MULTIDEGREES, FAMILIES, AND INTEGRAL DEPENDENCE

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ABSTRACT. We study the behavior of multidegrees in families and the existence of numerical criteria to detect integral dependence. We show that mixed multiplicities of modules are upper semicontinuous functions when taking fibers and that projective degrees of rational maps are lower semicontinuous under specialization. We investigate various aspects of the polar multiplicities and Segre numbers of an ideal and introduce a new invariant that we call polar-Segre multiplicities. In terms of polar multiplicities and our new invariants, we provide a new integral dependence criterion for certain families of ideals. By giving specific examples, we show that the Segre numbers are the only invariants among the ones we consider that can detect integral dependence. Finally, we generalize the result of Gaffney and Gassler regarding the lexicographic upper semicontinuity of Segre numbers.

1. INTRODUCTION

This paper is concerned with *the behavior of multidegrees in families* and with *the search for criteria to detect integral dependence*. Although these two themes are not typically studied together, the backbone of our work is the delicate interplay between them. Multidegrees provide the natural generalization of the degree of a projective variety to a multiprojective setting, and their study goes back to classical work of van der Waerden [62]. The notion of multidegrees (or mixed multiplicities) has become of importance in several areas of mathematics (e.g., algebraic geometry, commutative algebra and combinatorics; see [3, 5, 7, 8, 14, 15, 31, 33, 36, 40, 41, 46, 47, 54, 58, 59]). On the other hand, the idea of detecting integral dependence with numerical invariants was initiated with seminal work of Rees [50]. Considerable effort has been made to extend Rees' theorem to the case of arbitrary ideals, modules, and, more generally, algebras (see [4, 11, 22, 24, 25, 38, 39, 48, 51, 60, 61]).

Teissier's Principle of Specialization of Integral Dependence (PSID) can be seen as the first indication of a fruitful connection between the behavior of multiplicities in families and the detection of integral dependence (see [56], [55, Appendice I]). Indeed, in an analytic setting, the original PSID states that for a family of zero-dimensional ideals with constant Hilbert-Samuel multiplicity, a section is integrally dependent on the total family if and only if it is integrally dependent on the fibers corresponding to a Zariski-open dense subset of the base. For families of not necessarily zero-dimensional ideals (also in an analytic setting), the PSID was extended by Gaffney and Gassler [25] using Segre numbers. This paper continues the line of research traced by the aforementioned works of Teissier, Gaffney and Gassler.

We now describe the results of this paper more precisely.

1.1. **The behavior of multidegrees and projective degrees of rational maps in families.**

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Let $\Bbbk$ be a field and T be a finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded $\Bbbk$-algebra. Let $X = \text{MultiProj}(T)$ be the corresponding multiprojective scheme embedded in a product of projective spaces $\mathbb{P} = \mathbb{P}^{m_1}_{\Bbbk} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}^{m_p}_{\Bbbk}$. For each $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{Z}^p_{\geqslant 0}$ with $|\mathbf{n}| = n_1 + \cdots + n_p = \dim(X)$, one denotes by $\deg^{\mathbf{n}}_{\mathbb{P}}(X)$ the *multidegree of* X *of type* $\mathbf{n}$ *with respect to* $\mathbb{P}$. In classical geometric terms, if $\Bbbk$ is algebraically closed, then $\deg^{\mathbf{n}}_{\mathbb{P}}(X)$ is equal to the number of points (counted with multiplicities) in the intersection of X with the product $L_1 \times_{\Bbbk} \cdots \times_{\Bbbk} L_p \subset \mathbb{P}$, where $L_i \subseteq \mathbb{P}^{m_i}_{\Bbbk}$ is a general linear subspace of dimension $m_i - n_i$ for each $1 \leqslant i \leqslant p$. More generally, given a finitely generated $\mathbb{Z}^p$-graded T-module M, one denotes by $e(\mathbf{n}; M)$ the *mixed multiplicity of* M *of type* $\mathbf{n}$, for each $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$ with $|\mathbf{n}| = \dim(\text{Supp}_{++}(M))$. Let $\Psi : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ be a rational map and $\Gamma \subset \mathbb{P}^r_{\Bbbk} \times_{\Bbbk} \mathbb{P}^s_{\Bbbk}$ be the closure of the graph of $\mathcal{F}$. For each $0 \leqslant i \leqslant r$, the i-th *projective degree* of $\Psi : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ is given by the following multidegree

$$d_i(\Psi) = \deg^{i, r-i}_{\mathbb{P}^r_{\Bbbk} \times_{\Bbbk} \mathbb{P}^s_{\Bbbk}}(\Gamma).$$

For more details on these notions, see Section 2.

In Section 3, we study the behavior of mixed multiplicities under the process of taking fibers with respect to a base ring. The idea of studying the multiplicities of families is now classical (see, e.g., [43], [16, Chapter 5]), but the case of mixed multiplicities does not seem to have been considered before. Let A be a Noetherian ring and $\mathscr{T}$ be a finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded A-algebra. Denote by $\mathscr{X} = \text{MultiProj}(\mathscr{T})$ the corresponding multiprojective scheme. Let $\mathscr{M}$ be a finitely generated $\mathbb{Z}^p$-graded $\mathscr{T}$-module. For each $\mathfrak{p} \in \text{Spec}(A)$, let $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be the residue field of $\mathfrak{p}$, and consider the finitely generated $\mathbb{Z}^p$-graded module $\mathscr{M} \otimes_A \kappa(\mathfrak{p})$ over the finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded $\kappa(\mathfrak{p})$-algebra $\mathscr{T}(\mathfrak{p}) = \mathscr{T} \otimes_A \kappa(\mathfrak{p})$. Then, for all $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$, we introduce a function

$$e^{\mathscr{M}}_{\mathbf{n}} : \text{Spec}(A) \to \mathbb{Z} \cup \{\infty\}$$

that naturally measures the mixed multiplicities of the fibers $\mathscr{M} \otimes_A \kappa(\mathfrak{p})$ (see Definition 3.3). Our first main result is the following.

Theorem A (Theorem 3.4)**.** *For all* $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$*, the function* $e^{\mathscr{M}}_{\mathbf{n}} : \text{Spec}(A) \to \mathbb{Z} \cup \{\infty\}$ *is upper semicontinuous.*

A direct consequence of the above theorem is the upper semicontinuity of the respective functions

$$\deg^{\mathbf{n}}_{\mathscr{X}, \mathbb{P}_A} : \text{Spec}(A) \to \mathbb{Z} \cup \{\infty\}$$

measuring the multidegrees of the fibers $\mathscr{X} \times_{\text{Spec}(A)} \text{Spec}(\kappa(\mathfrak{p}))$ (see Definition 3.5 and Corollary 3.6). On the other hand, under certain conditions, in Corollary 3.7 we show that mixed multiplicities are lower semicontinuous under the process of taking specializations.

We study rational maps and their specializations in Section 4. In particular, we are interested in the behavior of projective degrees with respect to specializations. Since projective degrees are the mixed multiplicities of the corresponding Rees algebra (i.e., the multidegrees of the graph), this type of question can be traced back to the problem of specializing Rees algebras (see [20, 37]). More recently, specializations of rational maps were studied in [9, 13]. Let A be a Noetherian domain, $S = A[x_0, \ldots, x_r]$ be a standard graded polynomial ring and $\mathbb{P}^r_A = \text{Proj}(S)$. Let $\mathcal{F} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A$ be a rational map with representative $\mathbf{f} = (f_0 : \cdots : f_s)$ such that $\{f_0, \ldots, f_s\} \subset S$ are homogeneous elements of the same degree. Denote by $I = (f_0, \ldots, f_s) \subset S$ the base ideal of $\mathcal{F}$. For any $\mathfrak{p} \in \text{Spec}(A)$, we get the rational map

$\mathcal{F}(\mathfrak{p}) : \mathbb{P}^r_{\kappa(\mathfrak{p})} \dashrightarrow \mathbb{P}^s_{\kappa(\mathfrak{p})}$ with representative $\pi_\mathfrak{p}(\mathbf{f}) = (\pi_\mathfrak{p}(f_0) : \cdots : \pi_\mathfrak{p}(f_s))$ where $\pi_\mathfrak{p}(f_i)$ is the image of f_i under the natural map $\pi_\mathfrak{p} : S \to S(\mathfrak{p}) = S \otimes_A \kappa(\mathfrak{p})$. Then, we introduce the functions

$$\begin{aligned} \mathrm{degIm}^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z} \qquad & \text{(measures the degree of the image of } \mathcal{F}(\mathfrak{p})\text{)} \\ d_i^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z} \qquad & \text{(measures the projective degrees of } \mathcal{F}(\mathfrak{p})\text{)} \\ j^I : \mathrm{Spec}(A) \to \mathbb{Z} \qquad & \text{(measures the j-multiplicity of } I(\mathfrak{p}) \subset S(\mathfrak{p})\text{);} \end{aligned}$$

see Definition 4.2. Our second main result deals with the behavior of the last three functions.

Theorem B (Theorem 4.3)**.** *The following statements hold:*

(i) $\mathrm{degIm}^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function.*
(ii) $d_i^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function for all* $0 \leqslant i \leqslant r$.
(iii) $j^I : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function.*

In Corollary 4.5, we use Theorem B to give sharp upper bounds for the projective degrees of several families of rational maps (the list includes perfect ideals of height two and Gorenstein ideals of height three). The basic idea is that for several families of rational maps under generic conditions we can compute projective degrees, and then Theorem B yields an upper bound for any specialization.

1.2. **Polar multiplicities, Segre numbers and integral dependence.**

Our next objective is to study various aspects of the *polar multiplicities* and *Segre numbers* of an ideal and introduce a new invariant that plays an important role in our work. One technical goal of our work is to extend several of the results of Gaffney and Gassler [25] from their analytic setting to an algebraic one over a Noetherian local ring. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let $d = \dim(R)$, $X = \mathrm{Spec}(R)$ and $I \subset R$ be a proper ideal. Consider the blow-up $\pi : P = \mathrm{Proj}(\mathscr{R}(I)) \to X$ and the exceptional divisor $E = \mathrm{Proj}(\mathrm{gr}_I(R))$ of X along I. Following the general notion of polar multiplicities due to Kleiman and Thorup [38, 39], one defines

$$(\textit{polar multiplicity}) \;\; m_i(I,R) = m_d^{d-i}(\mathscr{R}(I)) \quad \text{and} \quad (\textit{Segre number}) \;\; c_i(I,R) = m_{d-1}^{d-i}(\mathrm{gr}_I(R))$$

as polar multiplicities of $\mathscr{R}(I)$ and $\mathrm{gr}_I(R)$, respectively. It should be mentioned that the polar multiplicities of a standard graded R-algebra can be seen as the multidegrees of a biprojective scheme over the residue field κ. We introduce the new invariant

$$\nu_i(I,R) = m_i(I,R) + c_i(I,R)$$

that we call *polar-Segre multiplicity*.

Section 5 is dedicated to establish several properties of the invariants $m_i(I,R)$, $c_i(I,R)$ and $\nu_i(I,R)$. Let $\delta = o(I)$ be the order of I (see Notation 5.7). In Proposition 5.8, we show the inequality

$$\delta \cdot m_{i-1}(I,R) \leqslant m_i(I,R) + c_i(I,R) = \nu_i(I,R),$$

and that equality holds for all $1 \leqslant i \leqslant d$ if and only if I satisfies the *$\mathscr{G}$-parameter condition generically* (see Notation 5.7). Next, we express all these numbers as the multiplicities of the push-forward via π of certain cycles obtained by making general cuttings; thus following general tradition when studying local invariants (see, e.g., [25], [38, §8]). Assume that κ is an infinite field, $\underline{H} = H_1, \ldots, H_d$ is a sequence of general hyperplanes, and denote by $\underline{g} = g_1, \ldots, g_d$ the associated sequence of elements in I (see Notation 5.5). We introduce the following objects:

$$\begin{aligned} &(\textit{polar scheme}) && P_i(I,X) \;=\; P_i^{\underline{H}}(I,X) \;=\; \pi(H_1 \cap \cdots \cap H_i) \\ &(\textit{Segre cycle}) && \Lambda_i(I,X) \;=\; \Lambda_i^{\underline{H}}(I,X) \;=\; \pi_*\left(\left[E \cap H_1 \cap \cdots \cap H_{i-1}\right]_{d-i}\right) \;\in\; Z_{d-i}(X) \\ &(\textit{polar-Segre cycle}) && V_i(I,X) \;=\; V_i^{\underline{H}}(I,X) \;=\; \pi_*\left(\left[H_1 \cap \cdots \cap H_{i-1} \cap \pi^* D_i\right]_{d-i}\right) \;\in\; Z_{d-i}(X); \end{aligned}$$

for more details, see Setup 5.11. We have the following unifying result.

Theorem C (Theorem 5.13)**.** (κ *infinite*). *The following statements hold:*

(i) $m_i(I,R) = e_{d-i}(P_i(I,X))$ *and* $P_i(I,X) \;=\; \mathrm{Spec}\left(R/(g_1,\ldots,g_i) :_R I^\infty\right)$.

(ii) $c_i(I,R) = e_{d-i}(\Lambda_i(I,X))$ *and*

$$\Lambda_i(I,X) \;=\; \sum_{\substack{\mathfrak{p} \in V((g_1,\ldots,g_{i-1}):_R I^\infty) \\ \mathfrak{p} \in V(I),\, \dim(R/\mathfrak{p}) = d-i}} e\big(I, R_\mathfrak{p}/(g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p}\big) \cdot [R/\mathfrak{p}] \;\in\; Z_{d-i}(X).$$

(iii) $\nu_i(I,R) = e_{d-i}(V_i(I,X))$ *and*

$$V_i(I,X) \;=\; [P_i(I,X)]_{d-i} + \Lambda_i(I,X) \;=\; \left[\mathrm{Spec}\left(R/(g_1,\ldots,g_{i-1}) :_R I^\infty + g_i R\right)\right]_{d-i} \;\in\; Z_{d-i}(X).$$

A particular consequence of the above theorem is that it recovers known formulas for the polar multiplicities and Segre numbers of an ideal (see Remark 5.14).

In Section 6, we introduce new numerical criteria for integral dependence. From the main result of [48], we know that Segre numbers detect integral dependence (also, in an analytic setting, see [25]). In Theorem 6.6, we prove a PSID that is similar to the ones in [25, Theorem 4.7] and [48, Theorem 4.4]. Then a driving question for our work is: *can we detect integral dependence with the invariants* $m_i(I,R)$ *and* $\nu_i(I,R)$*?* In the particular case of the polar multiplicities $m_i(I,R)$ this has been a folklore question for many years. Here we give a definitive and perhaps unfortunate answer:

- *"only Segre numbers detect integral dependence"*.

Indeed, in Example 6.9 and Example 6.10, we provide examples where the invariants $m_i(I,R)$ and $\nu_i(I,R)$ do not detect integral dependence. On the other hand, we have the following criterion where these two invariants can be used.

Theorem D (Theorem 6.7)**.** *Assume that* R *is equidimensional and universally catenary, and let* $I \subset J$ *be two* R*-ideals. Suppose the following two conditions:*

(a) $o(I) = o(J)$.

(b) I *satisfies the* $\mathscr{G}$*-parameter condition generically* (*see* Notation 5.7).

Then the following are equivalent:

(i) J *is integral over* I.

(ii) $m_i(I,R) = m_i(J,R)$ *for all* $0 \leqslant i \leqslant d-1$.

(iii) $\nu_i(I,R) = \nu_i(J,R)$ *for all* $0 \leqslant i \leqslant d$.

An interesting family of ideals where the above theorem applies is that of equigenerated ideals (see Corollary 6.15).

Finally, we study the behavior of Segre numbers in families. More precisely, we generalize the result of Gaffney and Gassler [25] regarding the lexicographic upper semicontinuity of Segre numbers. We now introduce a suitable algebraic notation to extend their original result expressed in an analytic setting. Let $\iota : A \hookrightarrow R$ be a flat injective homomorphism of finite type of Noetherian rings and assume that $\pi : R \twoheadrightarrow A$ is a section of ι. Let $Q = \mathrm{Ker}(\pi)$. For each $\mathfrak{p} \in \mathrm{Spec}(A)$, consider the Noetherian local ring $S(\mathfrak{p}) = R(\mathfrak{p})_{QR(\mathfrak{p})}$ that we call the *distinguished fiber* of $\mathfrak{p}$ (see §6.1). Our last main result is the following.

Theorem E (Theorem 6.12)**.** *Assume that for all* $\mathfrak{p} \in \mathrm{Spec}(A)$*, the fibers* $R(\mathfrak{p})$ *are equidimensional of the same dimension* d *and* $\mathrm{ht}(I(\mathfrak{p})) > 0$*. Then the function*

$$\mathfrak{p} \in \mathrm{Spec}(A) \;\mapsto\; \left(c_1(I, S(\mathfrak{p})), c_2(I, S(\mathfrak{p})), \ldots, c_d(I, S(\mathfrak{p}))\right) \in \mathbb{Z}^d_{\geqslant 0}$$

is upper semicontinuous with respect to the lexicographic order.

We give related lexicographic upper semicontinuity results for Segre numbers in Corollary 6.4 and Corollary 6.5.

2. Preliminaries and notation

Here we recall the concepts of mixed multiplicities, multidegrees and projective degrees. We also set up the notation that is used throughout the paper. Let $p \geqslant 1$ be a positive integer and, for each $1 \leqslant i \leqslant p$, let $\mathbf{e}_i \in \mathbb{Z}^p_{\geqslant 0}$ be the i-th elementary vector $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$. If $\mathbf{n} = (n_1, \ldots, n_p)$, $\mathbf{m} = (m_1, \ldots, m_p) \in \mathbb{Z}^p$ are two vectors, we write $\mathbf{n} \geqslant \mathbf{m}$ whenever $n_i \geqslant m_i$ for all $1 \leqslant i \leqslant p$, and $\mathbf{n} > \mathbf{m}$ whenever $n_j > m_j$ for all $1 \leqslant j \leqslant p$. We write $\mathbf{0} \in \mathbb{Z}^p_{\geqslant 0}$ for the zero vector $\mathbf{0} = (0, \ldots, 0)$.

Let $\Bbbk$ be a field and T be a finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded algebra over $\Bbbk$, that is, $[T]_{\mathbf{0}} = \Bbbk$ and T is finitely generated over $\Bbbk$ by elements of degree $\mathbf{e}_i$ with $1 \leqslant i \leqslant p$. The multiprojective scheme $X = \mathrm{MultiProj}(T)$ corresponding to T is given by the set of all multihomogeneous prime ideals in T not containing the irrelevant ideal $\mathfrak{N} := ([T]_{\mathbf{e}_1}) \cap \cdots \cap \left([T]_{\mathbf{e}_p}\right)$, that is,

$$X = \mathrm{MultiProj}(T) := \left\{\mathfrak{P} \in \mathrm{Spec}(T) \mid \mathfrak{P} \text{ is multihomogeneous and } \mathfrak{P} \not\supseteq \mathfrak{N}\right\},$$

and its scheme structure is obtained by using multihomogeneous localizations (see, e.g., [35, §1]). We embed X as a closed subscheme of a multiprojective space $\mathbb{P} := \mathbb{P}^{m_1}_{\Bbbk} \times_{\Bbbk} \cdots \times_{\Bbbk} \mathbb{P}^{m_p}_{\Bbbk}$.

