1 is a unit and $I \not\subset \mathfrak{n}^3$. In this case $\det(L') \notin \mathfrak{n}^3$. Moreover $\det(L') \in I$ by the definition of the matrix L. The determinant of L' is also the determinant of the 2×2 matrix $\begin{pmatrix} L_0 + L_1' & L_1' \end{pmatrix}$. If the entries of $L_0 + L_1'$ were linearly dependent modulo $\mathfrak{n}^2$, then the determinant would factor modulo $\mathfrak{n}^3$. Therefore $I + \mathfrak{n}^3 = (x^2) + \mathfrak{n}^3$ by part (a). But then, modulo $\mathfrak{n}^2$ the matrix L' must be

$$\begin{pmatrix} y & x \\ -x & 0 \end{pmatrix}.$$

Thus the entries of $L_0 + L_1'$ generate $\mathfrak{n}$ as claimed. ∎

The significance of the matrix L considered in Lemma 4.2 is that the columns of $\overline{L}$ are obviously a generating set of $\operatorname{syz}_1^R(\mathfrak{m})$, and even a minimal generating set by [20, Satz 5]. In particular $\mu(\operatorname{syz}_1^R(\mathfrak{m})) = \mu(I) + 1$, where $\mu(\cdot)$ denotes minimal number of generators.

Theorem 4.3. *Suppose that* $I \subset \mathfrak{n}^2$. *Write* $\operatorname{syz}_1^R(\mathfrak{m}) = N \oplus N'$ *where* $N' \cong k^a$ *and* N *has no* k*-summands. If* $\mathfrak{m}$ *is indecomposable then*:

(*a*) $a = \dim_k \left(\frac{\mathfrak{n}(I:\mathfrak{n})}{\mathfrak{n}I}\right)$ *and* $\mu(N) = \mu(I) + 1 - a \geq 1$.

(*b*) N *is indecomposable.*

Proof. We may assume that $I \neq 0$. We fix generators x, y of $\mathfrak{n}$ and the corresponding embedding $Z := \operatorname{syz}_1^R(\mathfrak{m}) \subset R^2$, and we use the notation introduced before Lemma 4.2.

(a): Since $\mu(Z) = \mu(I) + 1$, we have $\mu(N) = \mu(I) + 1 - a$.

Notice that

$$a = \dim_k \left(\frac{\operatorname{Soc} Z}{\mathfrak{m}Z \cap \operatorname{Soc} Z} \right).$$

Thus it suffices to prove that

$$\frac{\operatorname{Soc} Z}{\mathfrak{m}Z \cap \operatorname{Soc} Z} \cong \frac{\mathfrak{n}(I : \mathfrak{n})}{\mathfrak{n}I}.$$

To this end we define an R-linear map ψ as the composition of the maps

$$\operatorname{Soc} Z \longrightarrow \frac{Z}{RL_0} = H_1(x, y; R) \xrightarrow{\sim} \frac{I}{\mathfrak{n}I}.$$

Notice that $\psi((f, g)) = (xF + yG) + \mathfrak{n}I$, where F, G are preimages of f, g in S. As $\operatorname{Soc} Z = \operatorname{Soc} R^2$, it follows that $\operatorname{Im} \psi = \frac{\mathfrak{n}(I:\mathfrak{n})}{\mathfrak{n}I}$. Clearly $\operatorname{Ker} \psi = RL_0 \cap \operatorname{Soc} Z$.

Thus it remains to prove

$$RL_0 \cap \operatorname{Soc} Z = \mathfrak{m}Z \cap \operatorname{Soc} Z.$$

The right hand side is in the left hand side, because $\frac{Z}{RL_0} = H_1(x, y; R)$ and therefore $\mathfrak{m}Z \subset RL_0$. As to the converse, the indecomposability of $\mathfrak{m}$ implies that $I \neq \mathfrak{n}^2$, hence $\mathfrak{m}L_0 \neq 0$. Therefore $RL_0 \cap \operatorname{Soc} Z \subset \mathfrak{m}L_0 \subset \mathfrak{m}Z$.

