Contemporary Mathematics

Minimal Homogenous Liaison and Licci Ideals

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Abstract. We study the linkage classes of homogeneous ideals in polynomial rings. An ideal is said to be homogeneously licci if it can be linked to a complete intersection using only homogeneous regular sequences at each step. We ask a natural question: if I is homogeneously licci, then can it be linked to a complete intersection by linking using regular sequences of forms of smallest possible degree at each step (we call such ideals minimally homogeneously licci)? In this paper we answer this question in the negative. In particular, for every $n \geq 28$ we construct a set of n points in $\mathbb{P}^3$ which are homogeneously licci, but not minimally homogeneously licci. Moreover, we prove that one cannot distinguish between the classes of homogeneously licci and non-licci ideals based only on their Hilbert functions, nor distinguish between homogeneously licci and minimally homogeneously licci ideals based solely on the graded Betti numbers. Finally, by taking hypersurface sections, we show that the natural question has a negative answer whenever the height of the ideal is at least three.

1. Introduction

Let R be a commutative Noetherian ring, and let I and J be two proper ideals of height c in R. These ideals are said to be *directly linked* if there exists a regular sequence $f_1, \ldots, f_c$ such that $(f_1, \ldots, f_c) : I = J$ and $(f_1, \ldots, f_c) : J = I$. We say I and J are in the same *linkage class* (or *liaison class*) if there exists a sequence of ideals $I = I_0, \ldots, I_n = J$ such that I_j is directly linked to I_{j+1} for $0 \leq j \leq n-1$, the case $n = 2$ being referred to as *double linkage*. Such a sequence of links connecting I and J is far from unique. We call the ideal I *licci* if I is in the **li**nkage **c**lass of a **c**omplete **i**ntersection, i.e., of an ideal generated by a regular sequence.

Let I be a homogeneous ideal of $S = k[x_1, \ldots, x_d]$, a polynomial ring over a field with the standard grading. In considering the linkage class of I one must distinguish between allowing only homogeneous links, i.e., links where all the regular sequences are homogeneous, and non-homogeneous linkage. The corresponding linkage classes are not a priori the same. In trying to understand homogeneous

2000 *Mathematics Subject Classification.* 13C40, 14M06, 14M10.

The first and last authors were partially supported by the National Science Foundation, grants DMS-0244405 and DMS-0200858, respectively. The second and third authors were partially supported by the National Security Agency, grants H98230-07-1-0036 and H98230-07-1-0065, respectively.

linkage, an obviously critical question to address is to understand the sequence of links by which a homogeneously licci ideal is in the liaison class of a complete intersection. In the local case, in [**4**, 2.5] it was shown that if an ideal I in a Gorenstein local ring with infinite residue field is licci, then one can pick regular sequences consisting of general combinations of minimal generators at every step to reach a complete intersection. A corresponding question for homogeneous linkage would ask if I is homogeneously licci, then can it be linked to a complete intersection by linking using regular sequences of forms of smallest possible degree at each step? In this paper we answer this question in the negative.

Since homogeneous linkage by a regular sequence of minimal possible degree will be the chief topic of this paper, we give it a name; we say that I is *directly minimally homogeneously linked* to an ideal J if $J = L : I$ and $I = L : J$ where L is generated by a regular sequence in I of smallest possible degree. Note that contrary to usual linkage, this is not necessarily a symmetric relation! We say that I is *minimally homogeneously licci* if I is homogeneously licci and all the links in a linkage sequence from I to a complete intersection can be taken to be direct minimal homogeneous links. (Of course, this does not mean *all* paths from I to a complete intersection are minimal linkages.)

Intuitively, it seems that all homogeneously licci ideals should be minimally homogeneously licci, as is the case in height 2. However, in [**2**, pg. 61-62] an example was announced of a homogeneously licci ideal which is not minimally homogeneously licci. An $\mathfrak{m}$-primary homogeneous ideal I with h-vector $(1, 3, 6, 8, 7, 6, 2)$ was constructed which is homogeneously licci, and it was claimed that it is not minimally homogeneously licci. However, no proof was given for this assertion. We began the present paper in an attempt to understand this example. In Section 2 we will give examples of three ideals, each with h-vector $(1, 3, 6, 8, 7, 6, 2)$, such that one of them is not licci, one is minimally homogeneously licci, and the last is homogeneously licci, but not minimally homogeneously licci. Although none of these examples is the one from [**2**] (which was level, while ours are not), they show that no argument based only on the Hilbert function can distinguish such ideals. In the last two examples, even the graded Betti numbers are the same. This proves that there is no way to distinguish homogeneously licci and minimally homogeneously licci ideals by only using their graded Betti numbers. At this point, we do not know whether there exist two homogeneous ideals with the same graded Betti numbers, one homogeneously licci and the other not.