Let M be a finitely generated $\mathbb{Z}^p$-graded T-module. A homogeneous element is said to be *filter-regular on* M (with respect to $\mathfrak{N}$; see [53, Appendix]) if $z \notin \mathfrak{P}$ for all associated primes $\mathfrak{P} \in \mathrm{Ass}_T(M)$ of M such that $\mathfrak{P} \not\supseteq \mathfrak{N}$. In terms of the multiprojective scheme X, this means that $zT_{\mathfrak{P}}$ is a nonzerodivisor on $M_{\mathfrak{P}}$ for all $\mathfrak{P} \in X$. A sequence of homogeneous elements $z_1, \ldots, z_m$ in T is said to be *filter-regular on* M (with respect to $\mathfrak{N}$) if z_j is a filter-regular element on $M/\left(z_1, \ldots, z_{j-1}\right)M$ for all $1 \leqslant j \leqslant m$. The relevant support of M is given by $\mathrm{Supp}_{++}(M) := \mathrm{Supp}(M) \cap X$. There is a polynomial $P_M(\mathbf{t}) = P_M(t_1, \ldots, t_p) \in \mathbb{Q}[\mathbf{t}] = \mathbb{Q}[t_1, \ldots, t_p]$, called the *Hilbert polynomial* of M (see, e.g., [31, Theorem 4.1], [38, §4]), such that the degree of P_M is equal to $r = \dim\left(\mathrm{Supp}_{++}(M)\right)$ and

$$P_M(\nu) = \dim_{\Bbbk}\left([M]_\nu\right)$$

for all $v \in \mathbb{Z}^p$ such that $v \gg \mathbf{0}$. Furthermore, if we write

$$P_M(\mathbf{t}) = \sum_{n_1,\ldots,n_p \geqslant 0} e(n_1,\ldots,n_p)\binom{t_1+n_1}{n_1}\cdots\binom{t_p+n_p}{n_p},$$

then $e(n_1,\ldots,n_p) \in \mathbb{Z}_{\geqslant 0}$ for all $n_1+\cdots+n_p = r$. From this, we obtain the following invariants:

Definition 2.1. (i) For $\mathbf{n} = (n_1,\ldots,n_p) \in \mathbb{Z}^p_{\geqslant 0}$ with $|\mathbf{n}| = \dim\left(\mathrm{Supp}_{++}(M)\right)$, $e(\mathbf{n};M) := e(n_1,\ldots,n_p)$ is the *mixed multiplicity of* M *of type* $\mathbf{n}$.

(ii) For $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$ with $|\mathbf{n}| = \dim(X)$, $\deg^{\mathbf{n}}_{\mathbb{P}}(X) := e(\mathbf{n};T)$ is the *multidegree of* $X = \mathrm{MultiProj}(T) \subset \mathbb{P}$ *of type* $\mathbf{n}$ *with respect to* $\mathbb{P}$.

We recall the following basic concepts related to rational maps.

Definition 2.2. Let $\Psi : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ be a rational map, $Y \subset \mathbb{P}^s_{\Bbbk}$ be the closure of the image of Ψ, and $\Gamma \subset \mathbb{P}^r_{\Bbbk} \times_{\Bbbk} \mathbb{P}^s_{\Bbbk}$ be the closure of the graph of Ψ. The rational map Ψ is *generically finite* if one of the following equivalent conditions is satisfied:

(i) The field extension $K(Y) \hookrightarrow K(\mathbb{P}^r_{\Bbbk})$ is finite, where $K(\mathbb{P}^r_{\Bbbk})$ and $K(Y)$ denote the fields of rational functions of $\mathbb{P}^r_{\Bbbk}$ and Y, respectively.

(ii) $\dim(Y) = \dim(\mathbb{P}^r_{\Bbbk}) = r$.

The *degree* of Ψ is defined as $\deg(\Psi) := \left[K(\mathbb{P}^r_{\Bbbk}) : K(Y)\right]$ when Ψ is generically finite. Otherwise, by convention, we set $\deg(\Psi) := 0$. For each $0 \leqslant i \leqslant r$, the i-th *projective degree* of $\Psi : \mathbb{P}^r_{\Bbbk} \dashrightarrow \mathbb{P}^s_{\Bbbk}$ is given by

$$d_i(\Psi) \;:=\; \deg^{i,r-i}_{\mathbb{P}^r_{\Bbbk}\times_{\Bbbk}\mathbb{P}^s_{\Bbbk}}(\Gamma).$$

For more details on projective degrees, the reader is referred to [29, Example 19.4] and [17, §7.1.3]. Of particular interest is the 0-th projective degree $d_0(\Psi)$ as it is equal to the product of the degree of the map times the degree of the image

$$d_0(\Psi) \;=\; \deg(\Psi)\cdot \deg_{\mathbb{P}^s_{\Bbbk}}(Y) \tag{1}$$

(e.g., this follows from [10, Theorem 5.4] and [6, Theorem 2.4]).

When we work with families of ideals, we shall use the following notation.

Notation 2.3. Let A be a ring and R be an A-algebra. For any prime $\mathfrak{p} \in \mathrm{Spec}(A)$, let $\kappa(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be the residue field of $\mathfrak{p}$, and set $R_{\mathfrak{p}} := R \otimes_A A_{\mathfrak{p}}$, $R(\mathfrak{p}) := R \otimes_A \kappa(\mathfrak{p})$ and $I(\mathfrak{p}) := IR(\mathfrak{p}) \subset R(\mathfrak{p})$ for any ideal $I \subset R$.

The notion general element will be quite useful in our treatment, thus we recall the following definition.

Definition 2.4. Let R be a Noetherian local ring infinite residue field κ. Let $I \subset R$ be a proper ideal generated by elements $f_1,\ldots,f_m \in R$.

– We say that a property $\mathscr{P}$ holds for a *general element* $g \in I$ if there exists a dense Zariski-open subset $U \subset \kappa^e$ such that whenever $g = a_1f_1+\cdots+a_mf_m$ and the image of $(a_1,...,a_m)$ belongs to U, then the property $\mathscr{P}$ holds for g.

– We say that $\underline{g} = g_1, \ldots, g_k$ is a *sequence of general elements* in I if the image of g_i in the ideal $I/(g_1, \ldots, g_{i-1}) \subset R/(g_1, \ldots, g_{i-1})$ is a general element for all $1 \leqslant i \leqslant k$. We also say that $\underline{g} = g_1, \ldots, g_k$ are *sequentially general elements* in I.

Given a multihomogeneous ideal $\mathfrak{J} \subset T$, we say that an element $z \in T$ is general in $\mathfrak{J}$ if its image is general in the localization $\mathfrak{J}T_{\mathfrak{M}}$ where $\mathfrak{M} := ([T]_{\mathbf{e}_1}) + \cdots + ([T]_{\mathbf{e}_p})$.

3. The behavior of mixed multiplicities

In this section, we study the behavior of mixed multiplicities under the processes of taking fibers and performing specializations, both with respect to a base ring. This section revisits and generalizes some results from [16, Chapter 5] and [9, 13]. We fix the following setup throughout this section.

Setup 3.1. Let A be a Noetherian ring and $\mathscr{T}$ be a finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded algebra over A. Let $\mathscr{X} = \text{MultiProj}(\mathscr{T})$ be the corresponding multiprojective scheme.

Given a topological space Z and a totally ordered set $(\mathfrak{S}, <)$, we say that a function $f : Z \to \mathfrak{S}$ is *upper semicontinuous* if $\{z \in Z \mid f(z) \geqslant s\}$ is a closed subset of Z for each $s \in \mathfrak{S}$; on the other hand, a function $f : Z \to \mathfrak{S}$ is said to be *lower semicontinuous* if $\{z \in Z \mid f(z) \leqslant s\}$ is a closed subset of Z for each $s \in \mathfrak{S}$.

An important basic tool in this paper is the topological Nagata criterion (see, e.g., [44, Theorem 24.2]).

Remark 3.2 (topological Nagata criterion for openness)**.** A subset $U \subset \text{Spec}(A)$ is open if and only if the following two conditions are satisfied:

(i) If $\mathfrak{q} \in U$, then U contains a nonempty open subset of $V(\mathfrak{q}) \subset \text{Spec}(A)$.
(ii) If $\mathfrak{p}, \mathfrak{q} \in \text{Spec}(A)$, $\mathfrak{p} \in U$ and $\mathfrak{p} \supseteq \mathfrak{q}$, then $\mathfrak{q} \in U$.

Given a finitely generated $\mathbb{Z}^p$-graded $\mathscr{T}$-module $\mathscr{M}$, we seek to study the behavior of mixed multiplicities $e(\mathbf{n}; \bullet)$ when considering the family of modules $\mathscr{M} \otimes_A \kappa(\mathfrak{p})$ with $\mathfrak{p} \in \text{Spec}(A)$. Notice that $\mathscr{M} \otimes_A \kappa(\mathfrak{p})$ is a finitely generated $\mathbb{Z}^p$-graded $\mathscr{T}(\mathfrak{p})$-module and that $\mathscr{T}(\mathfrak{p})$ is a finitely generated standard $\mathbb{Z}^p_{\geqslant 0}$-graded algebra over the field $\kappa(\mathfrak{p})$.

Definition 3.3. We consider the functions

$$d^{\mathscr{M}}_{++} : \text{Spec}(A) \to \mathbb{Z}, \quad \mathfrak{p} \mapsto \dim\left(\text{Supp}_{++}(\mathscr{M} \otimes_A \kappa(\mathfrak{p}))\right)$$

and

$$e^{\mathscr{M}}_{\mathbf{n}} : \text{Spec}(A) \to \mathbb{Z} \cup \{\infty\}, \qquad \mathfrak{p} \mapsto \begin{cases} e(\mathbf{n}; \mathscr{M} \otimes_A \kappa(\mathfrak{p})) & \text{if } |\mathbf{n}| = d^{\mathscr{M}}_{++}(\mathfrak{p}) \\ 0 & \text{if } |\mathbf{n}| > d^{\mathscr{M}}_{++}(\mathfrak{p}) \\ \infty & \text{if } |\mathbf{n}| < d^{\mathscr{M}}_{++}(\mathfrak{p}) \end{cases}$$

for every $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$. We use the natural ordering on the set $\mathbb{Z} \cup \{\infty\}$.

The following result extends [16, Theorem 5.13] to a multigraded setting.

Theorem 3.4. *Assume Setup 3.1. Let $\mathscr{M}$ be a finitely generated $\mathbb{Z}^p$-graded $\mathscr{T}$-module. Then the following statements hold:*

(i) $d^{\mathscr{M}}_{++} : \text{Spec}(A) \to \mathbb{Z}$ *is an upper semicontinuous function.*
(ii) $e^{\mathscr{M}}_{\mathbf{n}} : \text{Spec}(A) \to \mathbb{Z} \cup \{\infty\}$ *is an upper semicontinuous function for every* $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$.

Proof. We prove both statements by utilizing the topological Nagata criterion (see Remark 3.2) and Grothendieck's Generic Freeness Lemma (see, e.g., [44, Theorem 24.1], [19, Theorem 14.4]).

Fix elements $d \in \mathbb{Z}$, $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbb{Z}^p_{\geqslant 0}$ and $e \in \mathbb{Z} \cup \{\infty\}$. We need to show that

$$U_d := \left\{\mathfrak{p} \in \operatorname{Spec}(A) \mid d^{\mathscr{M}}_{++}(\mathfrak{p}) \leqslant d\right\} \quad \text{and} \quad V_{\mathbf{n},e} := \left\{\mathfrak{p} \in \operatorname{Spec}(A) \mid e^{\mathscr{M}}_{\mathbf{n}}(\mathfrak{p}) \leqslant e\right\}$$

are open subsets of $\operatorname{Spec}(A)$.

First, we verify condition (i) of Remark 3.2 for both subsets U_d and $V_{\mathbf{n},e}$. Let $\mathfrak{q} \in \operatorname{Spec}(A)$ and $\overline{A} := A/\mathfrak{q}$. The Generic Freeness Lemma applied to the module $\overline{\mathscr{M}} := \mathscr{M}/\mathfrak{q}\mathscr{M}$ gives a nonzero element $0 \neq a \in \overline{A}$ such that each graded component of $\overline{\mathscr{M}}_a$ is a finitely generated free $\overline{A}_a$-module. It follows that $P_{\mathscr{M} \otimes_A \kappa(\mathfrak{p})} = P_{\mathscr{M} \otimes_A \kappa(\mathfrak{q})}$ for every $\mathfrak{p} \in D(a) \subset V(\mathfrak{q}) \subset \operatorname{Spec}(A)$, which verifies the validity of condition (i) of Remark 3.2 for both U_d and $V_{\mathbf{n},e}$.

Next, we show that condition (ii) of Remark 3.2 also holds. Due to Nakayama's lemma, for any two primes $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$, we have that

$$\dim_{\kappa(\mathfrak{p})}\left([\mathscr{M} \otimes_A \kappa(\mathfrak{p})]_\nu\right) = \mu_{A_\mathfrak{p}}\left([\mathscr{M} \otimes_A A_\mathfrak{p}]_\nu\right) \geqslant \mu_{A_\mathfrak{q}}\left([\mathscr{M} \otimes_A A_\mathfrak{q}]_\nu\right) = \dim_{\kappa(\mathfrak{q})}\left([\mathscr{M} \otimes_A \kappa(\mathfrak{q})]_\nu\right)$$

for all $\nu \in \mathbb{Z}^p$. The dimension of the relevant support equals the degree of the Hilbert polynomial, and the latter can be read-off from the Hilbert function. For any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$, it follows that $P_{\mathscr{M} \otimes_A \kappa(\mathfrak{p})}(\nu) \geqslant P_{\mathscr{M} \otimes_A \kappa(\mathfrak{q})}(\nu)$ for all $\nu \gg \mathbf{0}$, and so $d^{\mathscr{M}}_{++}(\mathfrak{p}) \geqslant d^{\mathscr{M}}_{++}(\mathfrak{q})$. This shows that condition (ii) of Remark 3.2 is satisfied for the subset U_d.

Given two primes $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$ and $d^{\mathscr{M}}_{++}(\mathfrak{p}) > d^{\mathscr{M}}_{++}(\mathfrak{q})$, we easily check from the definition of the function $e^{\mathscr{M}}_{\mathbf{n}}$ that $e^{\mathscr{M}}_{\mathbf{n}}(\mathfrak{p}) \geqslant e^{\mathscr{M}}_{\mathbf{n}}(\mathfrak{q})$.

Next, consider the case $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$ and $d^{\mathscr{M}}_{++}(\mathfrak{p}) = d^{\mathscr{M}}_{++}(\mathfrak{q})$. (That $P_{\mathscr{M} \otimes_A \kappa(\mathfrak{p})}(\nu) - P_{\mathscr{M} \otimes_A \kappa(\mathfrak{q})}(\nu) \geqslant 0$ for all $\nu \gg \mathbf{0}$ does not necessarily imply that the coefficients of the monomials of highest degree of $P_{\mathscr{M} \otimes_A \kappa(\mathfrak{p})}$ are bigger or equal than the ones of $P_{\mathscr{M} \otimes_A \kappa(\mathfrak{q})}$; for instance, $f(x,y) = (x-y)^2 = x^2 - 2xy + y^2 \in \mathbb{Q}[x,y]$.)

Let $r := |\mathbf{n}|$. We only need to consider the case where $r = d^{\mathscr{M}}_{++}(\mathfrak{p}) = d^{\mathscr{M}}_{++}(\mathfrak{q})$. Notice that we can reduce modulo $\mathfrak{q}$ and localize at $\mathfrak{p}$. Hence we assume A is a local domain with maximal ideal $\mathfrak{p}$ and $\mathfrak{q} = 0$. By utilizing the faithfully flat extension $A \to A[y]_{\mathfrak{p}A[y]}$, we may assume that the residue field of A is not an algebraic extension of a finite field. From [48, Lemma 2.6], for any multihomogeneous ideal $\mathfrak{J} \subset \mathscr{T}$, we can choose an element $z \in \mathfrak{J}$ whose image is general in both $\mathfrak{J}\mathscr{T}(\mathfrak{p})$ and $\mathfrak{J}\mathscr{T}(\mathfrak{q})$. Therefore, by prime avoidance, there exists a sequence of homogeneous elements $z_1, \ldots, z_r$ in $\mathscr{T}$ such that the following three conditions are satisfied:

(1) each $z_j \in \mathscr{T}$ has degree $\deg(z_j) = \mathbf{e}_{l_j} \in \mathbb{Z}^p_{\geqslant 0}$ where $1 \leqslant l_j \leqslant p$;
(2) n_i equals the number $|\{j \mid 1 \leqslant j \leqslant r \text{ and } l_j = i\}|$;
(3) $\{z_1, \ldots, z_r\}\mathscr{T}(\mathfrak{p})$ and $\{z_1, \ldots, z_r\}\mathscr{T}(\mathfrak{q})$ are filter-regular sequences on the modules $\mathscr{M} \otimes_A \kappa(\mathfrak{p})$ and $\mathscr{M} \otimes_A \kappa(\mathfrak{q})$, respectively.