(b): If R is Gorenstein, then Z is indecomposable. If $I = 0$ this is obvious and otherwise it follows from the fact that syzygies of indecomposable maximal Cohen-Macaulay modules over local Gorenstein rings are indecomposable. Thus the assertion of (b) holds and we may assume that R is not Gorenstein.

Since $Z = N \oplus N'$ and $\mathfrak{m}N' = 0$, we have $\mathfrak{m}N = \mathfrak{m}Z$. As shown above $\mathfrak{m}Z \subset RL_0$. But $\overline{L_0}$ is a minimal generator of Z, hence $\mathfrak{m}Z \subset \mathfrak{m}L_0$, and therefore $\mathfrak{m}Z = \mathfrak{m}L_0$. Finally, $RL_0 \cong R/(0 : \mathfrak{m})$, so $\mathfrak{m}L_0 \cong \mathfrak{m}/(0 : \mathfrak{m})$. Putting these facts together, we have

$$\mathfrak{m}\, N \cong \mathfrak{m}/(0 : \mathfrak{m}). \tag{4.3.1}$$

Since N does not have k as a direct summand, the indecomposability of N follows from the indecomposability of $\mathfrak{m}N \cong \mathfrak{m}/(0 : \mathfrak{m})$, so it suffices to treat the cases where the maximal ideal $\mathfrak{m}/(0 : \mathfrak{m})$ of $S/(I : \mathfrak{n})$ is decomposable. By Theorem 3.2(3) this is the case if and only if for a suitable choice of x and y one has $I : \mathfrak{n} = (xy, ux^\alpha, vy^\beta)$, where each of u, v is a unit or 0 and both of α, β are ≥ 2.

We first show that in this case the module N can be generated by 2 elements. Set $I' = \mathfrak{n}\,(I : \mathfrak{n})$. It suffices to prove that $\dim_k(I/I') \leq 1$ because part (a) gives $\mu(N) = 1 + \mu(I) - \dim_k(I'/\mathfrak{n}I) = 1 + \dim_k(I/I')$.

Suppose first that $\dim R = 1$. In this case we may assume that $I : \mathfrak{n} = (xy, ux^\alpha)$, hence $I \subset (xy, ux^\alpha)$. By Theorem 3.2(2) the ideal I contains no product of two elements that generate the maximal ideal of S, so no element of the form $xy + \lambda x^\alpha$ with $\lambda \in S$ can be in I. Thus $I \subset \mathfrak{n}(xy) + (ux^\alpha) = (x^2y, xy^2, ux^\alpha) =: I''$. As $I' = (x^2y, xy^2, ux^{\alpha+1})$, it follows that $\dim_k(I''/I') \leq 1$. Now the containments $I' \subset I \subset I''$ show that $\dim_k(I/I') \leq 1$.

Now assume that $\dim R = 0$, thus $I : \mathfrak{n} = (xy, x^\alpha, y^\beta)$, where α, β are ≥ 2. If $\alpha = \beta = 2$ then $I : \mathfrak{n} = \mathfrak{n}^2$ and hence $I' = \mathfrak{n}^3$. Therefore $\dim_k(I/I') \leq 1$ by Lemma 4.2(b). Finally, without loss of generality, we can assume that $\alpha \geq 3$. We have $I \subset I : \mathfrak{n} = (xy, x^\alpha, y^\beta)$. Since

$\alpha \geq 3$, Lemma 4.2(a) shows that I cannot contain an element of the form $xy + \lambda x^\alpha + \mu y^\beta$. Thus $I \subset \mathfrak{n}(xy) + (x^\alpha, y^\beta) = (x^2y, xy^2, x^\alpha, y^\beta) =: I''$. As $x^{\alpha-1} \notin I : \mathfrak{n}$ but $x^{\alpha-1} \in I'' : \mathfrak{n}$ because $\alpha \geq 3$, the ideal I cannot be equal to I''. On the other hand $I' = (x^2y, xy^2, x^{\alpha+1}, y^{\beta+1})$ and so $\dim_k(I''/I') \leq 2$. Since $I' \subset I \subsetneq I''$, we see that, again, $\dim_k(I/I') \leq 1$. This concludes the proof of the inequality $\mu(N) \leq 2$ and the present choice of the elements x, y.