In our last section we go much farther: we prove that for any $n \geq 28$ and $d \geq 3$ there exist reduced subschemes of degree n and codimension three in $\mathbb{P}^d_k$, for k an infinite field, that are homogeneously licci, but not minimally homogeneously licci. Moreover, these subschemes can be taken to be finite unions of reduced linear subspaces. In particular, for any $n \geq 28$ there exist sets of n k-rational points in $\mathbb{P}^3_k$ that are homogeneously licci, but are not minimally homogeneously licci. The examples given are quite explicit, and suggest that this phenomenon is in fact very common. Finally, by taking hypersurface sections, we show that in any codimension ≥ 3 there are reduced subschemes that are homogeneously licci, but not minimally homogeneously licci. Related results for Gorenstein liaison recently have been given in [**3**]. On the other hand, it is still conceivable that every homogeneously licci ideal can be linked to a complete intersection by using only homogeneous regular sequences that are part of a minimal homogeneous generating set of the respective ideals.

For unexplained terminology we refer to the book of Eisenbud, [**1**], while for additional information on liaison we refer to [**6**].

We would like to thank the referee of this article for a careful reading and many insightful comments.

2. Three $\mathfrak{m}$-Primary Examples

In this section we give three examples of $\mathfrak{m}$-primary homogeneous ideals, all with Hilbert function $(1,3,6,8,7,6,2)$, which exhibit different behavior in terms of their linkage classes. One is not licci at all, one is minimally homogeneously licci, and the third is homogeneously licci, but not minimally homogeneously licci. The third example takes considerably more work.

We begin by establishing some notation. We will always write $S = k[x_1, \dots, x_d]$ for a polynomial ring over a field k and $\mathfrak{m}$ for its homogeneous maximal ideal $(x_1, \dots, x_d)$. If I is a homogeneous ideal, we let $I_{\leq j}$ denote the ideal generated by all forms of degree at most j in I.

One of our examples is monomial. By a monomial in S we mean an element of the form $x_1^{a_1} \cdots x_d^{a_d}$. A *monomial ideal* is an ideal generated by monomials. Every $\mathfrak{m}$-primary monomial ideal I can be written uniquely in *standard form* $I = (x_1^{a_1}, \dots, x_d^{a_d}) + I^{\#}$, where $I^{\#}$ is generated by monomials that together with $\{x_1^{a_1}, \dots, x_d^{a_d}\}$ generate I minimally.

We will use the following lemma of [**5**, 2.5]:

LEMMA 2.1. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and let I be an $\mathfrak{m}$-primary monomial ideal. If $I^{\#} = x_1^{b_1} \cdots x_d^{b_d} K$ where $K \neq S$ is a monomial ideal, then the ideal $I' = (x_1^{a_1-b_1}, \dots, x_d^{a_d-b_d}) + K$ is obtained from I by a double link defined by the monomial regular sequences $x_1^{a_1}, \dots, x_d^{a_d}$ and $x_1^{a_1-b_1}, \dots, x_d^{a_d-b_d}$.*

EXAMPLE 2.2. Let $I = (z^3, xyz, x^3y, x^4, y^6, y^5z, xy^5) \subset S = k[x,y,z]$. Then I is $\mathfrak{m}$-primary and the Hilbert function of S/I is $(1,3,6,8,7,6,2)$. We claim that I is not licci. We compute $I^{\#} = (xyz, x^3y, y^5z, xy^5)$. This ideal has height one with greatest common divisor y. By Lemma 2.1, I is doubly linked to $I' = (xz, x^3, y^4z, xy^4, z^3, y^5)$, and $(I')^{\#} = (xz, y^4z, xy^4)$, which has height two. Now [**5**, 2.4] shows that I is not licci in any sense. We note that the minimal homogeneous free resolution of I is

$$0 \to S(-8)^2 \oplus S(-9)^2 \longrightarrow S(-5)^3 \oplus S(-7)^5 \oplus S(-8)^2 \longrightarrow S(-3)^2 \oplus S(-4)^2 \oplus S(-6)^3.$$

EXAMPLE 2.3. Let $I = (z^3, xyz, x^3y, x^4, y^6, y^5z + x^2y^4, xy^5) \subset S = k[x,y,z]$. Then I is an $\mathfrak{m}$-primary ideal with exactly the same Hilbert function as the ideal in Example 2.2. However, I is licci, and it is even minimally homogeneously licci. We prove this by simply constructing a sequence of minimal homogeneous links to a complete intersection. Let $L = (z^3, x^4, y^6)$. Then L is a complete intersection inside I of degrees $3,4,6$, which are the minimal possible ones since $I_{\leq 3}$ has height one, and $I_{\leq 5}$ has height two. A calculation gives that $I_1 := L : I = (z^3, x^3y, x^4, x^3z - xyz^2, y^5)$. The minimal possible degrees of a homogeneous regular sequence in I_1 are $3,4,5$. Let $L_1 = (z^3, x^4, y^5)$ and set $I_2 := L_1 : I_1 = (xz, x^3, z^3, xy^4, y^5, x^2y^3 + y^4z)$.