To simplify notation, let $\mathscr{M}_{j,\mathfrak{p}} := \mathscr{M}/(z_1, \ldots, z_j)\mathscr{M} \otimes_A \kappa(\mathfrak{p})$. By successively applying [10, Lemma 3.9], we obtain

$$e(\mathbf{n}; \mathscr{M} \otimes_A \kappa(\mathfrak{p})) = e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{p}}) \quad \text{and} \quad e(\mathbf{n}; \mathscr{M} \otimes_A \kappa(\mathfrak{q})) = e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{q}}).$$

We choose $\mathbf{0} \ll \nu \in \mathbb{Z}^p_{\geqslant 0}$ with the property that $e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})}\left([\mathscr{M}_{r,\mathfrak{p}}]_\nu\right)$ and $e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{q}}) = \dim_{\kappa(\mathfrak{q})}\left([\mathscr{M}_{r,\mathfrak{q}}]_\nu\right)$. Finally, Nakayama's lemma yields that

$$\begin{aligned} e(\mathbf{n}; \mathscr{M} \otimes_A \kappa(\mathfrak{p})) &= e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})}\left([\mathscr{M}_{r,\mathfrak{p}}]_\nu\right) = \mu_{A_\mathfrak{p}}\left([\mathscr{M}/(z_1,\ldots,z_r)\mathscr{M} \otimes_A A_\mathfrak{p}]_\nu\right) \\ &\geqslant \mu_{A_\mathfrak{q}}\left([\mathscr{M}/(z_1,\ldots,z_r)\mathscr{M} \otimes_A A_\mathfrak{q}]_\nu\right) = \dim_{\kappa(\mathfrak{q})}\left([\mathscr{M}_{r,\mathfrak{q}}]_\nu\right) = e(\mathbf{0}; \mathscr{M}_{r,\mathfrak{q}}) = e(\mathbf{n}; \mathscr{M} \otimes_A \kappa(\mathfrak{q})). \end{aligned}$$

Therefore, the subset $V_{\mathbf{n},e}$ satisfies condition (ii) of Remark 3.2. This completes the proof of the theorem. □

We restate the above theorem for the case of multidegrees. As before, we embed $\mathscr{X}$ as a closed subscheme of a multiprojective space $\mathbb{P}_A := \mathbb{P}^{m_1}_A \times_A \cdots \times_A \mathbb{P}^{m_p}_A$. We seek to study the multidegrees of the fibers $\mathscr{X}_\mathfrak{p} := \mathscr{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa(\mathfrak{p})) = \mathrm{MultiProj}(\mathscr{T}(\mathfrak{p})) \subset \mathbb{P}_\mathfrak{p} := \mathbb{P}_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa(\mathfrak{p}))$.

Definition 3.5. We define the functions

$$d^{\mathscr{X}} : \mathrm{Spec}(A) \to \mathbb{Z}, \quad \mathfrak{p} \mapsto \dim(\mathscr{X}_\mathfrak{p})$$

and

$$\deg^{\mathbf{n}}_{\mathscr{X},\mathbb{P}_A} : \mathrm{Spec}(A) \to \mathbb{Z} \cup \{\infty\}, \qquad \mathfrak{p} \mapsto \begin{cases} \deg^{\mathbf{n}}_{\mathbb{P}_\mathfrak{p}}(\mathscr{X}_\mathfrak{p}) & \text{if } |\mathbf{n}| = \dim(\mathscr{X}_\mathfrak{p}) \\ 0 & \text{if } |\mathbf{n}| > \dim(\mathscr{X}_\mathfrak{p}) \\ \infty & \text{if } |\mathbf{n}| < \dim(\mathscr{X}_\mathfrak{p}) \end{cases}$$

for every $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$.

We have the following direct consequence of Theorem 3.4.

Corollary 3.6. *Assume Setup 3.1. Then the following statements hold:*

(i) $d^{\mathscr{X}} : \mathrm{Spec}(A) \to \mathbb{Z}$ *is an upper semicontinuous function.*
(ii) $\deg^{\mathbf{n}}_{\mathscr{X},\mathbb{P}_A} : \mathrm{Spec}(A) \to \mathbb{Z} \cup \{\infty\}$ *is an upper semicontinuous function for every* $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$.

We quickly revisit the notion of specialization of a module. Here we follow the same setting and notations as in [9]. Let $\mathscr{M}$ be a finitely generated torsionless $\mathbb{Z}^p$-graded $\mathscr{T}$-module, with a fixed injection $\iota : \mathscr{M} \hookrightarrow \mathscr{F}$ into a free $\mathbb{Z}^p$-graded $\mathscr{T}$-module of finite rank. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the *specialization of* $\mathscr{M}$ *with respect to* $\mathfrak{p}$ is defined as

$$\mathbb{S}_\mathfrak{p}(\mathscr{M}) := \mathrm{Im}\Big(\iota \otimes_A \kappa(\mathfrak{p}) : \mathscr{M} \otimes_A \kappa(\mathfrak{p}) \to \mathscr{F} \otimes_A \kappa(\mathfrak{p})\Big).$$

We now consider the following function

$$\mathbb{S}e^{\mathscr{M}}_{\mathbf{n}} : \mathrm{Spec}(A) \to \mathbb{Z}, \qquad \mathfrak{p} \mapsto e(\mathbf{n}; \mathbb{S}_\mathfrak{p}(\mathscr{M}))$$

for every $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$ with $|\mathbf{n}| \geqslant \dim\left(\mathrm{Supp}_{++}(\mathbb{S}_\mathfrak{p}(\mathscr{M}))\right)$.

The following result deals with the behavior of mixed multiplicities with respect to specializations.

Corollary 3.7. *Assume Setup 3.1 and that each graded component of* $\mathscr{T}$ *is a free* A*-module. Let* $\mathscr{M}$ *be a finitely generated torsionless* $\mathbb{Z}^p$*-graded* $\mathscr{T}$*-module, with a fixed injection* $\iota : \mathscr{M} \hookrightarrow \mathscr{F}$ *into a free* $\mathbb{Z}^p$*-graded* $\mathscr{T}$*-module of finite rank. Let* r *be the common dimension* $\dim(\mathrm{Supp}_{++}(\mathscr{F} \otimes_A \kappa(\mathfrak{p})))$ *for all* $\mathfrak{p} \in \mathrm{Spec}(A)$*. Then the function*

$$\mathbb{S}e^{\mathscr{M}}_{\mathbf{n}} : \mathrm{Spec}(A) \to \mathbb{Z}$$

is lower semicontinuous for every $\mathbf{n} \in \mathbb{Z}^p_{\geqslant 0}$ *with* $|\mathbf{n}| \geqslant r$.

Proof. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, we have the short exact sequence

$$0 \to \mathbb{S}_{\mathfrak{p}}(\mathscr{M}) \to \mathscr{F} \otimes_A \kappa(\mathfrak{p}) \to \mathscr{F}/\mathscr{M} \otimes_A \kappa(\mathfrak{p}) \to 0.$$

From the additivity of mixed multiplicities, we get $e_{\mathbf{n}}^{\mathscr{F}}(\mathfrak{p}) = e_{\mathbf{n}}^{\mathscr{F}/\mathscr{M}}(\mathfrak{p}) + \mathbb{S}e_{\mathbf{n}}^{\mathscr{M}}(\mathfrak{p})$. Since $e_{\mathbf{n}}^{\mathscr{F}}$ is a constant function by assumption and $e_{\mathbf{n}}^{\mathscr{F}/\mathscr{M}}$ is upper semicontinuous by Theorem 3.4, the result of the corollary follows. □

4. Rational maps and their specializations

Here we concentrate on a specialization process of rational maps. The next setup is now in place.

Setup 4.1. Let $r < s$ be two positive integers. Let A be a Noetherian domain, $S = A[x_0, \ldots, x_r]$ be a standard graded polynomial ring, $\mathbb{P}_A^r = \mathrm{Proj}(S)$, and $\mathfrak{m} = (x_0, \ldots, x_r) \subset S$ be the graded irrelevant ideal. Let $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^s$ be a rational map with representative $\mathbf{f} = (f_0 : \cdots : f_s)$ such that $\{f_0, \ldots, f_s\} \subset S$ are homogeneous elements of degree $\delta > 0$.

We specialize this rational map as follows. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, we get the rational map

$$\mathcal{F}(\mathfrak{p}) : \mathbb{P}_{\kappa(\mathfrak{p})}^r \dashrightarrow \mathbb{P}_{\kappa(\mathfrak{p})}^s$$

with representative $\pi_{\mathfrak{p}}(\mathbf{f}) = (\pi_{\mathfrak{p}}(f_0) : \cdots : \pi_{\mathfrak{p}}(f_s))$ where $\pi_{\mathfrak{p}}(f_i)$ is the image of f_i under the natural map $\pi_{\mathfrak{p}} : S \to S(\mathfrak{p})$.

Let $I = (f_0, \ldots, f_s) \subset S$ be the base ideal of the rational map $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^s$. The closure of the graph of $\mathcal{F}$ is given as $\Gamma = \mathrm{BiProj}(\mathscr{R}(I)) \subset \mathbb{P}_A^r \times_A \mathbb{P}_A^s$ where $\mathscr{R}(I) := \bigoplus_{n=0}^{\infty} I^n T^n \subset S[T]$ is the Rees algebra of I. As customary, $\mathscr{R}(I)$ is presented as a quotient of a standard bigraded polynomial ring $\mathscr{T} := S \otimes_A A[y_0, \ldots, y_s]$ by using the A-algebra homomorphism

$$\mathscr{T} \twoheadrightarrow \mathscr{R}(I), \quad x_i \mapsto x_i,\ y_j \mapsto f_j t.$$

We have the equalities $I(\mathfrak{p}) = (\pi_{\mathfrak{p}}(f_0), \ldots, \pi_{\mathfrak{p}}(f_s)) \subset S(\mathfrak{p})$ and $(I^k)(\mathfrak{p}) = I(\mathfrak{p})^k \subset S(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ and $k \geqslant 0$. For any $\mathfrak{p} \in \mathrm{Spec}(A)$, let $\Gamma(\mathfrak{p}) \subset \mathbb{P}_{\kappa(\mathfrak{p})}^r \times_{\kappa(\mathfrak{p})} \mathbb{P}_{\kappa(\mathfrak{p})}^s$ and $Y(\mathfrak{p}) \subset \mathbb{P}_{\kappa(\mathfrak{p})}^s$ be the closures of the graph and the image of the rational map $\mathcal{F}(\mathfrak{p})$. We have that $\dim(\Gamma(\mathfrak{p})) \leqslant r$ and $\dim(Y(\mathfrak{p})) \leqslant r$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. We also consider the *j-multiplicity* of ideal $I(\mathfrak{p}) \subset S(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ (see [1], [21, §6.1]).

Definition 4.2. We have the following three functions: $j^I : \mathrm{Spec}(A) \to \mathbb{Z}$, $\mathfrak{p} \mapsto j(I(\mathfrak{p}))$,

$$\deg\mathrm{Im}^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}, \qquad \mathfrak{p} \mapsto \begin{cases} \deg_{\mathbb{P}_{\kappa(\mathfrak{p})}^s}(Y(\mathfrak{p})) & \text{if } \dim(Y(\mathfrak{p})) = r \\ 0 & \text{if } \dim(Y(\mathfrak{p})) < r \end{cases}$$

and $d_i^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}$, $d_i(\mathcal{F}(\mathfrak{p}))$ for all $0 \leqslant i \leqslant r$.

Our main result regarding these functions is the following theorem.

Theorem 4.3. *Assume Setup 4.1. Then the following statements hold:*

(i) $\deg\mathrm{Im}^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function.*
(ii) $d_i^{\mathcal{F}} : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function for all* $0 \leqslant i \leqslant r$.
(iii) $j^I : \mathrm{Spec}(A) \to \mathbb{Z}$ *is a lower semicontinuous function.*

Proof. By [42, Theorem 5.3], we have that $j^I(\mathfrak{p}) = \delta \cdot d_0^{\mathcal{F}}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Thus, we only need to prove parts (i) and (ii) of the theorem.

Fix $0 \leqslant i \leqslant r$, $e \in \mathbb{Z}$ and $h \in \mathbb{Z}$. It remains to show that

$$D_e := \left\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{degIm}^{\mathcal{F}}(\mathfrak{p}) \geqslant e\right\} \quad \text{and} \quad E_{i,h} := \left\{\mathfrak{p} \in \mathrm{Spec}(A) \mid d_i^{\mathcal{F}}(\mathfrak{p}) \geqslant h\right\}$$

are open subsets of $\mathrm{Spec}(A)$. Again, to prove this we utilize a combination of the topological Nagata criterion and the Generic Freeness Lemma.

First, we verify condition (i) of Remark 3.2 for both D_e and $E_{i,h}$. Let $\mathfrak{q} \in \mathrm{Spec}(A)$, and set $\overline{A} = A/\mathfrak{q}$, $\overline{S} = S/\mathfrak{q}S$ and $\overline{I} = I\overline{S} \subset \overline{S}$. We use the version of the Generic Freeness Lemma given in [32, Lemma 8.1] applied to the inclusion of algebras $\mathscr{R}_{\overline{S}}(\overline{I}) \hookrightarrow \overline{S}[t]$, and we find a nonzero element $0 \neq a \in \overline{A}$ such that each graded component of $\overline{S}[t]/\mathscr{R}_{\overline{S}}(\overline{I}) \otimes_{\overline{A}} \overline{A}_a$ is a finitely generated free $\overline{A}_a$-module (also, see [9, Theorem 3.5]). For every $\mathfrak{p} \in D(a) \subset V(\mathfrak{q}) \subset \mathrm{Spec}(A)$, we obtain that $\dim_{\kappa(\mathfrak{p})}([I(\mathfrak{p})^n]_\nu) = \dim_{\kappa(\mathfrak{q})}([I(\mathfrak{q})^n]_\nu)$ for all $n \geqslant 0$ and $\nu \in \mathbb{Z}$. Accordingly, condition (i) of Remark 3.2 holds for both D_e and $E_{i,h}$.

Next, we show that condition (ii) of Remark 3.2 also holds for D_e and $E_{i,h}$.

For any $\mathfrak{p} \in \mathrm{Spec}(A)$, we have that $\mathrm{degIm}^{\mathcal{F}}(\mathfrak{p}) = e_{r+1}(L_\mathfrak{p})$ and $d_i^{\mathcal{F}}(\mathfrak{p}) = e\left(i, r-i; \mathscr{R}_{S(\mathfrak{p})}(I(\mathfrak{p}))\right)$, where $L_\mathfrak{p} := \kappa(\mathfrak{p})[\pi_\mathfrak{p}(f_0), \ldots, \pi_\mathfrak{p}(f_s)] = \bigoplus_{n=0}^{\infty} [I(\mathfrak{p})^n]_{n\delta}$. For each $n \geqslant 0$, we have a short exact sequence

$$0 \to I(\mathfrak{p})^n \to S(\mathfrak{p}) \to S/I^n \otimes_A \kappa(\mathfrak{p}) \to 0.$$

Since we know that $\dim_{\kappa(\mathfrak{p})}([S/I^n \otimes_A \kappa(\mathfrak{p})]_\nu) \geqslant \dim_{\kappa(\mathfrak{q})}([S/I^n \otimes_A \kappa(\mathfrak{q})]_\nu)$ for all $n \geqslant 0$, $\nu \in \mathbb{Z}$ and $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$, it follows that D_e satisfies condition (ii) of Remark 3.2.

Fix two primes $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ with $\mathfrak{p} \supseteq \mathfrak{q}$. As in the proof of Theorem 3.4, we may assume that $\mathfrak{q} = 0$ and that A is a local domain with maximal ideal $\mathfrak{p}$. Moreover, by making a purely transcendental field extension, we may assume that for any multihomogeneous ideal $\mathfrak{J} \subset \mathscr{T}$, we can choose an element $z \in \mathfrak{J}$ whose image is general in both $\mathfrak{J}\mathscr{T}(\mathfrak{p})$ and $\mathfrak{J}\mathscr{T}(\mathfrak{q})$ (see [48, Lemma 2.6]).

By applying [10, Proposition 5.6], we find a sequence $\{z_1, \ldots, z_i\} \subset \mathfrak{m} \subset S \subset \mathscr{T}$ of homogeneous elements of degree one such that

$$d_i^{\mathcal{F}}(\mathfrak{p}) = e\left(0, r-i; \mathscr{R}_{R_\mathfrak{p}}(J_\mathfrak{p})\right) \quad \text{and} \quad d_i^{\mathcal{F}}(\mathfrak{q}) = e\left(0, r-i; \mathscr{R}_{R_\mathfrak{q}}(J_\mathfrak{q})\right)$$

where $R_\mathfrak{p} := S(\mathfrak{p})/(z_1, \ldots, z_i)S(\mathfrak{p})$, $R_\mathfrak{q} := S(\mathfrak{q})/(z_1, \ldots, z_i)S(\mathfrak{q})$, $J_\mathfrak{p} := IR_\mathfrak{p} \subset R_\mathfrak{p}$ and $J_\mathfrak{q} := IR_\mathfrak{q} \subset R_\mathfrak{q}$. Since $i \leqslant r$, we may further assume that $\{z_1, \ldots, z_i\}S(\mathfrak{p})$ and $\{z_1, \ldots, z_i\}S(\mathfrak{q})$ are regular sequences on the polynomial rings $S(\mathfrak{p})$ and $S(\mathfrak{q})$, respectively.

There exists a positive integer $m > 0$ such that

$$e\left(0, r-i; \mathscr{R}_{R_\mathfrak{p}}(J_\mathfrak{p})\right) = \lim_{n \to \infty} \frac{\dim_{\kappa(\mathfrak{p})}\left(\left[\mathscr{R}_{R_\mathfrak{p}}(J_\mathfrak{p})\right]_{(m,n)}\right)}{n^{r-i}/(r-i)!} = \lim_{n \to \infty} \frac{\dim_{\kappa(\mathfrak{p})}\left(\left[J_\mathfrak{p}^n\right]_{m+n\delta}\right)}{n^{r-i}/(r-i)!}$$

and

$$e\left(0, r-i; \mathscr{R}_{R_\mathfrak{q}}(J_\mathfrak{q})\right) = \lim_{n \to \infty} \frac{\dim_{\kappa(\mathfrak{q})}\left(\left[\mathscr{R}_{R_\mathfrak{q}}(J_\mathfrak{q})\right]_{(m,n)}\right)}{n^{r-i}/(r-i)!} = \lim_{n \to \infty} \frac{\dim_{\kappa(\mathfrak{q})}\left(\left[J_\mathfrak{q}^n\right]_{m+n\delta}\right)}{n^{r-i}/(r-i)!}.$$

So, to show that $d_i^{\mathcal{F}}(\mathfrak{p}) \leqslant d_i^{\mathcal{F}}(\mathfrak{q})$, it suffices to verify that $\dim_{\kappa(\mathfrak{p})}\left(\left[J_\mathfrak{p}^n\right]_\nu\right) \leqslant \dim_{\kappa(\mathfrak{q})}\left(\left[J_\mathfrak{q}^n\right]_\nu\right)$ for all $n \geqslant 0$ and $\nu \in \mathbb{Z}$. Notice that the Hilbert functions of $R_\mathfrak{p}$ and $R_\mathfrak{q}$ are equal. Then, from Nakayama's lemma we get

$$\begin{aligned}\dim_{\kappa(\mathfrak{p})}\left(\left[R_{\mathfrak{p}}/J_{\mathfrak{p}}^n\right]_\nu\right) &= \mu_{A_{\mathfrak{p}}}\left(\left[\frac{S}{(z_1,\ldots,z_i,I^n)S}\otimes_A A_{\mathfrak{p}}\right]_\nu\right)\\ &\geqslant \mu_{A_{\mathfrak{q}}}\left(\left[\frac{S}{(z_1,\ldots,z_i,I^n)S}\otimes_A A_{\mathfrak{q}}\right]_\nu\right) = \dim_{\kappa(\mathfrak{q})}\left(\left[R_{\mathfrak{q}}/J_{\mathfrak{q}}^n\right]_\nu\right)\end{aligned}$$

for all $n \geqslant 0$ and $\nu \in \mathbb{Z}$. Finally, this implies that $d_i^{\mathcal{F}}(\mathfrak{p}) \leqslant d_i^{\mathcal{F}}(\mathfrak{q})$, and so it follows that condition (ii) of Remark 3.2 holds for $E_{i,h}$. So, we are done with the proof of the theorem. □

We single out an important corollary of Theorem 4.3. When $r = s$, and we consider a rational map of the form $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^r$, we have some control over the degree of the specialized rational maps $\mathcal{F}(\mathfrak{p}) : \mathbb{P}^r_{\kappa(\mathfrak{p})} \dashrightarrow \mathbb{P}^r_{\kappa(\mathfrak{p})}$.