Now choose x, y and L as in Lemma 4.2(c). Since $\mathfrak{m}$ is indecomposable, $I \neq \mathfrak{n}^2$ and thus $0 : \mathfrak{m} \subsetneq \mathfrak{m}$. Every minimal set of generators of Z contains a unit times an element of the form $L_0 + \sum_{i>0} \lambda_i L_i$, whose annihilator is exactly $0 : \mathfrak{m}$ by Lemma 4.2(c). This generator cannot be among the minimal generators of N', so every minimal set of generators of N contains such an element.

If $N = A \oplus B$ with A, B not zero, then A and B must be cyclic because $\mu(N) \leq 2$. We may assume that A is minimally generated by an element of the form $L_0 + \sum_{i>0} \lambda_i L_i$ and thus $A \cong R/(0 : \mathfrak{m})$. In particular $\mathfrak{m}A \oplus \mathfrak{m}B = \mathfrak{m}N \cong \mathfrak{m}A$, where the last isomorphism holds by (4.3.1). This implies that $\mathfrak{m}B = 0$ because the number of generators of $\mathfrak{m}B$ is 0. Since N does not have k as a direct summand, $B = 0$, and we are done. ■

If in Theorem 4.3 the ring R is Gorenstein, that is, a complete intersection, then $a = 0$. Indeed, as explained in the proof above, $\operatorname{syz}_1^R(\mathfrak{m})$ is indecomposable. Thus $\operatorname{syz}_1^R(\mathfrak{m}) = N$ because $N \neq 0$. Alternatively, one can argue that $\mathfrak{n}(I : \mathfrak{n}) = \mathfrak{n}I$. We may assume that I is $\mathfrak{n}$-primary, hence generated by a regular sequence h_1, h_2 contained in $\mathfrak{n}^2$. As before we write $\begin{pmatrix} h_1 & h_2 \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} L$, where L is a 2×2 matrix with entries in $\mathfrak{n}$. Multiplying this equation with the adjoint of L, whose entries are again in $\mathfrak{n}$, one sees that $\mathfrak{n}\Delta \subset \mathfrak{n}I$ with $\Delta = \det(L)$. On the other hand $I : \mathfrak{n} = I + (\Delta)$, showing that $\mathfrak{n}(I : \mathfrak{n}) \subset \mathfrak{n}I$. For more general results along these lines see [8, Proposition 2.1 and the proof of Theorem 2.2].

Proof of Corollary 1.4. By construction the module N' of Corollary 1.4 is minimally generated by a elements in the socle of $\operatorname{syz}_1^R(\mathfrak{m})$ that form part of a minimal generating set of $\operatorname{syz}_1^R(\mathfrak{m})$. So N' is a direct summand of $\operatorname{syz}_1^R(\mathfrak{m})$ and $N' \cong k^a$, with a as in Theorem 4.3. Since the number of k-summands only depends on $\operatorname{syz}_1^R(\mathfrak{m})$, the quotient $\operatorname{syz}_1^R(\mathfrak{m})/N'$ cannot have any k-summands and hence is indecomposable by Theorem 4.3. ■

5. Proof of Theorem 1.1 and applications

Proof of Theorem 1.1. For $j \geq 1$ let A_j be R tensored with the j^{th} module in a minimal S-free resolution of R, and for $j \geq 2$ set $B_j = A_{j-1}$. Consider the graded free R-module $B = \oplus_{j=2}^{e+1} B_j$. Write $T = T_R(B)$ for the tensor algebra of B over R and $K = K(\mathfrak{m}; R)$ for the Koszul complex of $\mathfrak{m}$. As a graded R-module, the minimal R-free resolution of k is isomorphic to $F := K \otimes_R T$ because R is Golod. There are isomorphisms of R-modules $T \cong R \oplus T \otimes_R B$, and hence $F \cong K \oplus F \otimes_R B$. The description of the differential of F in terms of Massey operations shows that d_F is of the form

$$\begin{array}{ccccc} F & \cong & K & \oplus & F \otimes B \\ \big\downarrow{\scriptstyle d_F} & & \big\downarrow{\scriptstyle d_K} & \swarrow & \big\downarrow{\scriptstyle d_F \otimes B} \\ F & \cong & K & \oplus & F \otimes B\,. \end{array}$$