The minimal possible degrees of a homogeneous regular sequence in I_2 are $2, 3, 5$. Set $L_2 = (xz, x^3 + z^3, y^5)$. A calculation gives that $I_3 := L_2 : I_2 = (xz, yz, xy + z^2, x^3, y^5)$. The minimal possible degrees of a homogeneous regular sequence in I_3 are $2, 2, 5$. Let $L_3 = (yz, xy + z^2, y^5 + x^5)$ and set $I_4 := L_3 : I_3 = (yz, xy + z^2, y^2, x^4, x^2z^2)$. The minimal possible degrees of a homogeneous regular sequence in I_4 are $2, 2, 4$. Let $L_4 = (xy + z^2, y^2, x^4)$ and set $I_5 = L_4 : I_4 = (y, xz, z^2, x^4)$. The minimal possible degrees of a homogeneous regular sequence in I_5 are $1, 2, 4$. Set $L_5 = (y, z^2, x^4)$. Then $L_5 : I_5 = (y, z, x^3)$ is a complete intersection.

We note that the minimal homogeneous free resolution of I is

$$0 \to S(-8) \oplus S(-9)^2 \longrightarrow S(-5)^3 \oplus S(-7)^5 \oplus S(-8) \longrightarrow S(-3)^2 \oplus S(-4)^2 \oplus S(-6)^3.$$

The first example was easy to verify using the work of [**5**]. The second example was easily verified by a straightforward calculation. However, the ease of proving the second example belies the difficulty in finding it. We needed to find a licci ideal with the given Hilbert function where a certain Koszul relation on the generators is minimal. The latter feature is crucial in the construction, as it causes Betti numbers to decrease under double linkage.

Our last example is more delicate, requiring an understanding of how the heights of ideals generated by forms of low degree change in the linkage class of the ideal.

THEOREM 2.4. *Let $S = k[x, y, z]$, where k is an infinite field, and consider the ideal $I = (x^2y + y^3 - yz^2 - z^3, xy^2 - xz^2, x^3z - xyz^2, y^2z^2 - z^4, x^6, z^6, xz^5)$. Then I is an $\mathfrak{m}$-primary ideal with minimal homogeneous free resolution*

$$0 \to S(-8) \oplus S(-9)^2 \longrightarrow S(-5)^3 \oplus S(-7)^5 \oplus S(-8) \longrightarrow S(-3)^2 \oplus S(-4)^2 \oplus S(-6)^3,$$

the Hilbert function of S/I is $(1, 3, 6, 8, 7, 6, 2)$, and I is homogeneously licci, but not minimally homogeneously licci.

PROOF. We first prove that I is homogeneously licci by direct computation. Let L be generated by the homogeneous regular sequence $x^2y+y^3-yz^2-z^3, y^2z^2-z^4, x^6$. A computation shows that $I_1 := L : I = (x^2y + y^3 - yz^2 - z^3, z^4, y^2z^2, xz^3, x^5)$. We now let L_1 be the ideal generated by the homogeneous regular sequence $x^2y + y^3 - yz^2 - z^3, z^4, x^5$. One obtains the link $I_2 := L_1 : I_1 = (z^2, x^2y + y^3, x^2z + y^2z, y^5, xy^3z, x^5)$. Let L_2 be the ideal in I_2 generated by the homogeneous regular sequence $z^2, x^2y + y^3, x^5$. One can compute $I_3 := L_2 : I_2 = (xz, y^2, z^2, x^2y, x^5)$. We let L_3 be the ideal in I_3 generated by the regular sequence y^2, z^2, x^5, and calculate $I_4 := L_3 : I_3 = (y^2, yz, z^2, x^4, x^3z)$. Let L_4 be generated by the regular sequence y^2, z^2, x^4, and $I_5 := L_4 : I_4 = (z, xy, y^2, x^4)$. Set $L_5 = (z, y^2, x^4)$. Then $I_6 := L_5 : I_5 = (z, y, x^3)$ is generated by a regular sequence, so that I is licci. The very first link, however, is not minimal; it uses a homogeneous regular sequence of degrees $3, 4, 6$, whereas the minimal degrees of a homogeneous regular sequence inside I are $3, 3, 6$.