Corollary 4.4. *Assume Setup 4.1. Let $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^r$ be a rational map. Then the function*

$$\deg^{\mathcal{F}} : \operatorname{Spec}(A) \to \mathbb{Z}, \qquad \mathfrak{p} \mapsto \deg\left(\mathcal{F}(\mathfrak{p})\right)$$

is lower semicontinuous.

Proof. We have that $\deg^{\mathcal{F}}(\mathfrak{p}) = d_0^{\mathcal{F}}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Indeed, if $\dim(Y(\mathfrak{p})) < r$, both $\deg^{\mathcal{F}}(\mathfrak{p})$ and $d_0^{\mathcal{F}}(\mathfrak{p})$ are equal to zero; and if $\dim(Y(\mathfrak{p})) = r$, $\deg_{\mathbb{P}^r_{\kappa(\mathfrak{p})}}(Y(\mathfrak{p})) = 1$ and so $d_0(\mathcal{F}(\mathfrak{p})) = \deg(\mathcal{F}(\mathfrak{p}))$ according to (1). Thus, the result follows from Theorem 4.3. □

We now apply the above results to different families of rational maps. We obtain generalizations of [13, Theorems 6.3, 6.8], and we eliminate the conditions assumed there. The following corollary yields significant upper bounds for the projective degrees of certain families of rational maps. It should be mentioned that these inequalities are sharp for the general members of the considered families (see [10, Theorems 5.7, 5.8]).

Corollary 4.5. *Let $\mathbb{k}$ be a field, $R = \mathbb{k}[x_0,\ldots,x_r]$ be a standard graded polynomial ring, $\mathbb{P}^r_{\mathbb{k}} = \operatorname{Proj}(R)$, $\Psi : \mathbb{P}^r_{\mathbb{k}} \dashrightarrow \mathbb{P}^s_{\mathbb{k}}$ be a rational map with representative $\mathbf{g} = (g_0 : \cdots : g_s)$ and base ideal $J = (g_0,\ldots,g_s) \subset R$, and suppose that $\delta = \deg(g_j) > 0$. Then the following statements hold:*

(i) $d_i(\Psi) \leqslant \delta^{r-i}$ *for all* $0 \leqslant i \leqslant r$.

(ii) *Suppose that J is a perfect ideal of height two with Hilbert-Burch resolution of the form*

$$0 \to \bigoplus_{i=1}^{s} R(-\delta-\mu_i) \to R(-\delta)^{s+1} \to J \to 0.$$

Then, for all $0 \leqslant i \leqslant r$, we have

$$d_i(\Psi) \leqslant e_{r-i}(\mu_1,\ldots,\mu_s)$$

where $e_{r-i}(\mu_1,\ldots,\mu_s)$ denotes the elementary symmetric polynomial

$$e_{r-i}(\mu_1,\ldots,\mu_s) = \sum_{1\leqslant j_1<\cdots<j_{r-i}\leqslant s} \mu_{j_1}\cdots\mu_{j_{r-i}}.$$

In particular, if $r = s$, then $\deg(\Psi) \leqslant \mu_1\cdots\mu_r$.

(iii) *Suppose that* J *is a Gorenstein ideal of height three. Let* $D \geqslant 1$ *be the degree of every nonzero entry of an alternating minimal presentation matrix of* J*. Then, for all* $0 \leqslant i \leqslant r$*, we have*

$$d_i(\Psi) \leqslant \begin{cases} D^{r-i} \sum_{k=0}^{\lfloor \frac{s-r+i}{2} \rfloor} \binom{s-1-2k}{r-i-1} & \text{if } 0 \leqslant i \leqslant r-3 \\ \delta^{r-i} & \text{if } r-2 \leqslant i \leqslant r. \end{cases}$$

In particular, if $r = s$*, then* $\deg(\Psi) \leqslant D^r$.

Proof. (i) For $0 \leqslant j \leqslant s$, consider a set of variables $\mathbf{z}_j = \{z_{j,1}, \ldots, z_{j,m}\}$ over $\Bbbk$ with $m = \binom{\delta+r}{r}$, and set $\mathbf{z} = \mathbf{z}_0 \cup \cdots \cup \mathbf{z}_s$. Let $A = \Bbbk[\mathbf{z}]$, $S = A[x_0, \ldots, x_r]$ and consider the generic polynomials

$$G_j := z_{j,1}x_0^\delta + z_{j,2}x_0^{\delta-1}x_1 + \cdots + z_{j,m}x_r^\delta \in S.$$

Let $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^s$ be a rational map with base ideal $I = (G_0, \ldots, G_s) \subset S$. Let $\xi = (0) \subset \operatorname{Spec}(A)$ be the generic point and $\mathfrak{m}_\alpha = \left(\{z_{j,k} - \alpha_{j,k}\}_{j,k}\right) \in \operatorname{Spec}(A)$ be a rational maximal ideal such that $J = I(\mathfrak{m}_\alpha) \subset R$. It is known that the projective degrees of the morphism $\mathcal{F}(\xi) : \mathbb{P}^r_{\Bbbk(\mathbf{z})} \rightarrow \mathbb{P}^s_{\Bbbk(\mathbf{z})}$ (it is base point free as $I \otimes_A \Bbbk(\mathbf{z}) \subset S(\xi) = \Bbbk(\mathbf{z})[x_0, \ldots, x_r]$ is a zero-dimensional ideal) are equal to $d_i(\mathcal{F}(\xi)) = \delta^{r-i}$ (see, e.g., [42, Observation 3.2]). By using Theorem 4.3, we obtain $d_i(\Psi) = d_i^{\mathcal{F}}(\mathfrak{m}_\alpha) \leqslant d_i^{\mathcal{F}}(\xi) = \delta^{r-i}$ for all $0 \leqslant i \leqslant r$.

(ii) For $1 \leqslant j \leqslant s+1$ and $1 \leqslant k \leqslant s$, let $\mathbf{z}_{j,k} = \{z_{j,k,1}, z_{j,k,2}, \ldots, z_{j,k,m_k}\}$ denote a set of variables over $\Bbbk$ of cardinality $m_k = \binom{\mu_k + r}{r}$, and set $\mathbf{z} = \bigcup_{j,k} \mathbf{z}_{j,k}$. Let $A = \Bbbk[\mathbf{z}]$, $S = A[x_0, \ldots, x_r]$ and consider the generic $(s+1) \times s$ Hilbert-Burch matrix

$$\mathcal{M} = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,s} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,s} \\ \vdots & \vdots & & \vdots \\ p_{s+1,1} & p_{s+1,2} & \cdots & p_{s+1,s} \end{pmatrix}$$

where each polynomial $p_{j,k} \in S$ is given by

$$p_{j,k} = z_{j,k,1}x_0^{\mu_k} + z_{j,k,2}x_0^{\mu_k-1}x_1 + \cdots + z_{j,k,m_k}x_r^{\mu_k}.$$

Let $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^s$ be a rational map with base ideal $I = I_s(\mathcal{M}) \subset S$. Let $\varphi \in R^{(s+1)\times s}$ be the Hilbert-Burch presentation of J. Let $\xi = (0) \subset \operatorname{Spec}(A)$ be the generic point and $\mathfrak{m}_\alpha = \left(\{z_{j,k,l} - \alpha_{j,k,l}\}_{j,k,l}\right) \in \operatorname{Spec}(A)$ be a rational maximal ideal such that $\varphi \in R^{(s+1)\times s}$ is obtained by specializing $\mathcal{M} \in S^{(s+1)\times s}$ via the map $S \twoheadrightarrow S/\mathfrak{m}_\alpha \cong R$. From [13, Lemma 6.2] the generic ideal $I \otimes_A \Bbbk(\mathbf{z}) \subset S(\xi) = \Bbbk(\mathbf{z})[x_0, \ldots, x_r]$ is perfect of height two and satisfies the condition G_{r+1}, and so [10, Theorem 5.7] implies that the projective degrees of $\mathcal{F}(\xi) : \mathbb{P}^r_{\Bbbk(\mathbf{z})} \dashrightarrow \mathbb{P}^s_{\Bbbk(\mathbf{z})}$ are equal to $d_i(\mathcal{F}(\xi)) = e_{r-i}(\mu_1, \ldots, \mu_s)$. Finally, Theorem 4.3 yields that $d_i(\Psi) = d_i^{\mathcal{F}}(\mathfrak{m}_\alpha) \leqslant d_i^{\mathcal{F}}(\xi) = e_{r-i}(\mu_1, \ldots, \mu_s)$ for all $0 \leqslant i \leqslant r$.

(iii) This part follows verbatim as part (ii), except that we need to use [12, Lemma 2.12] and [10, Theorem 5.8]. □

From Theorem 4.3, we have that the projective degrees and the degree of the image of a rational map $\mathcal{F} : \mathbb{P}_A^r \dashrightarrow \mathbb{P}_A^s$ behave as lower semicontinuous functions under specialization; accordingly, these invariants cannot increase under specialization. However, it turns out that the degree of a rational map is a much more erratic invariant, and it seems that Corollary 4.4 is the most general result one can hope for. Indeed, the following two examples show that the degree of a rational map can either increase or decrease under specialization.

Example 4.6 (The degree of a rational map can increase)**.** Let $\mathbb{Q}$ be the field of rational numbers, and $A=\mathbb{Q}[a]$ and $S=A[x_0,x_1,x_2]$ be polynomial rings. Consider the following matrix

$$\mathcal{M}=\begin{pmatrix} x_0 & x_1 & x_0^2\\ x_1 & x_0 & x_1^2\\ x_0+a\,x_2 & x_1 & x_2^2\\ 0 & x_0 & 0\end{pmatrix},$$

and let $\mathcal{F}:\mathbb{P}^2_A \dashrightarrow \mathbb{P}^3_A$ be a rational map with base ideal $I=(f_0,f_1,f_2,f_3)\subset S$ given by

$$I=I_3(\mathcal{M})=\begin{pmatrix} x_0^2x_1^2+a\,x_0x_1^2x_2-x_0x_1x_2^2,\\ -x_0^4-a\,x_0^3x_2+x_0^2x_2^2,\\ x_0^3x_1-x_0^2x_1^2,\\ x_0^4-x_0^2x_1^2+a\,x_0^3x_2-a\,x_1^3x_2-x_0^2x_2^2+x_1^2x_2^2\end{pmatrix}.$$

Let $\mathscr{T}=S\otimes_A A[y_0,y_1,y_2,y_3]$ be a standard bigraded polynomial ring over A and write the Rees algebra $\mathscr{R}(I)$ as $\mathscr{R}(I)\cong\mathscr{T}/\mathcal{J}$. The ideal $\mathcal{J}\subset\mathscr{T}$ is equal to

$$\begin{pmatrix} x_1y_0+x_0y_1+x_1y_2+x_0y_3,\\ x_0y_0+x_1y_1+(x_0+a\,x_2)\,y_2,\\ \left(x_0x_1-x_1^2\right)y_1+\left(x_0^2+a\,x_0x_2-x_2^2\right)y_2,\\ x_2y_0^2+(-a\,x_0+a\,x_1-x_2)\,y_1^2+(a^2+2)\,x_2y_0y_2-a\,x_1y_1y_2+x_2y_2^2+(-a\,x_0+a\,x_1-x_2)\,y_1y_3,\\ y_0^4-2\,y_0^2y_1^2+y_1^4+\left(a^2+4\right)y_0^3y_2-4\,y_0y_1^2y_2+a^2y_1^3y_2+\left(2\,a^2+6\right)y_0^2y_2^2-2\,y_1^2y_2^2+\\ \left(a^2+4\right)y_0y_2^3+y_2^4-2\,y_0^2y_1y_3+2\,y_1^3y_3-4\,y_0y_1y_2y_3+2\,a^2y_1^2y_2y_3-2\,y_1y_2^2y_3+y_1^2y_3^2+a^2y_1y_2y_3^2\end{pmatrix}.$$

This can be computed in a computer algebra system like `Macaulay2` [27]. The ideal $\mathcal{J}\subset\mathscr{T}$ is generated by 5 bihomogeneous polynomials in $\mathscr{T}$. The first, second and fourth generators of $\mathcal{J}$ are linear in the variables x_i. Let $\mathbb{L}=\mathbb{Q}(a)=\mathrm{Quot}(A)$ and $\mathbb{G}:\mathbb{P}^2_{\mathbb{L}}\dashrightarrow\mathbb{P}^3_{\mathbb{L}}$ be the generic rational map with base ideal $I\otimes_A\mathbb{L}\subset\mathbb{L}[x_0,x_1,x_2]$. From [18, Theorem 2.18], we can check that $\mathbb{G}$ is birational, i.e., $\deg(\mathbb{G})=1$. Alternatively, we give a short direct argument. We may assume that $\mathbb{L}$ is algebraically closed. By considering the generator $f_2=x_0^3x_1-x_0^2x_1^2=x_0^2x_1(x_0-x_1)$ of I, we obtain the morphism $h:D(f_2)\to\mathbb{P}^3_{\mathbb{L}}$. Notice that $D(f_2)$ lies inside the affine patch $\mathbb{A}^2_{\mathbb{L}}\subset\mathbb{P}^2_{\mathbb{L}}$ with $x_0=1$. For any point $p=(p_0:p_1:p_2:p_3)\in\mathbb{P}^3_{\mathbb{L}}$ in the image of h, if $(1,\alpha_1,\alpha_2)\in h^{-1}(p)$, then we get the following linear system

$$\begin{array}{rcll} p_1\alpha_1 & + & p_2a\alpha_2 & =-(p_0+p_2)\\ (p_0+p_2)\alpha_1 & & & =-(p_1+p_3)\end{array}$$

that is derived from the two linear syzygies in $\mathcal{M}$; this system has a unique solution $(1,\alpha_1,\alpha_2)$ if the determinant $-ap_2(p_0+p_2)$ is non-zero. It follows that for any $\alpha=(1,\alpha_1,\alpha_2)\in D(f_2)\cap D(af_2(f_0+f_2))=D(f_2(f_0+f_2))$, the fiber $h^{-1}(h(\alpha))$ has one element. Thus $\mathbb{G}$ is birational.

On the other hand, we make the specialization $a=0$, which gives the matrix

$$M=\begin{pmatrix} x_0 & x_1 & x_0^2\\ x_1 & x_0 & x_1^2\\ x_0 & x_1 & x_2^2\\ 0 & x_0 & 0\end{pmatrix}.$$

Let $\mathfrak{g} : \mathbb{P}^2_{\Bbbk} \dashrightarrow \mathbb{P}^3_{\Bbbk}$ be a rational map with base ideal

$$J = I_3(M) = \left(-x_0^4 + x_0^2x_1^2 + x_0^2x_2^2 - x_1^2x_2^2,\, x_0^3x_1 - x_0^2x_1^2,\, x_0^4 - x_0^2x_2^2,\, x_0^2x_1^2 - x_0x_1x_2^2\right).$$

In this case $\mathfrak{g}$ is not birational, indeed $\deg(\mathfrak{g}) = 2$.

Therefore, under the above specialization, we obtain $\deg(\mathfrak{g}) = 2 > 1 = \deg(\mathbb{G})$.

Example 4.7 (The degree of a rational map usually decreases)**.** We recall an example [13, Example 6.5] where the degree of a rational map can decrease arbitrarily under specialization. Let $m \geqslant 1$ be an integer. Let $\Bbbk$ be a field, and $A = \Bbbk[a]$ and $S = A[x_0, x_1, x_2]$ be polynomial rings. Consider the matrix

$$\mathcal{M} = \begin{pmatrix} x & zy^{m-1} \\ -y & zx^{m-1} + y^m \\ az & zx^{m-1} \end{pmatrix}$$

with entries in S. Let $\mathcal{F} : \mathbb{P}^2_A \dashrightarrow \mathbb{P}^2_A$ be a rational map with base ideal $I = I_2(\mathcal{M}) \subset S$. Let $\mathbb{L} = \Bbbk(a) = \mathrm{Quot}(A)$ and $\mathbb{G} : \mathbb{P}^2_{\mathbb{L}} \dashrightarrow \mathbb{P}^2_{\mathbb{L}}$ be the generic rational map with base ideal $I \otimes_A \mathbb{L} \subset \mathbb{L}[x_0, x_1, x_2]$. For any $\beta \in \Bbbk$, let $\mathfrak{n}_\beta = (a - \beta) \in A$ and $\mathfrak{g}_\beta : \mathbb{P}^2_{\Bbbk} \dashrightarrow \mathbb{P}^2_{\Bbbk}$ be a rational map with base ideal $I(\mathfrak{n}_\beta) \subset \Bbbk[x_0, x_1, x_2]$. Then, we have that $\deg(\mathbb{G}) = m$ and

$$\deg(\mathfrak{g}_\beta) = \begin{cases} 1 & \text{if } \beta = 0 \\ m & \text{if } \beta \neq 0. \end{cases}$$

So, the specialization $a = 0$ gives an arbitrary decrease in degree $\deg(\mathfrak{g}_0) = 1 < m = \deg(\mathbb{G})$.

5. Polar multiplicities, Segre numbers and new set of invariants

In this section, we prove several results regarding polar multiplicities and Segre numbers of an ideal, and we introduce a new related invariant. These invariants are defined as a special case of the general notion of polar multiplicities due to Kleiman and Thorup [38, 39]. Here, an important goal for us is to extend several of the results of Gaffney and Gassler [25] from their analytic setting to an algebraic one over a Noetherian local ring. The following setup is used throughout this section.

