

WHAT IS THE REES ALGEBRA OF A MODULE?

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ABSTRACT. In this paper we show that the Rees algebra can be made into a functor on modules over a ring in a way that extends its classical definition for ideals. The Rees algebra of a module M may be computed in terms of a “maximal” map f from M to a free module as the image of the map induced by f on symmetric algebras. We show that the analytic spread and reductions of M can be determined from any embedding of M into a free module, and in characteristic 0—but not in positive characteristic!—the Rees algebra itself can be computed from any such embedding.

The Rees algebra of an ideal I in a ring R , namely $\mathcal{R} = \bigoplus_{n=0}^{\infty} I^n = R[It] \subset R[t]$, plays a major role in commutative algebra and in algebraic geometry since $\text{Proj}(\mathcal{R})$ is the blowup of $\text{Spec}(R)$ along the subscheme defined by I . Several authors have found it useful to generalize this construction from ideals to modules; see for instance Gaffney and Kleiman [1999], Katz [1995], Katz and Kodiyalam [1997], Kleiman and Thorup [1997], Kodiyalam [1995], Liu [1998], Rees [1987], Simis, Ulrich, and Vasconcelos [1998, 2000], and Vasconcelos [1994], who define the Rees algebra of a module satisfying one or another hypothesis.

The goal of this paper is to clarify the definition for arbitrary finitely generated modules over a Noetherian ring R . Even for ideals there is a problem: in the depth 0 case it is not clear from the definition above whether the Rees algebra depends on the embedding of I in R . For our project to succeed we must show that it is independent.

It is at first tempting to define the Rees algebra of a module as the symmetric algebra of the module modulo torsion (that is, modulo elements killed by nonzerodivisors of the ring). This does *not* work: it may give the wrong answer even for an ideal, if the ring is not a domain. (In general, it is a good definition only if the module M “has a rank” in the sense that M is free of constant rank locally at the associated primes of R .)

Another alternative is to consider a module M together with an embedding into a free module G , and define the Rees algebra of M to be the subalgebra of the symmetric algebra of G generated by M ; more generally, for any map $g : M \rightarrow N$ we define $\mathcal{R}(g)$, the Rees algebra of g , to be the graded R -algebra which is the image of the map $\text{Sym}(g) : \text{Sym}(M) \rightarrow \text{Sym}(N)$. One may then try to define the Rees algebra of M as $\mathcal{R}(g)$ for an embedding g of M into a free module G .

However, the result may depend on the chosen embedding g . In Section 1 we give as example a principal ideal I in an Artinian ring of characteristic $p > 0$, and an embedding $g : I \rightarrow R^2$ such that $\mathcal{R}(g)$ is not isomorphic to $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n$. Of course $\mathcal{R}(I)$ may also

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be expressed as $\mathcal{R}(i)$, where i is the inclusion of I as an ideal of R . Thus $\mathcal{R}(g)$ depends on g , not only on M .

Here we take a third alternative:

Definition 0.1. If R is a ring and M is an R -module, we define the *Rees algebra* of M to be

$$\mathcal{R}(M) = \text{Sym}(M)/(\cap_g L_g)$$

where the intersection is taken over all maps g from M to free R -modules, and L_g denotes the kernel of $\text{Sym}(g)$.

Although the definition may at first appear mysterious, it is at least obviously functorial: if $h : M \rightarrow N$ is a homomorphism of R -modules then for every map from N to a free module $g : N \rightarrow G$ the map gh is a map from M to a free module, so $\text{Sym}(h) : \text{Sym}(M) \rightarrow \text{Sym}(N)$ induces a map $\mathcal{R}(M) \rightarrow \mathcal{R}(N)$. As the symmetric algebra functor preserves epimorphisms, so does the Rees algebra functor.

In Section 1 we solve the problem of computing $\mathcal{R}(M)$ by showing that (for finitely generated M) $\mathcal{R}(M) = \mathcal{R}(f)$ for any map from M to a free module F such that the dual map $F^* \rightarrow M^*$ is surjective. This implies that forming the Rees algebra of a finitely generated module over a Noetherian ring commutes with flat base change. We also show that if $g : M \rightarrow R$ is an embedding in a free module of rank 1, then $\mathcal{R}(M) = \mathcal{R}(g)$, so that $\mathcal{R}(M)$ agrees with the classical definition for ideals (in particular, this shows that the classical definition is independent of the choice of representation of M as an ideal). Moreover, we show that in many cases $\mathcal{R}(M)$ can be computed from any embedding. The following is a special case of what we prove:

Theorem 0.2. *Let R be a Noetherian ring and let M be a finitely generated R -module. If R is torsion free over \mathbb{Z} , or R is unmixed and generically Gorenstein, or M is free locally at each associated prime of R , then $\mathcal{R}(M) \cong \mathcal{R}(g)$ for any embedding $g : M \rightarrow G$ of M into a free module G .*

In Section 2 we use our Rees algebra construction to introduce analytic spread and integral dependence for arbitrary modules. We prove that for any embedding g of M into a free module, the natural map $\mathcal{R}(M) \rightarrow \mathcal{R}(g)$ has nilpotent kernel. It follows that analytic spread and integral dependence can be computed in $\mathcal{R}(g)$.