The claims concerning the Hilbert function of I and the resolution of I are easy to check on any computer algebra program. The remainder of the proof is to show that I is not minimally homogeneously licci.

We consider any $\mathfrak{m}$-primary homogeneous ideal I' in $S = k[x, y, z]$ satisfying condition $(\star)$; by this latter condition we mean that I' contains a homogeneous regular sequence of degrees $3, 3, 6$ and has a (not necessarily minimal) homogeneous free resolution of the form

$$0 \to S(-8) \oplus S(-9)^2 \longrightarrow S(-5)^3 \oplus S(-7)^5 \oplus S(-8) \longrightarrow S(-3)^2 \oplus S(-4)^2 \oplus S(-6)^3.$$

The ideal I satisfies ($\star$). Also notice that any ideal satisfying ($\star$) requires 7 generators, hence cannot be a complete intersection or directly homogeneously linked to a complete intersection. We are going to prove that any minimal homogeneous double link reproduces condition ($\star$). Therefore no ideal satisfying ($\star$) can be minimally homogeneously licci.

Thus assume that I' satisfies ($\star$). We first prove that H, the subideal of I' generated by all forms of degree at most 5, has height at most 2. Indeed if H has height three, then H contains an ideal L generated by a homogeneous regular sequence of degrees $3, 3, 4$. The k-dimension of L_7 is 35. On the other hand the resolution of I' shows that the k-dimension of H_7 is at most $2\dim_k(S(-3)_7) + 2\dim_k(S(-4)_7) - 3\dim_k(S(-5)_7) = 32$. This contradiction proves that H has height at most two. Thus the smallest degrees of a homogeneous regular sequence contained in I' is $3, 3, 6$.

Let J be a link of I' using a homogeneous regular sequence of degrees $3, 3, 6$. We obtain its resolution from Ferrand's mapping cone construction (cf. [**7**]). Since $3, 3, 6$ are the minimal possible degrees of a homogeneous regular sequence inside I', any such regular sequence is part of a minimal homogeneous generating set of I'; in particular, we can assume that this regular sequence appears in the generating set given by the above resolution of I'. This accounts for splitting, in the mapping cone, of $S(-3)^2 \oplus S(-6)$. Thus we obtain the following (not necessarily minimal) homogeneous free resolution of J:

$$0 \to S(-6)^2 \oplus S(-8)^2 \longrightarrow S(-4) \oplus S(-5)^5 \oplus S(-7)^3 \longrightarrow S(-3)^4 \oplus S(-4) \oplus S(-6).$$

If $3, 3, 6$ are again the minimal degrees of a homogeneous regular sequence contained in J, then linking once more we reproduce an ideal satisfying condition ($\star$) as asserted. Otherwise J contains a homogeneous regular sequence of degrees $3, 3, 4$. Linking J with respect to such a complete intersection would lead to this minimal homogeneous free resolution of the linked ideal K

$$\ldots \longrightarrow S(-3)^{a+1} \oplus S(-5)^5 \oplus S(-6)^b \longrightarrow S(-2)^2 \oplus S(-3)^a \oplus S(-4)^3 \longrightarrow K \to 0,$$

where $a \geq 0$ and $b \geq 0$. Since the two linearly independent quadrics in K must have exactly one generating syzygy, it follows that $a + 1 = 1$, hence $a = 0$. Thus $K_{\leq 3}$ is generated by these two quadrics. Furthermore, having a nontrivial cubic relation, the two quadrics are forced to have a linear factor in common. We conclude that $K_{\leq 3}$ has height one and hence cannot contain two cubics forming a regular sequence. Thus there exists no homogeneous regular sequence of degrees $3, 3, 4$ inside J. This finishes the proof. □

One crucial difference between the examples in (2.3) and (2.4) is that in the first the cubics in the ideal have height one, while in the second they have height two.

3. Minimal Homogeneous Linkage in $\mathbb{P}^d$

In this section we continue our discussion of homogeneously licci, but not minimally homogeneously licci ideals, by constructing for any $n \geq 28$ a reduced subscheme of $\mathbb{P}^d$ of degree n whose defining ideal is homogeneously licci, but not minimally homogeneously licci. We begin with a general remark.