Setup 5.1. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ. Let $d := \dim(R)$ and $X := \mathrm{Spec}(R)$. Let $I \subset R$ be a proper ideal generated by elements $f_1, \ldots, f_m \in R$. We consider the Rees algebra $B := \mathscr{R}(I) := R[IT] = \bigoplus_{\nu \geqslant 0} I^\nu T^\nu \subset R[T]$ of the ideal I. We have a natural homogeneous presentation $W := R[y_1, \ldots, y_m] \twoheadrightarrow \mathscr{R}(I)$, $y_i \mapsto f_i T$, where W is a standard graded polynomial ring over R. Let $P := \mathrm{Bl}_I(X) = \mathrm{Proj}(B) \subset \mathrm{Proj}(W) = \mathbb{P}^{m-1}_R$ be the blowup of X along I and consider the natural projection

$$\pi : P \subset \mathbb{P}^{m-1}_R \to X.$$

Let $E := \pi^{-1}(V(I)) \cong \mathrm{Proj}(G) \subset P$ be the exceptional divisor and $G := \mathrm{gr}_I(R) := \bigoplus_{\nu \geqslant 0} I^\nu / I^{\nu+1}$ be the corresponding associated graded ring. The blowup P has a natural affine open cover $P = \bigcup_{i=1}^m U_i$ where $U_i = \mathrm{Spec}\left(\left[B_{y_i}\right]_0\right)$. More precisely, we can write

$$U_i = \mathrm{Spec}\left(R[I/f_i]\right),$$

where $R[I/f_i]$ denotes the R-subalgebra of R_{f_i} generated by all f/f_i with $f \in I$. Notice that the local equation of E on U_i is given by $f_i \in R[I/f_i]$.

We now briefly recall the general notion of polar multiplicities due to Kleiman and Thorup [38, 39]. Here we shall freely use the results from the references [11, 38, 39] regarding polar multiplicities. Let M be a finitely generated graded B-module and $\mathcal{F} = \widetilde{M}$ the corresponding coherent $\mathcal{O}_P$-module. The function $(\nu, n) \mapsto \text{length}_R\left(M_\nu/\mathfrak{m}^{n+1}M_\nu\right)$ eventually coincides with a bivariate polynomial $P_M(\nu, n)$ of degree equal to $\dim(\text{Supp}(\mathcal{F}))$. Then, for all $r \geqslant \dim(\text{Supp}(\mathcal{F}))$, we can write

$$P_M(\nu, n) \;=\; \sum_{i=0}^{r} \frac{m_r^i(M)}{i!\,(r-i)!}\, \nu^{r-i} n^i \;+\; \text{(lower degree terms)}.$$

We say that the invariants $m_r^i(M)$ are the *polar multiplicities* of M. Recall that $\dim(P) \leqslant d$ and $\dim(E) \leqslant d-1$. Our main interest is on the following invariants:

Definition 5.2. (i) For all $0 \leqslant i \leqslant d$, we say that $m_i(I, R) := m_d^{d-i}(B)$ is the i*-th polar multiplicity* of the ideal $I \subset R$.

(ii) For all $1 \leqslant i \leqslant d$, we say that $c_i(I, R) := m_{d-1}^{d-i}(G)$ is the i*-th Segre number* of the ideal $I \subset R$.

(iii) For all $1 \leqslant i \leqslant d$, we say that $\nu_i(I, R) := m_i(I, R) + c_i(I, R)$ is the i*-th polar-Segre multiplicity* of the ideal $I \subset R$. By convention, we also set $\nu_0(I, R) = m_0(I, R)$.

By [11, Proposition 2.10], we get $m_d(I, R) = m_d^0(B) = j_{d+1}(B)$, and since $\dim(B/\mathfrak{m}B) \leqslant d$, it follows that $m_d(I, R) = j_{d+1}(B) = 0$ (see [21, §6.1]). For all $0 \leqslant i \leqslant d-1$, the polar multiplicity $m_i(I, R)$ is also referred to as the *mixed multiplicity* $e_i(\mathfrak{m} \mid I)$ (see, e.g., [58]).

We shall need some very basic rudiments from intersection theory. Since we are working over our Noetherian local ring R (and not over a field), the usual developments from Fulton's book [23] do not suffice. In terms of a suitable *dimension function*, we could use available extensions of intersection theory (see, e.g., [23, Chapter 20], [52, Chapter 02P3], [57]). However, as we shall not require a notion of rational equivalence, we present our results in terms of cycles and quickly develop the necessary concepts.

Notation 5.3. (i) Let Y be a Noetherian scheme. We denote by $Z_k(Y)$ the free group of k-dimensional cycles. For a coherent sheaf $\mathcal{F}$ on Y and an integer $k \geqslant \dim(\text{Supp}(\mathcal{F}))$, we denote by $\left[\mathcal{F}\right]_k \in Z_k(Y)$ the associated k-cycle. For a closed subscheme $Z \subset Y$ and an integer $k \geqslant \dim(Z)$, we denote by $\left[Z\right]_k = \left[\mathcal{O}_Z\right]_k \in Z_k(Y)$ the associated k-cycle.

(ii) Given k-cycle $\xi = \sum_i l_i [R/\mathfrak{p}_i] \in Z_k(X)$, its multiplicity is given by $e_k(\xi) := \sum_i l_i e_k(R/\mathfrak{p}_i)$.

Below we give a short self-contained result regarding the push-forward of cycles along a projection (cf., [52, Lemma 02R6], [57, Proposition 4.3]).

Definition-Proposition 5.4. Let S be a standard graded R-algebra and consider the projective morphism $\eta : Y = \text{Proj}(S) \to X = \text{Spec}(R)$. Given an integral closed subscheme $Z \subset Y$, the push-forward is defined as

$$\eta_*([Z]) \;:=\; \begin{cases} \left[K(Z) : K(Z')\right] \cdot [Z'] & \text{if } \dim(Z) = \dim(Z') \\ 0 & \text{otherwise,} \end{cases}$$

where $Z' = \eta(Z)$, and $K(Z)$ and $K(Z')$ denote the function fields of Z and Z', respectively. The push-forward map $\eta_* : Z_k(Y) \to Z_k(X)$ is then determined by linearity. Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module and $k \geqslant \dim(\text{Supp}(\mathcal{F}))$. Then $\eta_*\left(\left[\mathcal{F}\right]_k\right) = \left[\eta_*(\mathcal{F})\right]_k \in Z_k(X)$.

Proof. Let $Z = \mathrm{Proj}(S/\mathfrak{P})$ where $\mathfrak{P} \subset S$ is a relevant prime ideal, and set $Z' = \eta(Z) = \mathrm{Spec}(R/\mathfrak{p})$ where $\mathfrak{p} = \mathfrak{P} \cap R$. From [63, Lemma 1.2.2], we get $\dim(S/\mathfrak{P}) = \dim(R/\mathfrak{p}) + \mathrm{trdeg}_{R/\mathfrak{p}}(S/\mathfrak{P})$. Since we have $\mathrm{trdeg}_{R/\mathfrak{p}}(S/\mathfrak{P}) \geqslant 1$, it follows that $\dim(Z) \geqslant \dim(Z')$. Let M be a finitely generated graded S-module with $\mathcal{F} \cong \widetilde{M}$ and $H^0_{S_+}(M) = 0$. Notice that we may substitute S by $S/\mathrm{Ann}_S(M)$ and R by $R/(R \cap \mathrm{Ann}_S(M))$. Therefore we assume that $k \geqslant \dim(Y)$ and $k \geqslant \dim(R)$, and that any minimal prime of S is relevant.

Let $\mathfrak{p} \subset R$ be a minimal prime of $N = H^0(X, \eta_*(\mathcal{F}))$ of dimension k. Notice that any relevant prime of S contracting to $\mathfrak{p}$ should be minimal. Thus the fiber $\eta^{-1}(\mathfrak{p})$ is finite, and so we can find an affine open neighborhood $V \subset X$ of $\mathfrak{p}$ such that $\eta^{-1}(V) \to V$ is finite (see [52, Lemma 02NW], [30, Exercise II.3.7]). We set $\eta^{-1}(V) = \mathrm{Spec}(A)$ and choose a finitely generated A-module L such that $\widetilde{L} \cong \mathcal{F}|_{\eta^{-1}(V)}$. Since $R_\mathfrak{p}$ is an Artinian local ring and $A_\mathfrak{p} = A \otimes_R R_\mathfrak{p}$ is module-finite over $R_\mathfrak{p}$, it follows that $A_\mathfrak{p}$ is Artinian. We have the equality

$$\sum_{\mathfrak{q}} [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \cdot \mathrm{length}_{A_\mathfrak{q}}(L_\mathfrak{q}) \;=\; \mathrm{length}_{R_\mathfrak{p}}(N_\mathfrak{p}),$$

where the sum runs through the minimal primes of A contracting to $\mathfrak{p}$. Finally, notice that the right-hand side of the above equality is the coefficient in $[\eta_*(\mathcal{F})]_k$ corresponding to $\mathfrak{p}$ and that the left-hand side is the coefficient in $\eta_*([\mathcal{F}]_k)$ corresponding to $\mathfrak{p}$. □

Notation 5.5. Given elements $a_1, \ldots, a_m$ in R, say that $g = a_1 f_1 + \cdots + a_m f_m$ is the associated element in I, that $\ell = a_1 y_1 + \cdots + a_m y_m \in B_1$ is the associated linear form, that $D = V(g) \subset X$ is the associated hypersurface in X, and that $H = V_+(\ell) \subset P$ is the associated hyperplane in P. We denote by $\pi^* D := V_+(gB) = \mathrm{Proj}(B/gB)$ the pullback of the hypersurface D. When the residue field κ is infinite, we use the following conventions:

– We say that $D \subset X$ is general (equivalently $H \subset P$ is general) if g is a general element in I (equivalently ℓ is a general element in B_+).
– We say that a sequence of hyperplanes $\underline{H} = H_1, \ldots, H_k$ is *a sequence of general hyperplanes* in P if the associated sequence $\underline{g} = g_1, \ldots, g_k$ is a sequence of general elements in I.

When κ is infinite and $g \in I$ is a general element, the following remark shows that $\pi^* D$ is an effective Cartier divisor on P even if $D = V(g)$ is not a divisor on X.

Remark 5.6. (κ infinite). Let $\overline{R} = R/(0 :_R I^\infty)$ and $\overline{X} = \mathrm{Spec}(\overline{R})$. Since $\mathrm{Bl}_I(X) \cong \mathrm{Bl}_I(\overline{X})$, we may assume that $(0 :_R I^\infty) = 0$, and so by prime avoidance we get that g is a nonzerodivisor when $g \in I$ is general. This shows that $\pi^* D$ is an effective Cartier divisor when $g \in I$ is general.

The next notation includes an inequality that will be useful in our approach. This inequality is related to when a general element of I is a $\mathcal{G}$-parameter on the Rees algebra $B = \mathcal{R}(I)$ (in the sense of [11, Definition 2.7]).

Notation 5.7. We say that the *order* of the ideal I is given by $o(I) := \sup\{\beta \in \mathbb{Z}_{\geqslant 0} \mid I \subseteq \mathfrak{m}^\beta\}$. Let $\delta = o(I)$. Consider the standard bigraded algebra $\mathcal{G} := \mathrm{gr}_\mathfrak{m}(B)$ with bigraded parts $[\mathcal{G}]_{(\nu,n)} = \mathfrak{m}^n B_\nu / \mathfrak{m}^{n+1} B_\nu$. Take the κ-vector subspace $\mathfrak{b} = I/\mathfrak{m}^{\delta+1} \subset \mathfrak{m}^\delta/\mathfrak{m}^{\delta+1} = [\mathcal{G}]_{(0,\delta)}$. Let

$$\mathrm{in}^\delta(I) \;:=\; \mathfrak{b} \cdot \mathcal{G} \;=\; (\mathrm{in}(f) \mid f \in I \text{ and } o(f) = \delta) \subset \mathcal{G}$$

be the ideal generated by the initial forms of elements in I of order δ. If the following strict inequality

$$\dim\left(\mathrm{BiProj}\left(\mathscr{G}/\mathrm{in}^{\delta}(I)\right)\right) \;<\; d-1$$

holds, we say that I satisfies the *$\mathscr{G}$-parameter condition generically*.

The following proposition is inspired by one of the technical steps in Fulton's proof of the commutativity of intersecting with Cartier divisors: indeed, similarly to [23, Lemma 2.4], we express the pullback π^*D as the sum of the exceptional divisor E and a hyperplane H in the blowup $P = \mathrm{Bl}_I(X)$. As a consequence, we get inequalities relating the polar multiplicities and the Segre numbers of I.

Proposition 5.8. (κ *infinite*). *Let $H \subset P$ be a general hyperplane, $D = V(g)$ be the associated hypersurface and $\ell \in B_+$ be the associated linear form. Then the following statements hold:*

(i) *π^*D and H are effective Cartier divisors.*
(ii) $\pi^*D = E + H$.
(iii) *Let $\delta = o(I)$. Then we have the inequality*

$$\delta \cdot m_{i-1}(I,R) \;\leqslant\; m_i(I,R) + c_i(I,R) \;=\; \nu_i(I,R),$$

and equality holds for all $1 \leqslant i \leqslant d$ if and only if I satisfies the $\mathscr{G}$-parameter condition generically (see Notation 5.7).

Proof. (i) From Remark 5.6, we get that π^*D is an effective Cartier divisor. We have that H is also an effective Cartier divisor by prime avoidance.

(ii) We write $g = a_1 f_1 + \cdots + a_m f_m$ and $\ell = a_1 y_1 + \cdots + a_m y_m$. Consider the affine open subscheme $U_i = \mathrm{Spec}(R[I/f_i])$. As we mentioned before, the local equation of E on U_i is given by $f_i \in R[I/f_i]$. On the other hand, the local equations of π^*D and H on U_i are given by

$$a_1 f_1 + \cdots + a_m f_m \;\in\; R[I/f_i] \qquad \text{and} \qquad \frac{a_1 f_1 + \cdots + a_m f_m}{f_i} \;\in\; R[I/f_i],$$

respectively. This shows the equality $\pi^*D = E + H$ on each U_i, and so the equality holds globally on the whole blowup $P = \mathrm{Bl}_I(X)$.

(iii) First, we check that in the vacuous case $\dim(P) < d$, we have that $m_i(I,R) = 0$, $\nu_i(I,R) = 0$, $\dim(B) \leqslant d$ and $\dim(\mathrm{BiProj}(\mathscr{G})) \leqslant d-2$ (hence the equivalence statement holds trivially). Therefore, we assume $\dim(P) = d$.

As $g \in I$ is general, we may assume that $g \in \mathfrak{m}^{\delta} \setminus \mathfrak{m}^{\delta+1}$. Notice that $[\pi^*D]_{d-1} = [\widetilde{B/gB}]_{d-1}$ and $[H]_{d-1} = [\widetilde{B/\ell B}]_{d-1}$, and thus part (ii) yields the equality of cycles

$$[\widetilde{B/gB}]_{d-1} \;=\; [\widetilde{B/\ell B}]_{d-1} + [\widetilde{G}]_{d-1} \;\in\; Z_{d-1}(P);$$

see, e.g., [23, Lemma A.2.5]. Hence the additivity of polar multiplicities (see [11, Corollary 2.6]) implies that

$$m_{d-1}^i(B/gB) \;=\; m_{d-1}^i(B/\ell B) + m_{d-1}^i(G).$$

Due to [11, Theorem 2.8], we have $m_{d-1}^i(B/gB) \geqslant \delta \cdot m_d^{i+1}(B)$ and an equality holds for all $0 \leqslant i \leqslant d-1$ if and only if g is a $\mathscr{G}$-parameter on B (in the sense of [11, Definition 2.7]). Since $\ell \in B_+$ is a general element, by utilizing [11, Proposition 2.10], we obtain $m_{d-1}^i(B/\ell B) = m_d^i(B)$. Finally, by combining everything we get the inequality

$$\delta \cdot m_{i-1}(I,R) \;=\; \delta \cdot m_d^{d-i+1}(B) \;\leqslant\; m_d^{d-i}(B) + m_{d-1}^{d-i}(G) \;=\; m_i(I,R) + c_i(I,R),$$

and an equality holds for all $1 \leqslant i \leqslant d$ if and only if g is a $\mathscr{G}$-parameter on B. Since g is a general element of I, it follows that g is a $\mathscr{G}$-parameter on B if and only if I satisfies the $\mathscr{G}$-parameter condition generically. □

Remark 5.9. The equality of Proposition 5.8(ii) can also be derived as follows. Consider the extended Rees algebra $\mathscr{R}^+(I) := R[IT, T^{-1}]$. Notice that the equality $g = \ell \cdot T^{-1}$ holds in $\mathscr{R}^+(I)$ and recall the isomorphism $\text{gr}_I(R) \cong \mathscr{R}^+(I)/T^{-1}\mathscr{R}^+(I)$.

Remark 5.10. By [2, Theorem 4.1], the degree of Stückrad–Vogel cycles can computed as Segre numbers. Hence Proposition 5.8(iii) yields a generalization of the formula for the degree of Stückrad–Vogel cycles given in [58, Theorem 4.6].

Next, we introduce the notion of polar schemes and Segre cycles. We also introduce a new type of cycle that will be fundamental in our treatment. The following additional data is fixed for the rest of the section.

Setup 5.11. Assume Setup 5.1 and that the residue field κ is infinite. Let $\underline{H} = H_1, \ldots, H_d$ be a sequence of general hyperplanes, and denote by $\underline{g} = g_1, \ldots, g_d$ the associated sequence of elements in I and by $\underline{\ell} = \ell_1, \ldots, \ell_d$ the associated sequence of linear forms in B_+. We also set $D_i = V(g_i) \subset X$ and recall that the pullback $\pi^* D_i$ is an effective Cartier divisor on P (see Remark 5.6). We introduce the following objects:

(i) For $1 \leqslant i \leqslant d$, we say that the i-*th polar scheme* (with respect to $\underline{H}$) is given by the following schematic-image
$$P_i(I,X) \;=\; P_i^{\underline{H}}(I,X) \;:=\; \pi(H_1 \cap \cdots \cap H_i).$$

(ii) For $1 \leqslant i \leqslant d$, we say that the i-*th Segre cycle* (with respect to $\underline{H}$) is given by
$$\Lambda_i(I,X) \;=\; \Lambda_i^{\underline{H}}(I,X) \;:=\; \pi_*\Big(\big[E \cap H_1 \cap \cdots \cap H_{i-1}\big]_{d-i}\Big) \;\in\; Z_{d-i}(X).$$

(iii) For $1 \leqslant i \leqslant d$, we say that the i-*th polar-Segre cycle* (with respect to $\underline{H}$) is given by
$$V_i(I,X) \;=\; V_i^{\underline{H}}(I,X) \;:=\; \pi_*\Big(\big[H_1 \cap \cdots \cap H_{i-1} \cap \pi^* D_i\big]_{d-i}\Big) \;\in\; Z_{d-i}(X).$$

By convention, we set $P_0(I,X) := \pi(P)$ and $V_0(I,X) := [P_0(I,X)]_d \in Z_d(X)$.

Remark 5.12. By prime avoidance, we can assume that $g_1, \ldots, g_i$ is a regular sequence on $R_\mathfrak{p}$ for each $\mathfrak{p} \in V(g_1, \ldots, g_i) \setminus V(I)$ (we say that $g_1, \ldots, g_i$ are a filter-regular sequence with respect to I; see [53, Appendix]). Therefore, we have that $(g_1, \ldots, g_i) :_R I^\infty$ either equals R or it has height i, and so we obtain $\dim(R/(g_1, \ldots, g_i) :_R I^\infty) \leqslant d - i$. Similarly, we get $\dim(R/(g_1, \ldots, g_{i-1}) :_R I^\infty + g_i R) \leqslant d - i$.