(see, for instance, [6, Theorem 5.2.2] and its proof). Since K is concentrated in degrees $\leq e$, we obtain, for $i > e$, an isomorphism of complexes

$$F_{\geq i} \cong F_{\geq i-e-1} \otimes B_{e+1} \oplus \;\ldots\; \oplus F_{\geq i-2} \otimes B_2\,.$$

The assertion now follows because B_j is a free R-module of rank h_{j-1}. ■

Corollary 5.1. *Let $(R,\mathfrak{m},k)$ be a Noetherian local ring of embedding dimension 2. If R is neither regular nor a zero-dimensional complete intersection, then*

$$\operatorname{syz}_3^R(k) \cong k^{\mu(I)-1} \oplus \mathfrak{m}^{\mu(I)}.$$

Proof. Since R is Golod, we may apply Theorem 1.1. The result follows because the dimensions of the Koszul homology are the Betti numbers of R as an S-module: $h_1 = \mu(I)$ and $h_2 = \mu(I) - 1$. ■

Theorem 5.2. *Let $(R,\mathfrak{m},k)$ be a Noetherian local ring. If R has embedding dimension ≤ 2 and is not a zero-dimensional complete intersection, then every minimal R-syzygy of k is a direct sum of copies of k, $\mathfrak{m}$, and $\mathfrak{m}^* := \operatorname{Hom}_R(\mathfrak{m}, R) = \operatorname{syz}_1^R(\mathfrak{m})$. Moreover, copies of at most 3 indecomposable modules are required to build all the syzygies of k as direct sums.*

Proof. The first assertion follows from Theorem 1.1 and Theorem 2.1, and the second assertion is a consequence of Theorem 3.2, Proposition 4.1, and Theorem 4.3. ■

Theorem 5.3. *Let $(S,\mathfrak{n},k)$ be a regular local ring of dimension 2, and let I be an ideal contained in $\mathfrak{n}^2$. Write $R = S/I$ and $\mathfrak{m} = \mathfrak{n}R$.*

(*a*) $\operatorname{syz}_1^R(k) = \mathfrak{m}$ *is indecomposable if and only if $xy \notin I$ for any x, y with $\mathfrak{n} = (x, y)$ or R is a zero-dimensional complete intersection.*

(*b*) $\operatorname{syz}_2^R(k)$ *is indecomposable if and only if $\mathfrak{m}$ is indecomposable and $(I : \mathfrak{n})\mathfrak{n} = I\mathfrak{n}$.*

(*c*) $\operatorname{syz}_3^R(k)$ *is indecomposable if and only if I is a principal ideal such that $xy \notin I$ for any x, y with $\mathfrak{n} = (x, y)$ or R is a zero-dimensional complete intersection.*

(*d*) $\operatorname{syz}_i^R(k)$ *is indecomposable for every $i \geq 0$ if and only if $\operatorname{syz}_3^R(k)$ is indecomposable.*

Proof. Part (a) follows from Theorem 3.2. For part (b), notice that if $\mathfrak{m}$ is decomposable then $\operatorname{syz}_1^R(\mathfrak{m})$ is decomposable. Now the assertion follows from Theorem 4.3.

For parts (c) and (d), we may assume that R is neither regular nor a zero-dimensional complete intersection, since otherwise all syzygy modules of k are indecomposable. By Corollary 5.1 $\operatorname{syz}_3^R(k)$ is indecomposable if and only if I is principal and $\mathfrak{m}$ is indecomposable. Since then R is Gorenstein and $\mathfrak{m}$ is a maximal Cohen-Macaulay R-module, $\mathfrak{m}$ is indecomposable if and only if one or all of its syzygies are indecomposable. Now part (d) follows and part (c) is a consequence of (a). ■

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