Remark 3.1. Let $S = k[x_1, \dots, x_d]$ be a polynomial ring over a field k, and let I be a homogeneous ideal in S of height c such that S/I is Cohen-Macaulay. Let

$$0 \to \oplus_i S(-i)^{b_{c,i}} \longrightarrow \oplus_i S(-i)^{b_{c-1,i}} \longrightarrow \dots \longrightarrow \oplus_i S(-i)^{b_{1,i}} \longrightarrow I \to 0$$

be a minimal homogeneous free resolution of I. Suppose that I contains an ideal generated by a homogeneous regular sequence, say $f_1, \dots, f_c$, of degrees $d_1, \dots, d_c$, respectively. Then

$$d_1 + \cdots + d_c \geq \max\{i \mid b_{c,i} \neq 0\}.$$

Furthermore, if I is not equal to $(f_1, \dots, f_c)$, then the inequality above is strict.

Proof. Without loss of generality we may assume that k is infinite. Choose a sequence of linear forms $l_1, \dots, l_{d-c}$ which form a regular sequence on both S/I and $S/(f_1, \dots, f_c)$. The image of I in $T = S/(l_1, \dots, l_{d-c})$ has the same graded Betti numbers as I, and the images of $f_1, \dots, f_c$ in T form a regular sequence. Since both T/IT and $T/(f_1, \dots, f_c)T$ have finite length, it follows from [**1**, Exercises 20.18 and 20.19] that the regularity is given by the formulas

$$\operatorname{reg}(T/(f_1, \dots, f_c)T) = d_1 + \cdots + d_c - c = \max\{n \mid ((T/(f_1, \dots, f_c)T)_n \neq 0\}$$

and

$$\operatorname{reg}(T/IT) = \max_i\{i \mid b_{c,i} \neq 0\} - c = \max\{n \mid (T/IT)_n \neq 0\}.$$

Clearly the largest n for which $(T/IT)_n \neq 0$ must be at most the largest n for which $(T/(f_1, \dots, f_c)T)_n \neq 0$. Thus, $d_1 + \cdots + d_c \geq \max\{i \mid b_{c,i} \neq 0\}$. Moreover, if $I \neq (f_1, \dots, f_c)$, then this remains true after passing to T, since $l_1, \dots, l_{d-c}$ form a regular sequence modulo I. In this case, since $T/(f_1, \dots, f_c)T$ is Gorenstein, $(f_1, \dots, f_c)T : \mathfrak{m}T \subset IT$, and therefore the largest n for which $(T/IT)_n \neq 0$ must be strictly smaller than the largest n for which $(T/(f_1, \dots, f_c)T)_n \neq 0$. Hence $d_1 + \cdots + d_c > \max_i\{i \mid b_{c,i} \neq 0\}$. □

The next theorem provides a large class of examples of subschemes in $\mathbb{P}^d$ which are homogeneously licci, but not minimally homogeneously licci. The subschemes we obtain have codimension 3, which is the smallest possible codimension for such examples.

Theorem 3.2. *Let $S = k[x_0, \dots, x_d]$ with $d \geq 3$, where k is an infinite field. Let $4 \leq a_1 + 3 \leq a_2 < a_3 < a_4$ be integers so that $a_1 \neq 2$ and $a_2 + a_3 \leq \min\{2, a_1\} + a_4$. Choose $F_1, F_4 \in S$ homogeneous elements of degrees a_1, a_4, respectively, and a linear form L_1 such that L_1, F_1, F_4 is a regular sequence. Define $I_1 = (L_1, F_1, F_4)$. Choose $F_2, F_3 \in I_1$ homogeneous elements of degrees a_2, a_3, respectively, so that F_2, F_3 and L_1, F_2 form regular sequences. (This is possible since (L_1, F_1) has height 2. Necessarily $(F_2, F_3) \subset (L_1, F_1)$.) Let L_2 be a linear form such that L_2, F_2, F_3 is a regular sequence. Define $I = L_2 \cdot I_1 + (F_2, F_3)$. Then $Z = V(I)$ is homogeneously licci, but not minimally homogeneously licci.*

Proof. We begin by verifying the assertion that I is licci. In fact it is doubly linked to I_1, a complete intersection. To see this, choose any homogeneous $G \in I_1$ such that G, F_2, F_3 form a regular sequence. Since the linear form L_2 is regular modulo (F_2, F_3) one sees that $J := (G, F_2, F_3) : I_1 = (GL_2, F_2, F_3) : I$. Moreover, as the calculation of a free resolution of I done below proves, S/I is Cohen-Macaulay, hence I is unmixed and $(GL_2, F_2, F_3) : J = I$.