The theorem below gives an important description of the invariants $m_i(I,R)$, $c_i(I,R)$ and $\nu_i(I,R)$. It shows that these invariants are naturally the multiplicities of the cycles introduced in Setup 5.11.

Theorem 5.13. *Assume Setup 5.11. Then the following statements hold:*

(i) *For all* $0 \leqslant i \leqslant d$*, we have the equalities* $m_i(I,R) = e_{d-i}(P_i(I,X))$ *and*

$$P_i(I,X) \;=\; \operatorname{Spec}(R/(g_1,\ldots,g_i):_R I^\infty).$$

(ii) *For all* $1 \leqslant i \leqslant d$*, we have the equalities* $c_i(I,R) = e_{d-i}(\Lambda_i(I,X))$ *and*

$$\Lambda_i(I,X) \;=\; \sum_{\substack{\mathfrak{p}\in V((g_1,\ldots,g_{i-1}):_R I^\infty)\\ \mathfrak{p}\in V(I),\, \dim(R/\mathfrak{p})=d-i}} e\big(I, R_\mathfrak{p}/(g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p}\big)\cdot[R/\mathfrak{p}] \;\in\; Z_{d-i}(X).$$

(iii) *For all* $1 \leqslant i \leqslant d$*, we have the equalities* $v_i(I,R) = e_{d-i}(V_i(I,X))$ *and*

$$V_i(I,X) \;=\; [P_i(I,X)]_{d-i} + \Lambda_i(I,X) \;=\; \big[\operatorname{Spec}(R/(g_1,\ldots,g_{i-1}):_R I^\infty + g_iR)\big]_{d-i} \in Z_{d-i}(X).$$

Proof. Let $\mathfrak{a}_i := (g_1,\ldots,g_i) \subset R$, $R_i := R/\mathfrak{a}_i$, $X_i := \operatorname{Spec}(R_i)$ and $\overline{X}_i := \operatorname{Spec}(R/\mathfrak{a}_i :_R I^\infty)$.

(i) First, set $i=0$. By [11, Proposition 2.10], we have $m_0(I,R) = m_d^d(B) = e_d\left(H^0(P,\mathcal{O}_P)\right)$. We have that $P_0(I,X) = \pi(P) = \operatorname{Spec}(R/\mathfrak{a})$ where $\mathfrak{a} \subset R$ is the kernel of the natural map $R \xrightarrow{\text{nat}} H^0(P,\mathcal{O}_P)$ (see, e.g., [26, Proposition 10.30], [30, Exercise II.3.11]). On the other hand, we have a four-term exact sequence

$$0 \longrightarrow \left[H^0_{B_+}(B)\right]_0 \longrightarrow R = B_0 \xrightarrow{\text{nat}} H^0(P,\mathcal{O}_P) \longrightarrow \left[H^1_{B_+}(B)\right]_0 \longrightarrow 0.$$

Since $B = R[IT]$ is the Rees algebra of I, it follows that $\left[H^0_{B_+}(B)\right]_0 = 0:_R I^\infty$. Hence we have $\mathfrak{a} = 0:_R I^\infty$ and $P_0(I,X) = \pi(P) = \operatorname{Spec}(R/0:_R I^\infty)$. Moreover, we get the short exact sequence

$$0 \to R/0:_R I^\infty \to H^0(P,\mathcal{O}_P) \to \left[H^1_{B_+}(B)\right]_0 \to 0,$$

and so to prove $m_0(I,R) = e_d(P_0(I,X))$, it suffices to show that $\dim\left(\left[H^1_{B_+}(B)\right]_0\right) < d$. Let $\overline{B} := B/H^0_{B_+}(B)$. As $H^1_{B_+}(B) \cong H^1_{B_+}(\overline{B})$ and $\left[\overline{B}\right]_0 = R/0:_R I^\infty$, we have that $H^1_{B_+}(B)\otimes_R R_\mathfrak{p} = 0$ for any minimal prime $\mathfrak{p} \in \operatorname{Min}(R)$ that contains I. However, if a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ does not contain I, we obtain

$$\left[H^1_{B_+}(B)\right]_0 \otimes_R R_\mathfrak{p} \;\cong\; \left[H^1_{B_+}(B\otimes_R R_\mathfrak{p})\right]_0 \;\cong\; \left[H^1_T(R_\mathfrak{p}[T])\right]_0 = 0.$$

This settles the claim that $\dim\left(\left[H^1_{B_+}(B)\right]_0\right) < d$, and so the proof is complete for the case $i=0$.

Since $\ell_1,\ldots,\ell_i$ is a sequence of general elements in B_+, [11, Proposition 2.10] yields the equalities $m_i(I,R) = m_d^{d-i}(B) = m_{d-i}^{d-i}(B/(\ell_1,\ldots,\ell_i)B) = e_{d-i}\left(H^0(P,\mathcal{O}_{H_1\cap\cdots\cap H_i})\right)$. Consider the natural specialization map

$$\mathfrak{s} \,:\, B/(\ell_1,\ldots,\ell_i)B \;\cong\; \bigoplus_{\nu\geqslant 0} I^\nu/\mathfrak{a}_i I^{\nu-1} \quad\twoheadrightarrow\quad \mathscr{R}_{R_i}(IR_i) \;\cong\; \bigoplus_{\nu\geqslant 0} I^\nu/(I^\nu\cap\mathfrak{a}_i).$$

Since $g_1,\ldots,g_i$ is a sequence of general elements in I, we may assume that they form a superficial sequence for I (see [34, Proposition 8.5.7]), and so [34, Lemma 8.5.11] yields the equality $\mathfrak{a}_i I^{\nu-1} = I^\nu \cap \mathfrak{a}_i$ for $\nu \gg 0$. Therefore, we get the following equality (as schemes)

$$H_1\cap\cdots\cap H_i \;=\; \operatorname{Proj}(B/(\ell_1,\ldots,\ell_i)B) \;=\; \operatorname{Proj}(\mathscr{R}_{R_i}(IR_i)) \;=\; \operatorname{Bl}_I(X_i) \;=:\; P_i. \tag{2}$$

Since we already dealt with the initial case, by substituting X by X_i and setting $i=0$, we obtain

$$P_i(I,X) \;=\; \pi(H_1\cap\cdots\cap H_i) \;=\; \pi(P_i) \;=\; P_0(I,X_i) \;=\; \operatorname{Spec}(R/(g_1,\ldots,g_i):_R I^\infty)$$

and

$$m_i(I,R) \;=\; e_{d-i}\left(H^0(P,\mathcal{O}_{H_1\cap\cdots\cap H_i})\right) \;=\; e_{d-i}\left(H^0(P_i,\mathcal{O}_{P_i})\right) \;=\; e_{d-i}(P_0(I,X_i)).$$

This completes the proof of part (i).

(ii) First, set $i = 1$. By [11, Proposition 2.10], we get $c_1(I,R) = m_{d-1}^{d-1}(G) = e_{d-1}\left(H^0(E,\mathcal{O}_E)\right)$. From Definition-Proposition 5.4, we have the equality of cycles

$$\Lambda_1(I,X) \;=\; \left[\pi_*(\mathcal{O}_E)\right]_{d-1} \;=\; \left[H^0(X,\pi_*(\mathcal{O}_E))\right]_{d-1} \;=\; \left[H^0(E,\mathcal{O}_E)\right]_{d-1} \;\in\; Z_{d-1}(X),$$

and so it follows that $c_1(I,R) = e_{d-1}(\Lambda_1(I,X))$. Since we proved $\pi(P) = \mathrm{Spec}(R/0:_R I^\infty)$ in part (i), it follows that $\mathrm{Supp}\left(H^0(E,\mathcal{O}_E)\right) \subset V(I, 0:_R I^\infty)$. Then we get the following equality

$$\left[H^0(E,\mathcal{O}_E)\right]_{d-1} \;=\; \sum_{\substack{\mathfrak{p}\in V(I,0:_R I^\infty)\\ \dim(R/\mathfrak{p})=d-1}} \mathrm{length}_{R_\mathfrak{p}}\left(H^0(E,\mathcal{O}_E)\otimes_R R_\mathfrak{p}\right)\cdot[R/\mathfrak{p}] \;\in\; Z_{d-1}(X).$$

Fix a prime $\mathfrak{p}$ in the above summation. Let $E_\mathfrak{p} := \mathrm{Proj}\left(\mathrm{gr}_{IR_\mathfrak{p}}(R_\mathfrak{p})\right)$ be the exceptional divisor of $\mathrm{Spec}(R_\mathfrak{p})$ along $IR_\mathfrak{p}$. Notice that $\dim(R_\mathfrak{p}) = 1$ and $H^0(E,\mathcal{O}_E)\otimes_R R_\mathfrak{p} \cong H^0(E_\mathfrak{p},\mathcal{O}_{E_\mathfrak{p}})$. As a consequence, for $\nu \gg 0$, we obtain

$$\mathrm{length}_{R_\mathfrak{p}}\left(H^0(E_\mathfrak{p},\mathcal{O}_{E_\mathfrak{p}})\right) \;=\; \mathrm{length}_{R_\mathfrak{p}}\left(H^0(E_\mathfrak{p},\mathcal{O}_{E_\mathfrak{p}}(\nu))\right) \;=\; \mathrm{length}_{R_\mathfrak{p}}\left(I^\nu R_\mathfrak{p}/I^{\nu+1}R_\mathfrak{p}\right);$$

the first equality holds for any $\nu \in \mathbb{Z}$ because $\dim(E_\mathfrak{p}) \leqslant 0$. On the other hand, we have

$$e\left(I, R_\mathfrak{p}/0:_{R_\mathfrak{p}} I^\infty R_\mathfrak{p}\right) \;=\; \mathrm{length}_{R_\mathfrak{p}}\left(I^\nu R_\mathfrak{p}/I^{\nu+1}R_\mathfrak{p}\right) \quad \text{for} \quad \nu \gg 0.$$

This completes the proof for the case $i = 1$.

Since $\ell_1,\ldots,\ell_{i-1}$ is a sequence of general elements in B_+, [11, Proposition 2.10] gives $c_i(I,R) = m_{d-1}^{d-i}(G) = m_{d-i}^{d-i}(G/(\ell_1,\ldots,\ell_{i-1})G) = e_{d-i}\left(H^0(P,\mathcal{O}_{E\cap H_1\cap\cdots\cap H_{i-1}})\right)$. From (2), we obtain the following equality (as schemes)

$$E\cap H_1\cap\cdots\cap H_{i-1} \;=\; (H_1\cap\cdots\cap H_{i-1})\times_X \mathrm{Spec}(R/I) \;=\; P_{i-1}\times_X \mathrm{Spec}(R/I) \;=\; E_{i-1} \tag{3}$$

where $E_{i-1} = \mathrm{Proj}(\mathrm{gr}_{IR_{i-1}}(R_{i-1}))$ is the exceptional divisor of X_{i-1} along IR_{i-1}. Then the result of part (ii) follows.

(iii) We have the following equalities

$$\begin{aligned}
V_i(I,X) &= \pi_*\left([H_1\cap\cdots\cap H_{i-1}\cap\pi^*D_i]_{d-i}\right) && \text{by definition of } V_i(I,X)\\
&= \pi_*\left([P_{i-1}\cap\pi^*D_i]_{d-i}\right) && \text{by (2)}\\
&= \left[V_1(I,\overline{X}_{i-1})\right]_{d-i} && \text{by definition of } V_1(I,\overline{X}_{i-1})\\
&= \left[P_1(I,\overline{X}_{i-1})\right]_{d-i} + \left[\Lambda_1(I,\overline{X}_{i-1})\right]_{d-i} && \text{by applying Proposition 5.8 to } \overline{X}_{i-1}\\
&= [P_i(I,X)]_{d-i} + \Lambda_i(I,X) && \text{by the formulas of part (i) and (ii).}
\end{aligned}$$

Therefore, to complete the proof of part (iii), we substitute X by $\overline{X}_{i-1}$ and assume that $i = 1$. In particular, we may assume that $(0:_R I^\infty) = 0$ and that g_1 is a nonzerodivisor. Set $g := g_1$ and $D := D_1$. By Definition-Proposition 5.4, we obtain

$$V_1(I,X) = \left[\pi_*(\mathcal{O}_{\pi^*D})\right]_{d-1} = \sum_{\substack{\mathfrak{p}\in V(g)\\ \dim(R/\mathfrak{p})=d-1}} \mathrm{length}_{R_\mathfrak{p}}\left(H^0(P,\mathcal{O}_P/g\mathcal{O}_P)\otimes_R R_\mathfrak{p}\right)\cdot[R/\mathfrak{p}] \;\in\; Z_{d-1}(X).$$

Let $\mathfrak{p}$ be a prime in the above summation, and let $P_\mathfrak{p} := \mathrm{Proj}(B \otimes_R R_\mathfrak{p})$ the blowup of $\mathrm{Spec}(R_\mathfrak{p})$ along $IR_\mathfrak{p}$. Then the coefficient corresponding to $\mathfrak{p}$ is given by

$$\mathrm{length}_{R_\mathfrak{p}}\left(\mathrm{H}^0(P_\mathfrak{p}, \mathcal{O}_{P_\mathfrak{p}}/g\mathcal{O}_{P_\mathfrak{p}})\right) = \mathrm{length}_{R_\mathfrak{p}}\left(\mathrm{H}^0(P_\mathfrak{p}, \mathcal{O}_{P_\mathfrak{p}}/g\mathcal{O}_{P_\mathfrak{p}}(\nu))\right) = \mathrm{length}_{R_\mathfrak{p}}\left(I^\nu R_\mathfrak{p}/gI^\nu R_\mathfrak{p}\right)$$

for $\nu \gg 0$. Expressing length as a multiplicity (see [21, Corollary 1.2.14]) and utilizing the additivity of multiplicities give the following

$$\mathrm{length}_{R_\mathfrak{p}}\left(I^\nu R_\mathfrak{p}/gI^\nu R_\mathfrak{p}\right) = e((g), I^\nu R_\mathfrak{p}) = e((g), R_\mathfrak{p}) = \mathrm{length}_{R_\mathfrak{p}}\left(R_\mathfrak{p}/gR_\mathfrak{p}\right).$$

Hence the required equality $V_1(I,X) = [\mathrm{Spec}(R/gR)]_{d-1}$ follows. This completes the proof of the last part of the theorem. □

Remark 5.14. Due to Theorem 5.13, we obtain the following consequences:

(i) We have the equations
  (a) $\left[P_i(I,X)\right]_{d-i} = \left[P_1(I, P_{i-1}(I,X))\right]_{d-i}$.
  (b) $\Lambda_i(I,X) = \left[\Lambda_1(I, P_{i-1}(I,X))\right]_{d-i}$.
  (c) $V_i(I,X) = \left[V_1(I, P_{i-1}(I,X))\right]_{d-i}$.

(ii) We recover the formula of [58, Theorem 3.4] for the mixed multiplicities of an ideal

$$m_i(I,R) = e_{d-i}(R/(g_1,\ldots,g_i) :_R I^\infty).$$

(iii) The new invariants (i.e., polar-Segre multiplicities) have the following formula

$$\nu_i(I,R) = e_{d-i}(R/(g_1,\ldots,g_{i-1}) :_R I^\infty + g_i R).$$

(iv) We obtain the following formula for Segre numbers

$$c_i(I,R) = \sum_{\substack{\mathfrak{p}\in V((g_1,\ldots,g_{i-1}):_R I^\infty) \\ \mathfrak{p}\in V(I),\, \dim(R/\mathfrak{p})=d-i}} e\left(I, R_\mathfrak{p}/(g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p}\right)\cdot e(R/\mathfrak{p}).$$

On the other hand, the formulas of part (i) and (iii) yield

$$\begin{aligned} c_i(I,R) &= e_{d-i}(R/(g_1,\ldots,g_{i-1}) :_R I^\infty + g_i R) - e_{d-i}(R/(g_1,\ldots,g_i) :_R I^\infty) \\ &= e_{d-i}\left(\mathrm{H}^0_I(R/(g_1,\ldots,g_{i-1}) :_R I^\infty + g_i R)\right) \\ &= \sum_{\substack{\mathfrak{p}\in V((g_1,\ldots,g_{i-1}):_R I^\infty) \\ \mathfrak{p}\in V(I),\, \dim(R/\mathfrak{p})=d-i}} \mathrm{length}_{R_\mathfrak{p}}\left(\frac{R_\mathfrak{p}}{(g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p} + g_i R_\mathfrak{p}}\right)\cdot e(R/\mathfrak{p}). \end{aligned}$$

This recovers the length formula of [48, Proposition 2.1] for the Segre numbers of an ideal. Therefore, for any prime $\mathfrak{p}$ in the summation above, it follows that

$$e(I, R_\mathfrak{p}/(g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p}) = \mathrm{length}_{R_\mathfrak{p}}\left(R_\mathfrak{p}/\left((g_1,\ldots,g_{i-1})R_\mathfrak{p} :_{R_\mathfrak{p}} I^\infty R_\mathfrak{p} + g_i R_\mathfrak{p}\right)\right).$$

In any case, this is expected: if the residue field κ is not an algebraic extension of a finite field, we may assume that $g_i \in I$ is general in $IR_\mathfrak{p}$ for each prime $\mathfrak{p}$ above (see [48, Lemma 2.6]).

6. Integral dependence and specialization

In this section, we discuss how integral dependence can be detected by utilizing the invariants that we study in Section 5. We also discuss when integral closure and polar schemes specialize. Moreover, we generalize the result of Gaffney and Gassler [25] regarding the lexicographic upper semicontinuity of Segre numbers (see Theorem 6.12). Throughout this section, we continue using Setup 5.1.

To deal with the case where $\mathrm{ht}(I) = 0$, we also need to consider an additional number that is denoted as $c_0(I,R)$. Let $\mathscr{H} := \mathrm{gr}_{\mathfrak{m}}(G) = \mathrm{gr}_{\mathfrak{m}}(\mathrm{gr}_I(R))$ with standard bigrading $[\mathscr{H}]_{(\nu,n)} = \mathfrak{m}^n G_\nu/\mathfrak{m}^{n+1}G_\nu$. The first sum transform of the Hilbert function of $\mathscr{H}$ with respect to n is equal to

$$H^1_{\mathscr{H}}(\nu,n) := \sum_{k=0}^{n} \dim_\kappa\left([\mathscr{H}]_{(\nu,n)}\right) = \mathrm{length}_R\left(G_\nu/\mathfrak{m}^{n+1}G_\nu\right),$$

and it encodes the polar multiplicities of the standard graded algebra $G = \mathrm{gr}_I(R)$ (i.e., the Segre numbers $c_1(I,R),\ldots,c_d(I,R)$; see Section 5). If we further consider the second sum transform $H^2_{\mathscr{H}}(\nu,n) := \sum_{k=0}^{\nu} H^1_{\mathscr{H}}(k,n)$, for $\nu \gg 0$ and $n \gg 0$, we get a polynomial

$$H^2_{\mathscr{H}}(\nu,n) \;=\; \sum_{i=0}^{d} \frac{c_i(I,R)}{i!\,(d-i!)}\,\nu^i n^{d-i} \;+\; \text{(lower degree terms)},$$

that encodes the Segre numbers $c_1(I,R),\ldots,c_d(I,R)$ defined in Section 5, but also the new number $c_0(I,R)$. This leads to the following definition.