Next we find a free resolution of I. The definition of I immediately gives that $I : L_2 = I_1$, again as the linear form L_2 is regular modulo (F_2, F_3). Thus there is an exact sequence

$$0 \to S/(L_1, F_1, F_4)(-1) \longrightarrow S/I \longrightarrow S/(L_2, F_2, F_3) \to 0\,.$$

Using a Koszul resolution of the first and the last module in this sequence, applying the horseshoe lemma, and splitting off a summand we obtain a (not necessarily minimal) homogeneous free resolution of I:

$$0 \to \begin{matrix} S(-a_1-a_4-2) \\ \oplus \\ S(-a_2-a_3-1) \end{matrix} \longrightarrow \begin{matrix} S(-a_1-2) \\ \oplus \\ S(-a_4-2) \\ \oplus \\ S(-a_1-a_4-1) \\ \oplus \\ S(-a_2-a_3) \\ \oplus \\ S(-a_2-1) \\ \oplus \\ S(-a_3-1) \end{matrix} \longrightarrow \begin{matrix} S(-2) \\ \oplus \\ S(-a_1-1) \\ \oplus \\ S(-a_4-1) \\ \oplus \\ S(-a_2) \\ \oplus \\ S(-a_3) \end{matrix} \longrightarrow I \to 0\,.$$

Observe that the degrees of the generators are ordered by $2 \leq a_1 + 1 < a_2 < a_3 < a_4 + 1$.

Let I' be an ideal satisfying the following three conditions, which we denote by $(\star)$: I' has height three, contains a homogeneous regular sequence of degrees $2, a_2, a_4 + 1$, and has a (not necessarily minimal) homogeneous free resolution of the form

$$0 \to \begin{matrix} S(-a_1-a_4-2) \\ \oplus \\ S(-a_2-a_3-1) \end{matrix} \longrightarrow \begin{matrix} S(-a_1-2) \\ \oplus \\ S(-a_4-2) \\ \oplus \\ S(-a_1-a_4-1) \\ \oplus \\ S(-a_2-a_3) \\ \oplus \\ S(-a_2-1) \\ \oplus \\ S(-a_3-1) \end{matrix} \longrightarrow \begin{matrix} S(-2) \\ \oplus \\ S(-a_1-1) \\ \oplus \\ S(-a_4-1) \\ \oplus \\ S(-a_2) \\ \oplus \\ S(-a_3) \end{matrix} \longrightarrow I' \to 0\,.$$

The ideal I satisfies $(\star)$ since L_1L_2, F_2 form a homogeneous regular sequence in I of degrees $2, a_2$ and since I is generated in degrees at most $a_4 + 1$.

We will show that no ideal I' satisfying $(\star)$ is a complete intersection, nor is any direct minimal homogeneous link of I'. We will also prove that minimal homogeneous double linkage reproduces condition ($\star$). Therefore no ideal satisfying ($\star$) can be minimally homogeneously licci.

We first argue that no ideal I' satisfying ($\star$) can be generated by a regular sequence. Indeed, the two smallest degrees of minimal generators of I' are $2, a_1+1$, and there is a minimal syzygy in degree a_1+2. The fact that the syzygy in degree a_1+2 is minimal follows from our assumptions, most notably the inequality $a_1+3 \leq a_2$, which imply that there is no cancelation between this syzygy and a generator or a second syzygy of I'.

Next we claim that the minimal possible degrees of a homogeneous regular sequence in I' are $2, a_2, a_4+1$. By assumption I' contains a homogeneous regular sequence of these degrees. We need to show that I' cannot have a smaller regular sequence. First suppose that the ideal $I'_{\leq a_1+1}$ has height two. As noted above, there is a minimal first syzygy of degree a_1+2, which can only come from the two elements in I' of degrees 2 and a_1+1. But then these elements cannot form a regular sequence, proving the claim. It remains to show that I' does not contain a homogeneous regular sequence of degrees $2, a_2, a_3$. Suppose that such a regular sequence exists. By Remark 3.1, it follows that

$$2 + a_2 + a_3 > \max\{a_1 + a_4 + 2, a_2 + a_3 + 1\} = a_1 + a_4 + 2$$

(recall that by assumption $a_2 + a_3 \leq a_1 + a_4$). But then $a_2 + a_3 > a_1 + a_4$, a contradiction.

We now compute a homogeneous free resolution of any link J of I' with respect to a homogenous regular sequence of degrees $2, a_2, a_4+1$. Since such a sequence is part of a minimal homogeneous generating set of I', copies of $S(-2)$, $S(-a_2)$ and $S(-a_4-1)$ split off in the mapping cone construction (cf. [**7**]). Thus we obtain the following homogeneous free resolution of J:

$$0 \to \begin{array}{c} S(a_3-a_2-a_4-3) \\ \oplus \\ S(a_1-a_2-a_4-2) \end{array} \to \begin{array}{c} S(a_1-a_2-a_4-1) \\ \oplus \\ S(-a_2-1) \\ \oplus \\ S(a_1-a_2-2) \\ \oplus \\ S(a_3-a_4-3) \\ \oplus \\ S(-a_4-2) \\ \oplus \\ S(a_3-a_2-a_4-2) \end{array} \to \begin{array}{c} S(a_1-a_2-1) \\ \oplus \\ S(a_3-a_4-2) \\ \oplus \\ S(-a_4-1) \\ \oplus \\ S(-a_2) \\ \oplus \\ S(-2) \end{array} \to J \to 0\,.$$

We first argue that J cannot be a complete intersection. Indeed, the summands $S(a_1-a_2-a_4-2)$ and $S(a_1-a_2-a_4-1)$ in the last and next to last step of the above resolution are still present after passing to a minimal homogeneous free resolution, due to our numerical conditions. Hence the minimal homogeneous free resolution of S/J cannot be symmetric as J does not contain a linear form. It follows that S/J is not even Gorenstein.

Notice that the degrees of generators are a_2-a_1+1, a_4-a_3+2, a_4+1, a_2, 2, and that $2 < a_2-a_1+1 \leq a_2 \leq a_4-a_3+2 < a_4+1$ according to our numerical assumptions. By construction, J contains a homogeneous regular sequence of degrees $2, a_2, a_4+1$. We claim that these are the smallest possible degrees of a homogeneous regular sequence in J. Suppose the contrary. In this case either $a_1 > 1$ and J contains a homogeneous regular sequence of degrees $2, a_2 - a_1 + 1$, or else J contains a homogeneous regular sequence of degrees $2, a_2, a_4 - a_3 + 2$.

To rule out the first possibility notice that the two generators of degrees 2 and $a_2 - a_1 + 1$ are the only generators of degrees $< a_2 - a_1 + 2$ since $a_2 - a_1 + 1 < a_2$. Furthermore, the syzygy of degree $a_2 - a_1 + 2$ cannot cancel against any second syzygy or against any generator of J, because $a_1 > 2$. Thus this syzygy is a minimal syzygy, and hence gives a nontrivial relation of degree $a_2 - a_1 + 2$ among the two generators of degrees 2 and $a_2 - a_1 + 1$. It follows that these generators could not form a regular sequence.

To rule out the second possibility from above, suppose that J contains a homogeneous regular sequence of degrees $2, a_2, a_4 - a_3 + 2$. Now Remark 3.1 implies that

$$2 + a_2 + a_4 - a_3 + 2 > \max\{a_2 + a_4 - a_3 + 3, a_2 + a_4 - a_1 + 2\} = a_2 + a_4 - a_1 + 2.$$

But this is impossible by our numerical conditions.

Thus, when we perform a minimal homogeneous link for J, the residual again satisfies condition $(\star)$, proving the theorem. $\square$

Corollary 3.3. *Let k be an infinite field. For any $n \geq 28$ and $d \geq 3$ there exist reduced subschemes of degree n and codimension three in $\mathbb{P}^d_k$ that are homogeneously licci, but not minimally homogeneously licci. Moreover, the subschemes can be chosen to be a finite union of reduced linear subspaces. In particular, for any $n \geq 28$ there exists a set of n k-rational points in $\mathbb{P}^3_k$ that is homogeneously licci, but not minimally homogeneously licci.*

Proof. The last statement follows at once from the case $d = 3$.

We will use Theorem 3.2 with $a_1 = 1$. Let $l_2, \ldots, l_{a_2}, m_2, \ldots, m_{a_3}, n_1, \ldots, n_{a_4}$, L_1, L_2 be linear forms in S, which together with x_0 have the property that any four are linearly independent. Such linear forms exist because $d \geq 3$ and k is infinite. Setting

$$F_1 = x_0, F_2 = x_0 \cdot l_2 \cdots l_{a_2}, F_3 = L_1 \cdot m_2 \cdots m_{a_3}, F_4 = n_1 \cdots n_{a_4}$$

we obtain homogeneous elements in S that satisfy the assumptions of Theorem 3.2. Moreover, L_2 is regular modulo I_1. Thus the definition of I shows that

$$I = (L_2, F_2, F_3) \cap I_1 = (L_2, F_2, F_3) \cap (L_1, F_1, F_4),$$

and each primary component of this ideal is generated by linear forms, as can be seen from our linear independence condition. Therefore $V(I)$ is a finite union of reduced linear subspaces, of degree $a_2 a_3 + a_4$. Moreover, $V(I)$ is homogeneously licci, but not minimally homogeneously licci if the numerical conditions of the theorem are satisfied. Choosing $a_1 = 1, a_2 = 4, a_3 = 5, a_4 = 8$ gives a subscheme of degree 28, and simply increasing the value of a_4 gives all other possible n. $\square$

We were able to prove a stronger result by replacing the lower bound 28 for the number of points by the value 10, but the proof was more complicated and we opted to present the simpler proof with a slightly worse lower bound. We do not know what the best lower bound is.