Definition 6.1 (Achilles–Manaresi [2]). The *multiplicity sequence* of the ideal $I \subset R$ is given by

$$\left(c_0(I,R), c_1(I,R),\ldots,c_d(I,R)\right) \;\in\; \mathbb{Z}^{d+1}_{\geqslant 0}.$$

In Section 5, we restricted ourselves to considering only the first sum transform $H^1_{\mathscr{H}}$ (i.e., the polar multiplicities of G) because then we could use many desirable properties of polar multiplicities (see [11], [38, §8]).

We now discuss when the polar schemes specialize module an element $t \in \mathfrak{m}$ that is part of a system of parameters of R (i.e., $\dim(R/tR) = d-1$). We say that the i-th polar scheme $P_i(I,X)$ *specializes* modulo the element $t \in \mathfrak{m}$ if $\dim\left(P_i(I,X) \cap V(t)\right) \leqslant d-i-1$ and we have the equality of cycles

$$\left[P_i(I,X) \cap V(t)\right]_{d-i-1} \;=\; \left[P_i\left(I, X \cap V(t)\right)\right]_{d-i-1} \;\in\; Z_{d-i-1}(X).$$

We shall see that the question of whether polar schemes specialize is governed by the Segre numbers $c_i(I,R)$ and $c_i(I,R/tR)$. First, we have the following basic but useful observation.

Lemma 6.2. *Assume Setup 5.11. Let* $t \in \mathfrak{m}$ *be such that* $\dim(R/tR) = d-1$. *If* $\mathrm{ht}\left(I, (g_1,\ldots,g_i) :_R I^\infty, t\right) \geqslant i+2$, *then* $P_i(I,X)$ *specializes modulo* t.

Proof. Let $\overline{R} := R/\left((g_1,\ldots,g_i) :_R I^\infty, t\right)$ and notice that $I^k\overline{R} \cdot \left((t,g_1,\ldots,g_i) :_R I^\infty\right)\overline{R} = 0$ for some $k > 0$. Thus the result follows as a direct consequence of Theorem 5.13(i). ☐

A very important technical result is the following theorem. Our arguments below follow closely the ones in the proof of [48, Theorem 3.3].

Theorem 6.3. *Assume Setup 5.11 and that* R *is equidimensional and catenary. Let* $1 \leqslant i \leqslant d-1$. *Let* $t \in \mathfrak{m}$ *be such that* $\mathrm{ht}\left((g_1,\ldots,g_{i-1}) :_R I^\infty, t\right) \geqslant i$ *and* $\mathrm{ht}\left(I, (g_1,\ldots,g_{i-1}) :_R I^\infty, t\right) \geqslant i+1$. *Then the following statements hold:*

(i) $\mathrm{ht}\left((g_1,\ldots,g_i) :_R I^\infty, t\right) \geqslant i+1$.

(ii) $c_i(I,R/tR) \geqslant c_i(I,R)$.

(iii) *If* $c_i(I,R/tR) = c_i(I,R)$, *then* $\operatorname{ht}(I,(g_1,\ldots,g_i):_R I^\infty, t) \geqslant i+2$.

Proof. We may assume that $\dim(R/(g_1,\ldots,g_{i-1}):_R I^\infty) = d-i+1$ because R is equidimensional and catenary (see Remark 5.12). By utilizing Theorem 5.13(ii), we may assume $i=1$ and substitute R by $R/(g_1,\ldots,g_{i-1}):_R I^\infty$. So, we are free to assume the conditions $i=1$, $\dim(R/tR) = d-1$, $0:_R I^\infty = 0$ and $\operatorname{ht}(I,t) \geqslant 2$. Hence g_1 is a nonzerodivisor since it is a general element of I. Set $g := g_1$. Since g is a general element of I, it follows that $\operatorname{ht}((g):_R I^\infty, t) \geqslant \operatorname{ht}(g,t) \geqslant 2$; thus settling part (i). We need to verify $c_1(I,R/tR) \geqslant c_1(I,R)$ to prove part (ii). Part (iii) follows by showing that if $c_1(I,R/tR) = c_1(I,R)$ then $\operatorname{ht}(I,(g):_R I^\infty, t) \geqslant 3$.

The rest of the proof follows verbatim the arguments in [48, page 962, proof of Theorem 3.3]. Consider the finite sets of primes $\Lambda := \{\mathfrak{p} \in V(I,t) \mid \operatorname{ht}(\mathfrak{p}) = 2\}$ and $\Gamma := \{\mathfrak{q} \in V(I) \mid \operatorname{ht}(\mathfrak{q}) = 1\}$. Thus, to prove the inequality $\operatorname{ht}(I,(g):_R I^\infty, t) \geqslant 3$, it suffices to show that for each $\mathfrak{p} \in \Lambda$ we have $\mathfrak{p} \not\supset (g):_R I^\infty$ or, equivalently, $IR_\mathfrak{p} \subset \sqrt{gR_\mathfrak{p}}$. For each $\mathfrak{p} \in \Lambda$, we define the finite sets

$$\Sigma_\mathfrak{p} := \{\mathfrak{q} \in \operatorname{Min}(g) \mid \mathfrak{q} \subset \mathfrak{p}\} \quad \text{and} \quad \Gamma_\mathfrak{p} := \{\mathfrak{q} \in V(I) \mid \mathfrak{q} \subset \mathfrak{p} \text{ and } \operatorname{ht}(\mathfrak{q}) = 1\}.$$

Notice that $\Gamma_\mathfrak{p} \subseteq \Sigma_\mathfrak{p}$ and that an equality holds if and only if $IR_\mathfrak{p} \subset \sqrt{gR_\mathfrak{p}}$. By utilizing the faithfully flat extension $R \to R[y]_{\mathfrak{m}R[y]}$, we may assume that the residue field of R is not an algebraic extension of a finite field, and thus we may assume that g is a general element of $IR_\mathfrak{p}$ for each $\mathfrak{p} \in \Lambda$ (see [48, Lemma 2.6]).

We have the following

$$\begin{aligned}
c_1(I,R/tR) &= \sum_{\mathfrak{p}\in\Lambda} e(I,R_\mathfrak{p}/tR_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{by [48, Proposition 2.4(a)] or Theorem 5.13(ii)}\\
&= \sum_{\mathfrak{p}\in\Lambda} e((g),R_\mathfrak{p}/tR_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{since } g \text{ is general in } IR_\mathfrak{p}\\
&\geqslant \sum_{\mathfrak{p}\in\Lambda} e((t),R_\mathfrak{p}/gR_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{by [48, Lemma 3.1] since } g \text{ is a nonzerodivisor}\\
&= \sum_{\mathfrak{p}\in\Lambda} \sum_{\mathfrak{q}\in\Sigma_\mathfrak{p}} \operatorname{length}_{R_\mathfrak{q}}(R_\mathfrak{q}/gR_\mathfrak{q}) \cdot e((t),R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{by the associativity formula}\\
&\geqslant \sum_{\mathfrak{p}\in\Lambda} \sum_{\mathfrak{q}\in\Gamma_\mathfrak{p}} \operatorname{length}_{R_\mathfrak{q}}(R_\mathfrak{q}/gR_\mathfrak{q}) \cdot e((t),R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{since } \Gamma_\mathfrak{p} \subseteq \Sigma_\mathfrak{p}\\
&= \sum_{\mathfrak{q}\in\Gamma} \operatorname{length}_{R_\mathfrak{q}}(R_\mathfrak{q}/gR_\mathfrak{q}) \sum_{\mathfrak{p}\in\Lambda,\mathfrak{p}\supset\mathfrak{q}} e((t),R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \cdot e(R/\mathfrak{p}) && \text{by switching the summation}\\
&\geqslant \sum_{\mathfrak{q}\in\Gamma} \operatorname{length}_{R_\mathfrak{q}}(R_\mathfrak{q}/gR_\mathfrak{q}) \cdot e(R/\mathfrak{q}) && \text{by [48, Lemma 3.2]}\\
&\geqslant \sum_{\mathfrak{q}\in\Gamma} e((g),R_\mathfrak{q}) \cdot e(R/\mathfrak{q}) && \text{by [21, Corollary 1.2.12]}\\
&\geqslant \sum_{\mathfrak{q}\in\Gamma} e(IR_\mathfrak{q},R_\mathfrak{q}) \cdot e(R/\mathfrak{q}) && \text{since } g \in I\\
&= c_1(I,R) && \text{by [48, Proposition 2.4(a)] or Theorem 5.13(ii).}
\end{aligned}$$

Therefore, we obtain the inequality $c_1(I,R/tR) \geqslant c_1(I,R)$ and that $\Gamma_\mathfrak{p} = \Sigma_\mathfrak{p}$ if an equality holds. This concludes the proof of the theorem. ☐

Important consequences of Theorem 5.13 and Theorem 6.3 are the following lexicographical upper-semicontinuity results.

Corollary 6.4. *Assume that* R *is equidimensional and catenary, and let* $t \in \mathfrak{m}$ *be such that* $\dim(R/tR) = d-1$. *Suppose that* $\operatorname{ht}(I, 0 :_R I^\infty, t) \geqslant 2$. *Then we get the inequality*

$$\big(c_1(I, R/tR), c_2(I, R/tR), \ldots, c_{d-1}(I, R/tR)\big) \geqslant_{\text{lex}} \big(c_1(I, R), c_2(I, R), \ldots, c_{d-1}(I, R)\big)$$

under the lexicographical order.

Corollary 6.5. *Assume that* R *is equidimensional and universally catenary. Let* $I \subset J$ *be two* R*-ideals. Then we get the inequality*

$$\big(c_0(I, R), c_1(I, R), \ldots, c_d(I, R)\big) \geqslant_{\text{lex}} \big(c_0(J, R), c_1(J, R), \ldots, c_d(J, R)\big)$$

under the lexicographical order.

Proof. By replacing R by $R[y]_{(\mathfrak{m},y)}$ (where y is a new variable) and the ideals I and J by (I, y) and (J, y), we may assume that $\operatorname{ht}(I) \geqslant 1$ and $\operatorname{ht}(J) \geqslant 1$ and that $c_0(I, R) = 0$ and $c_0(J, R) = 0$; see [48, Corollary 2.2(d)].

Consider the local ring $S = R[t]_{(\mathfrak{m},t)}$ (where t is a new variable). Let $H = IS + tJS \subset S$. Notice that $\operatorname{ht}(H, 0 :_S H^\infty, t) \geqslant \operatorname{ht}(IS, t) \geqslant 2$, $H \cdot S/tS = I$ and $HS_{\mathfrak{m}S} = JR[t]_{\mathfrak{m}R[t]}$. Then we obtain the following inequalities

$$\begin{aligned}
\big(c_1(I, R), c_2(I, R), \ldots, c_d(I, R)\big) &= \big(c_1(H, S/tS), c_2(H, S/tS), \ldots, c_d(H, S/tS)\big) \\
&\geqslant_{\text{lex}} \big(c_1(H, S), c_2(H, S/tS), \ldots, c_d(H, S)\big) && \text{by Corollary 6.4} \\
&\geqslant \big(c_1(H, S_{\mathfrak{m}S}), c_2(H, S_{\mathfrak{m}S}), \ldots, c_d(H, S_{\mathfrak{m}S})\big) && \text{by [48, Proposition 2.7]} \\
&= \big(c_1(J, R), c_2(J, R), \ldots, c_d(J, R)\big)
\end{aligned}$$

that settle the claim of the corollary. □

We now present a principle of specialization of integral dependence that we enunciate similarly to the ones in [25, Theorem 4.7] and [48, Theorem 4.4].

Theorem 6.6 (Principle of Specialization of Integral Dependence – PSID)**.** *Assume Setup 5.1 and that* R *is equidimensional and universally catenary. Let* $t \in \mathfrak{m}$ *be such that* $\dim(R/tR) = d-1$. *Suppose that* $\operatorname{ht}(I, t) \geqslant 2$ *and that* $c_i(I, R/tR) = c_i(I, R)$ *for all* $1 \leqslant i \leqslant d-1$. *Then, for any element* $h \in R$, *the following two conditions are equivalent:*

(i) $h \in \overline{I}$.
(ii) $hR_\mathfrak{p} \in \overline{IR_\mathfrak{p}}$ *for all primes* $\mathfrak{p} \in \operatorname{Spec}(R)$ *such that* $t \notin \mathfrak{p}$.

Proof. Assume momentarily that $\dim(G/tG) < d$. As G is equidimensional of dimension d (see [49, Theorem 3.8], [51, Lemma 2.2]), it follows that the contraction of a minimal prime of G to R does not contain t. This implies that no prime in the set $L(I) := \{\mathfrak{p} \in V(I) \mid \dim(R_\mathfrak{p}) = \ell(IR_\mathfrak{p})\}$ contains t. Since every associated prime of $\overline{I}$ belongs to $L(I)$ (see [45, 3.9 and 4.1]), the result of the theorem follows. Hence it remains to show $\dim(G/tG) < d$.

Since $\operatorname{ht}(I) \geqslant 1$, it follows that $\dim(G/G_+) = \dim(R/I) \leqslant d-1$ and so no minimal prime of G contains G_+. Set $E' := \operatorname{Proj}(G/tG)$. Thus $\dim(E') < d-1$ if and only if $\dim(G/tG) < d$. From [11, Proposition 2.10], we obtain $m_{d-1}^{d-1}(G/tG) = e_{d-1}(H^0(E', \mathcal{O}_{E'}))$ and so it follows that $m_{d-1}^{d-1}(G/tG) =$

0 because $\mathrm{Supp}(H^0(E',\mathcal{O}_{E'})) \subset V(I,t)$ and by assumption we have the inequality $\mathrm{ht}(I,t) \geqslant 2$. We now assume the notation and assumptions of Setup 5.11. From [11, Proposition 2.10] and (3), we get

$$m_{d-1}^{d-1-i}(G/tG) \;=\; m_{d-1-i}^{d-1-i}(G/(\ell_1,\ldots,\ell_i,t)G) \;=\; m_{d-1-i}^{d-1-i}(G_i/tG_i),$$

where $G_i = \mathrm{gr}_{IR_i}(R_i)$ and $R_i = R/(g_1,\ldots,g_i)R$. Therefore, by utilizing Theorem 5.13(ii) and Theorem 6.3, we can prove inductively that

$$m_{d-1}^{d-1-i}(G/tG) = 0 \quad \text{for all} \quad 0 \leqslant i \leqslant d-1.$$

Due to [11, Lemma 2.3], we obtain $\dim(E') < d-1$, and so it follows that $\dim(G/tG) < d$. □

The following theorem contains our new criteria for integral dependence in terms of the invariants $m_i(I,R)$ and $\nu_i(I,R)$.

Theorem 6.7. *Assume Setup 5.1 and that R is equidimensional and universally catenary. Let $I \subset J$ be two R-ideals. Suppose the following two conditions:*

(a) $o(I) = o(J)$.

(b) I *satisfies the $\mathcal{G}$-parameter condition generically* (*see Notation 5.7*).

Then the following are equivalent:

(i) J *is integral over* I.

(ii) $m_i(I,R) = m_i(J,R)$ *for all* $0 \leqslant i \leqslant d-1$.

(iii) $\nu_i(I,R) = \nu_i(J,R)$ *for all* $0 \leqslant i \leqslant d$.

Proof. Let $\delta := o(I) = o(J)$ be the order of I and J.

(i) $\Rightarrow$ (ii) & (iii): If J is integral over I, then it is known that $m_i(I,R) = m_i(J,R)$ and $c_i(I,R) = c_i(J,R)$ (see [58, Corollary 3.8] and [48, Theorem 4.2]). This settles both implications (without assuming the conditions (a) and (b)).

(ii) $\Rightarrow$ (i): Suppose that $m_i(I,R) = m_i(J,R)$ for all $0 \leqslant i \leqslant d-1$. Then Proposition 5.8 yields

$$\nu_i(I,R) \;=\; \delta \cdot m_{i-1}(I,R) \;=\; \delta \cdot m_{i-1}(J,R) \;\leqslant\; \nu_i(J,R)$$

for all $1 \leqslant i \leqslant d$. This implies that $c_i(I,R) = \nu_i(I,R) - m_i(I,R) \leqslant \nu_i(J,R) - m_i(J,R) = c_i(J,R)$ for all $1 \leqslant i \leqslant d$. On the other hand, we have $c_0(I,R) = e(R) - m_0(I,R) = e(R) - m_0(J,R) = c_0(J,R)$. Therefore, J is integral over I due to [48, Theorem 4.2].

(iii) $\Rightarrow$ (i): Suppose that $\nu_i(I,R) = \nu_i(J,R)$ for all $0 \leqslant i \leqslant d$. From Proposition 5.8, we obtain

$$\delta \cdot m_{i-1}(I,R) \;=\; \nu_i(I,R) \;=\; \nu_i(J,R) \;\geqslant\; \delta \cdot m_{i-1}(J,R)$$

for all $1 \leqslant i \leqslant d$. It follows that $c_i(I,R) = \nu_i(I,R) - m_i(I,R) \leqslant \nu_i(J,R) - m_i(J,R) = c_i(J,R)$ for all $0 \leqslant i \leqslant d-1$. We also have $c_d(I,R) = \nu_d(I,R) = \nu_d(J,R) = c_d(J,R)$. Finally, J is integral over I by invoking [48, Theorem 4.2] once again. □

The next remark complements [58, Corollary 4.3].

Remark 6.8. Let R be a regular local ring of dimension at least two and let I be an ideal of height one. Write $I = aH$ where H is an ideal with $\mathrm{ht}(H) \geqslant 2$. Then $m_1(I,R) = o(H)$.

Proof. We may assume that the residue field of R is infinite. Let g be a general element of H. According to Theorem 5.13(i), we have $m_1(I,R) = e(R/agR :_R I^\infty)$. On the other hand,

$$agR : I^\infty = (agR :_R aH) :_R (aH)^\infty = (gR : H) :_R (aH)^\infty = (gR :_R H^\infty) :_R (aR)^\infty = gR.$$

Finally, $e(R/gR) = o(g) = o(H)$. □

The remark above shows that the polar multiplicities of an ideal of height one in a two-dimensional regular local ring only depend on the order of the saturation of the ideal with respect to its unmixed part, and hence cannot detect integral dependence.