Using the above result we are now going to show that in any codimension ≥ 3, there exist homogeneously licci subschemes that are not minimally homogeneously licci. We continue to assume that k is infinite. If $C \subset I$ are ideals we call the colon ideal $C : I$ the *residual* of I with respect to C.

Proposition 3.4. *Consider a class $\mathcal{C}$ of proper homogeneous ideals of height c in $S = k[x_0, \dots, x_d]$ that is closed under taking residuals with respect to homogeneous complete intersections of type $(d_1, \dots, d_c)$. Assume further that $(d_1, \dots, d_c)$ is the minimal possible type of a homogeneous complete intersection of height c contained in any ideal in $\mathcal{C}$.*

Let I be a fixed homogeneously licci ideal in $\mathcal{C}$ and let $F \in S$ be a general form of sufficiently large degree D that is regular modulo I. Then the ideal (I, F) is homogeneously licci, but not minimally homogeneously licci.

Moreover, consider the class $\mathcal{C}'$ of ideals of the form (I', F'), where $I' \in \mathcal{C}$ and $F' \in S$ is a form of degree D that is regular modulo I'. Then $\mathcal{C}'$ is closed under taking residuals with respect to homogeneous complete intersections of type $(d_1, \dots, d_c, D)$, and $(d_1, \dots, d_c, D)$ is the minimal possible type of a homogeneous complete intersection of height $c + 1$ contained in any ideal in $\mathcal{C}'$.

Proof. Since I is homogeneously licci, there is a finite sequence of homogeneous links taking I to a complete intersection. Since F is general and k is infinite, F is not only regular modulo I but also modulo all these intermediate ideals. It then follows that (I, F) is homogeneously licci (see [**6**, 5.2.17]).

The asserted properties of $\mathcal{C}'$ imply that a complete intersection of height $c+1$ in $\mathcal{C}'$ must be of type $(d_1, \dots, d_c, D)$, hence its residual with respect to itself is not a proper ideal and could not belong to $\mathcal{C}'$. It follows that no ideal in $\mathcal{C}'$ can be a complete intersection. Now again the asserted properties of $\mathcal{C}'$ give that (I, F) is not minimally homogeneously licci.

It remains to prove the two claims about $\mathcal{C}'$. Choose D to be any integer with $D > d_1 + \cdots + d_c - c + 1$. By Remark 3.1 the minimal homogeneous generators of every ideal in $\mathcal{C}$ have degree less than D. Now let (I', F') be an ideal in $\mathcal{C}'$ and let C be a homogeneous complete intersection of smallest type inside (I', F'), of height $c+1$. Since D exceeds the generator degrees of I', we can write $C = (C', G)$, where C' is a homogeneous complete intersection in I' whose generators have minimal degree, namely $d_1, \dots, d_c$, and G is a form in (I', F') with $\deg G = \deg F' = D$. Note that $G \notin I'$, so $(I', F') = (I', G)$. Let $J = C' : I'$ be the residual of I' with respect to C'. Then $J \in \mathcal{C}$ and $C : (I', F') = (C', G) : (I', G) = (J, G) \in \mathcal{C}'$, as claimed. $\square$

The proof of Theorem 3.2 shows that the ideals considered there generate a class $\mathcal{C}$ satisfying the assumptions of Proposition 3.4. Thus repeated application of the proposition immediately gives the following consequence:

Corollary 3.5. *Let k be an infinite field. Then there are subschemes of any codimension $c \geq 3$ in any projective space $\mathbb{P}^d_k$ with $d \geq c$ that are homogeneously licci, but not minimally homogeneously licci.*

We do not know if the result of Proposition 3.4 is still true if we remove the condition that $\deg F \gg 0$.

Remark 3.6. From the proof of Proposition 3.4 we can be more precise about what we mean by the hypothesis $\deg F \gg 0$. Indeed, the numbers $d_1, \dots, d_c$ cannot exceed the Castelnuovo-Mumford regularity of I. Hence it is certainly enough to assume $\deg F \geq (\text{codim } I) \cdot (\text{reg } I)$.

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