Example 6.9 (Polar multiplicities do not detect integral dependence)**.** Let $R = \Bbbk[x_0, x_1]$ be a polynomial ring over a field. Consider the ideals $I = (x_0^2, x_0x_1^2) \subsetneq J = (x_0^2, x_0x_1)$. We have the following table of values:

$c_0(I,R) = 0$	$m_0(I,R) = 1$	$v_0(I,R) = 1$	$c_0(J,R) = 0$	$m_0(J,R) = 1$	$v_0(J,R) = 1$
$c_1(I,R) = 1$	$m_1(I,R) = 1$	$v_1(I,R) = 2$	$c_1(J,R) = 1$	$m_1(J,R) = 1$	$v_1(I,R) = 2$
$c_2(I,R) = 4$	$m_2(I,R) = 0$	$v_2(I,R) = 4$	$c_2(J,R) = 2$	$m_2(J,R) = 0$	$v_2(I,R) = 2.$

Here the invariants m_i fail to detect the fact that $\overline{I} \subsetneq \overline{J}$. However, both invariants c_i and v_i do detect this.

From the previous example, we see that the invariants v_i seem to carry more information related to the integral closure of an ideal. Nevertheless, we have the following more delicate example where the invariants v_i fail to detect integral dependence.

Example 6.10 (Polar-Segre multiplicities do not detect integral dependence)**.** Let $\mathbb{P}^1_{\Bbbk} = \mathrm{Proj}(\Bbbk[s,t]) \to \mathbb{P}^3_{\Bbbk} = \mathrm{Proj}(\Bbbk[x_0, \ldots, x_3])$, $(s : t) \mapsto (s^4 : s^3t : st^3 : t^4)$ be the parametrization of the rational quartic $\mathcal{C} \subset \mathbb{P}^3_{\Bbbk}$. Let $R = \Bbbk[x_0, \ldots, x_3]/\mathfrak{P}$ be the homogeneous coordinate ring of $\mathcal{C}$. Consider the ideals $I = (x_0^2, x_0x_2x_3, x_1^2x_3^2)R \subsetneq J = (x_0^2, x_0x_2, x_0x_3, x_1x_3^2)R$. We get the following table of values:

$c_0(I,R) = 0$	$m_0(I,R) = 4$	$v_0(I,R) = 4$	$c_0(J,R) = 0$	$m_0(J,R) = 4$	$v_0(J,R) = 4$
$c_1(I,R) = 5$	$m_1(I,R) = 3$	$v_1(I,R) = 8$	$c_1(J,R) = 3$	$m_1(J,R) = 5$	$v_1(I,R) = 8$
$c_2(I,R) = 14$	$m_2(I,R) = 0$	$v_2(I,R) = 14$	$c_2(J,R) = 14$	$m_2(J,R) = 0$	$v_2(I,R) = 14.$

Therefore the invariants v_i fail to detect the fact that $\overline{I} \subsetneq \overline{J}$.

6.1. **Lexicographic upper semicontinuity of Segre numbers.**

In this subsection, we show that Segre numbers have a lexicographic upper semicontinuous behavior in families. This generalizes a result due to Gaffney and Gassler [25]. We now use the following setup.

Setup 6.11. Let $\iota : A \hookrightarrow R$ be a flat injective homomorphism of finite type of Noetherian rings. Suppose that the inclusion ι has a section; i.e., there is a homomorphism $\pi : R \twoheadrightarrow A$ such that $\pi \circ \iota = \mathrm{id}_A$. Consider the ideal $Q := \mathrm{Ker}(\pi) \subset R$.

Notice that given prime $\mathfrak{p} \in \mathrm{Spec}(A)$, we always have that $QR(\mathfrak{p})$ is a prime ideal in the fiber $R(\mathfrak{p})$. Indeed, we get the inclusion $\iota_{\mathfrak{p}} : A/\mathfrak{p} \hookrightarrow R/\mathfrak{p}R$ with section $\pi_{\mathfrak{p}} : R/\mathfrak{p}R \twoheadrightarrow A/\mathfrak{p}$, and since $\mathrm{Ker}(\pi_{\mathfrak{p}}) = Q \cdot R/\mathfrak{p}R$, it follows that $Q \cdot R/\mathfrak{p}R$ is a prime ideal satisfying $(R/\mathfrak{p}R)_{Q \cdot R/\mathfrak{p}R} = R(\mathfrak{p})_{QR(\mathfrak{p})}$. Moreover $QR(\mathfrak{p})$ is a maximal ideal in $R(\mathfrak{p})$. Thus, although the fiber $R(\mathfrak{p})$ might not be a local ring, we do get a distinguished local ring $R(\mathfrak{p})_{QR(\mathfrak{p})}$. To simplify notation, we say that $S(\mathfrak{p}) := R(\mathfrak{p})_{QR(\mathfrak{p})}$ is the *distinguished fiber* of $\mathfrak{p}$.

Our main result regarding the behavior of Segre numbers in families is the following.

Theorem 6.12. *Assume Setup 6.11 and let $I \subset R$ be an ideal. Assume that for all $\mathfrak{p} \in \mathrm{Spec}(A)$, the fibers $R(\mathfrak{p})$ are equidimensional of the same dimension d and $\mathrm{ht}(I(\mathfrak{p})) > 0$. Then the function*

$$\mathfrak{p} \in \mathrm{Spec}(A) \;\mapsto\; \big(c_1(I, S(\mathfrak{p})), c_2(I, S(\mathfrak{p})), \ldots, c_d(I, S(\mathfrak{p}))\big) \in \mathbb{Z}^d_{\geqslant 0}$$

is upper semicontinuous with respect to the lexicographic order.

Proof. By the topological Nagata criterion (see Remark 3.2), it suffices to show the following two conditions:

(i) Under the assumption that A is a domain, there is a dense open subset $U \subset \mathrm{Spec}(A)$ where the above function is constant.

(ii) If $\mathfrak{p} \supset \mathfrak{q}$, then $\big(c_1(I, S(\mathfrak{p})), \ldots, S(\mathfrak{p})\big) \geqslant_{\mathrm{lex}} \big(c_1(I, S(\mathfrak{q})), \ldots, c_d(I, S(\mathfrak{q}))\big)$.

First, we prove the claim (i). We assume A is a domain, thus by Grothendieck's Generic Freeness Lemma (see, e.g., [44, Theorem 24.1], [19, Theorem 14.4]) there is an element $0 \neq a \in A$ such that the bigraded components of $\mathrm{gr}_Q(\mathrm{gr}_I(R)) \otimes_A A_a$ and the graded components of $\mathrm{gr}_I(R) \otimes_A A_a$ are free A_a-modules. Set $U := D(a) \subset \mathrm{Spec}(A)$. Then, for any $\mathfrak{p} \in U$, we have the isomorphisms

$$\mathrm{gr}_Q(\mathrm{gr}_I(R)) \otimes_A \kappa(\mathfrak{p}) \;\cong\; \mathrm{gr}_Q(\mathrm{gr}_I(R) \otimes_A \kappa(\mathfrak{p})) \;\cong\; \mathrm{gr}_Q(\mathrm{gr}_I(R(\mathfrak{p}))).$$

This shows that the above function is constant on U.

Next, we prove the claim (ii). Notice that we can reduce modulo $\mathfrak{q}$ and localize at $\mathfrak{p}$. Thus we assume A is a local domain with maximal ideal $\mathfrak{p}$ and $\mathfrak{q} = 0$. Let $K := \mathrm{Quot}(A)$. By [30, Exercise II.4.11] or [28, Proposition 7.1.7], there is a discrete valuation ring V of K that dominates A; that is, $A \subset V$ and $\mathfrak{p} = \mathfrak{n} \cap A$ where $\mathfrak{n} = (t) \subset V$ is the closed point of $\mathrm{Spec}(V)$. Let $R_V := R \otimes_A V$. Notice that $\iota' : V \hookrightarrow R_V$ is flat and has a section $\pi' : R_V \twoheadrightarrow V$ with $\mathrm{Ker}(\pi') = QR_V$. We get a field extension $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{n})$ and the following isomorphisms

$$R_V(\mathfrak{n}) \;\cong\; (R \otimes_A V) \otimes_V \kappa(\mathfrak{n}) \;\cong\; R \otimes_A \kappa(\mathfrak{n}) \;\cong\; (R \otimes_A \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{n}) \;\cong\; R(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{n}).$$

This gives the isomorphisms $I^\nu R_V(\mathfrak{n}) \cong I^\nu R(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{n})$ and so it follows

$$\mathrm{gr}_Q(\mathrm{gr}_I(R_V(\mathfrak{n}))) \;\cong\; \mathrm{gr}_Q(\mathrm{gr}_I(R(\mathfrak{p}))) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{n}). \tag{4}$$

Let $T := (R_V)_{(t,Q)R_V}$. From (4), we have

$$c_i(I, S(\mathfrak{p})) \;=\; c_i(I, T/tT). \tag{5}$$

Since $T_{QT} = R(\mathfrak{q})_{QR(\mathfrak{q})} = S(\mathfrak{q})$, we also obtain

$$c_i(I, S(\mathfrak{q})) \;=\; c_i(I, T_{QT}). \tag{6}$$

To use Corollary 6.4, we now verify that the ideal $IT \subset T$ satisfies the required assumptions. By applying Lemma 6.14(i) to the ring $R(\mathfrak{p})$, we conclude that the fibers of R_V over V are equidimensional of dimension d. Hence due to Lemma 6.14(ii), T is equidimensional of dimension $d+1$ and universally catenary and $\dim(T/tT) = \dim(T) - 1$. Since $R(\mathfrak{p}) \hookrightarrow T/tT$ is a flat homomorphism, $\mathrm{ht}((I,t)T/tT) \geqslant$

$\mathrm{ht}(I(\mathfrak{p})) \geqslant 1$, hence $\mathrm{ht}(t, IT) \geqslant 2$ because t is regular on T. Now (5) and Corollary 6.4 yield

$$\big(c_1(I,S(\mathfrak{p})),\ldots,c_d(I,S(\mathfrak{p}))\big) \;=\; \big(c_1(I,T/tT),\ldots,c_d(I,T/tT)\big) \;\geqslant_{\mathrm{lex}}\; \big(c_1(I,T),\ldots,c_d(I,T)\big).$$

Finally, from [48, Proposition 2.7] and (6), we get

$$\big(c_1(I,T),\ldots,c_d(I,T)\big) \;\geqslant\; \big(c_1(I,T_{QT}),\ldots,c_d(I,T_{QT})\big) \;=\; \big(c_1(I,S(\mathfrak{q})),\ldots,c_d(I,S(\mathfrak{q}))\big).$$

This concludes the proof of the theorem. □

Remark 6.13. A basic situation where the assumptions of Setup 6.11 hold is the following. Assume $R = A[x_1,\ldots,x_d]$ is a positively graded polynomial ring over A, and let $\mathfrak{m} = (x_1,\ldots,x_d)$ be the graded irrelevant ideal. In this case, the natural projection $\pi : R \twoheadrightarrow A \cong R/\mathfrak{m}$ is a section of the inclusion $\iota : A \to R$. Therefore, given a prime $\mathfrak{p} \in \mathrm{Spec}(A)$, the distinguished fiber of $\mathfrak{p}$ is the localization $R(\mathfrak{p})_{\mathfrak{m}R(\mathfrak{p})}$ of the fiber $R(\mathfrak{p}) \cong \kappa(\mathfrak{p})[x_1,\ldots,x_d]$ at the graded irrelevant ideal $\mathfrak{m}R(\mathfrak{p})$.

The following technical lemma was used in the proof of Theorem 6.12.

Lemma 6.14. *The following statements hold:*

(i) *Let B be a finitely generated algebra over a field $\Bbbk$. If B is equidimensional, then $B \otimes_\Bbbk L$ is equidimensional of the same dimension for any field extension L of $\Bbbk$.*

(ii) *Let V be a discrete valuation ring with uniformizing parameter t. Let $j : V \hookrightarrow B$ be a flat injective homomorphism of finite type that has a section $\tau : B \twoheadrightarrow V$, and let $\mathfrak{Q} = \mathrm{Ker}(\tau)$. Let $T = B_{(t,\mathfrak{Q})}$. Suppose that the fibers of j are equidimensional of dimension e. Then T is equidimensional of dimension $e+1$ and universally catenary, and $\dim(T/tT) = e$.*

Proof. (i) By [44, Theorem 23.2], we have $\mathrm{Ass}\,(B \otimes_\Bbbk L) = \bigcup_{\mathfrak{p}\in\mathrm{Ass}(B)} \mathrm{Ass}\,(B/\mathfrak{p} \otimes_\Bbbk L)$. The result follows because each minimal prime of $B/\mathfrak{p} \otimes_\Bbbk \mathbb{L}$ has the same dimension as $B/\mathfrak{p}$ (see [30, Exercise II.3.20(f)]).

(ii) Let $K = \mathrm{Quot}(V)$. Since B is V-flat, it follows that t is a nonzerodivisor on B. Let $\mathfrak{P} \in \mathrm{Spec}(B)$ be a minimal prime contained in $(t,\mathfrak{Q})$. Such $\mathfrak{P}$ does not contain t, hence it corresponds to a minimal in the fiber $B \otimes_V K$. By applying the dimension formula [44, Theorems 15.5, 15.6] to the inclusion $V \hookrightarrow B/\mathfrak{P}$, we obtain

$$\dim\big((B/\mathfrak{P})_{(t,\mathfrak{Q})}\big) \;=\; \dim(V) + \mathrm{trdeg}_V(B/\mathfrak{P}) = 1 + e.$$

This implies that T is equidimensional of dimension $e+1$. As T is finitely generated over a discrete valuation ring, it is universally catenary. Observe that $\dim(T/tT) = e$ since t is regular on T. □

6.2. **The case of equigenerated ideals.**

This brief subsection is dedicated to the family of equigenerated ideals, where our results are particularly satisfying.

Corollary 6.15. *Let $\Bbbk$ be a field, R be a finitely generated standard graded $\Bbbk$-algebra of dimension d and $\mathfrak{m} = R_+$ be the graded irrelevant ideal. Let $I \subset R$ be a homogeneous ideal generated by elements of the same degree $\delta \geqslant 1$. Then the following statements hold:*

(1) $\delta \cdot m_{i-1}(I,R) = \nu_i(I,R)$ *for all* $1 \leqslant i \leqslant d$.

(2) *For any homogeneous ideal $J \subset R$ containing I such that $o(J) = \delta$, the following are equivalent:*

(i) J *is integral over* I.
(ii) $m_i(I,R) = m_i(J,R)$ *for all* $0 \leqslant i \leqslant d-1$.
(iii) $\nu_i(I,R) = \nu_i(J,R)$ *for all* $0 \leqslant i \leqslant d$.

First proof. We may assume that $\mathbb{k}$ is infinite. Notice that part (1) implies part (2) (indeed, by Proposition 5.8(iii), the statement (1) is equivalent to the condition (b) of Theorem 6.7). Since I is equigenerated in degree δ, we see the Rees algebra $\mathscr{R}(I)$ as a standard bigraded algebra with $[\mathscr{R}(I)]_{(\nu,n)} = [I^\nu]_{n+\nu\delta}$. Thus Nakayama's lemma yields the isomorphisms

$$[\mathrm{gr}_{\mathfrak{m}}(\mathscr{R}(I))]_{(\nu,n)} \;=\; \frac{\mathfrak{m}^n I^\nu}{\mathfrak{m}^{n+1} I^\nu} \;\cong\; [I^\nu]_{n+\nu\delta} \;=\; [\mathscr{R}(I)]_{(\nu,n)}.$$

Let g be a general element of I. Since $\left[\left(0 :_{\mathscr{R}(I)} g\right)\right]_{(\nu,*)} = 0$ for $\nu \gg 0$ (see Remark 5.6), it follows that $\left[\left(0 :_{\mathrm{gr}_{\mathfrak{m}}(\mathscr{R}(I))} \mathrm{in}(g)\right)\right]_{(\nu,*)} = 0$ for $\nu \gg 0$. The result follows from Proposition 5.8(iii). ☐

Second proof. Again, we see $\mathscr{R}(I)$ as a standard bigraded algebra with $[\mathscr{R}(I)]_{(\nu,n)} = [I^\nu]_{n+\nu\delta}$. Therefore $\mathrm{gr}_I(I) \cong \mathscr{R}(I)/I\mathscr{R}(I)$ is also naturally a standard bigraded algebra. Notice that we have a short exact sequence

$$0 \to \mathscr{R}(I)(1,-\delta) \to \mathscr{R}(I) \to \mathrm{gr}_I(R) \to 0.$$

Then a basic computation with the polynomial functions $P_{\mathscr{R}(I)}(\nu,n) = \sum_{k=0}^{n} \dim_{\mathbb{k}}\left([\mathscr{R}(I)]_{(\nu,k)}\right)$ and $P_{\mathrm{gr}_I(R)}(\nu,n) = \sum_{k=0}^{n} \dim_{\mathbb{k}}\left([\mathrm{gr}_I(R)]_{(\nu,k)}\right)$ gives the equality

$$\delta \cdot m_{i-1}(I,R) \;=\; m_i(I,R) + c_i(I,R) \;=\; \nu_i(I,R)$$

for all $1 \leqslant i \leqslant d$ (see [11, Remark 2.9]). ☐

As a consequence, in the equigenerated case, we get an alternative proof of the lexicographic upper semicontinuity of Segre numbers (see Theorem 6.12).

Remark 6.16. Let A be a Noetherian domain and $R = A[x_0,\ldots,x_r]$ be a standard graded polynomial ring. Let $\mathcal{F} : \mathbb{P}^r_A \dashrightarrow \mathbb{P}^s_A$ be a rational map with representative $\mathbf{f} = (f_0 : \cdots : f_s)$ such that $f_0,\ldots,f_s$ are homogeneous elements of degree $\delta > 0$. Let $I = (f_0,\ldots,f_s) \subset R$ and assume that $I(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Notice that $d_i(\mathcal{F}(\mathfrak{p})) = m_{r-i}(I,R(\mathfrak{p}))$ for all $0 \leqslant i \leqslant r$ and $\mathfrak{p} \in \mathrm{Spec}(A)$. By Theorem 4.3, each function $\mathfrak{p} \mapsto m_i(I,R(\mathfrak{p}))$ is lower semicontinuous. Notice that we always have $m_0(I,R(\mathfrak{p})) = 1$. Therefore, the equality $c_i(I,R(\mathfrak{p})) = \delta \cdot m_{i-1}(I,R(\mathfrak{p})) - m_i(I,R(\mathfrak{p}))$ implies that the function

$$\mathfrak{p} \in \mathrm{Spec}(A) \;\mapsto\; \left(c_1(I,R(\mathfrak{p})), c_2(I,R(\mathfrak{p})),\ldots,c_{r+1}(I,R(\mathfrak{p}))\right) \in \mathbb{Z}^{r+1}_{\geqslant 0}$$

is upper semicontinuous with respect to the lexicographic order.

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