In two future papers [2000a, 2000b] we will apply the notions developed here to obtain new generalized principal ideal theorems and results on heights of ideals of minors of a matrix.

1. Rees Algebras

We begin with an example showing that the Rees algebra of a module cannot be defined from an arbitrary embedding in a free module, even when the module is an ideal:

Example 1.1. Let k be a field of characteristic p , let

$$R = k[X, Y, Z]/((X^p, Y^p) + (X, Y, Z)^{p+1}),$$

write x, y, z for the images of X, Y, Z in R , and take M to be the ideal $M = Rz \cong R/(x, y, z)^p$. Write $g_1 : M \rightarrow R$ for the inclusion $M = Rz \subset R$, and let $g_2 : M \rightarrow R^2 = Rt_1 \oplus Rt_2$ be the map sending z to $xt_1 + yt_2$. It is easy to see that g_2 is also an embedding. The algebra $\mathcal{R}(g_1)$ is the same as the classical Rees algebra $\mathcal{R} = \bigoplus_{n=0}^{\infty} (z^n)$, and has p^{th} graded component $(z^p) \neq 0$. On the other hand

$$\mathcal{R}(g_2)_p = R(x^p t_1^p + y^p t_2^p) = 0,$$

and it follows that $\mathcal{R}(g_2)$ cannot surject onto the classical $\mathcal{R} = \mathcal{R}(g_1)$ by any graded homomorphism, so $\mathcal{R}(g_2) \not\cong \mathcal{R}(M)$ as graded rings. (Computation using the computer algebra system Macaulay2 shows that (at least in low characteristics) the vector space dimension of $\mathcal{R}(M)$ is 33, while that of $\mathcal{R}(g_2)$ is 42, so they are not abstractly isomorphic either.)

To compute the Rees algebra of a module, we use the following notion.

Definition 1.2. Let R be a ring and let M be an R -module. We say that a map f from M to a free R -module F is *versal* if every map from M to a free module factors through f .

It follows at once from the definition that if $f : M \rightarrow F$ is versal then the kernel of $\text{Sym}(f)$ maps to zero in $\mathcal{R}(M)$; and thus if $f : M \rightarrow F$ is a versal map to a free R -module F , then $\mathcal{R}(M) = \mathcal{R}(f)$. With a finiteness assumption it is easy to find such a map:

Proposition 1.3. *Let R be a ring, let M be a finitely generated R -module, and let $f : M \rightarrow F$ be a map from M to a free R -module F . If the dual map $F^* \rightarrow M^*$ is an epimorphism then f is versal and $\mathcal{R}(M) = \mathcal{R}(f)$. In particular, formation of the the Rees algebra of a finitely generated module over a Noetherian ring commutes with flat base change.*

Proof. Let $g : M \rightarrow G$ be a map from M to a free R -module. We must show that g factors through f . Since M is finitely generated, g factors through a finitely generated free summand of G , and we may assume that G is finitely generated. It follows that the dual G^* is free. Consequently we may write $g^* = f^*h$. We may also suppose that F is finitely generated. Since F and G are reflexive the desired factorization is $g = h^*f$. The last statement follows from the argument given just before the Proposition.

If R is Noetherian and M is a finitely generated R -module, then the dual f of any epimorphism from a finitely generated free module onto $\text{Hom}_R(M, R) = M^*$ will satisfy the hypothesis of the Proposition. If now S is a flat R -algebra and f is such a map, then because M is finitely presented $S \otimes_R \text{Hom}_R(M, R) = \text{Hom}_S(M, S)$, so $\mathcal{R}(S \otimes_R M) = \mathcal{R}(S \otimes_R f) = S \otimes_R \mathcal{R}(M)$ as required. \square

Example 1.1 continued. By Proposition 1.3 the map g_2 is not versal. It is easy to check that M^* requires 3 generators and a versal map from M to R^3 may be written as

$$f : M = Rz \rightarrow R^3 = Rt_1 \oplus Rt_2 \oplus Rt_3; \quad z \mapsto xt_1 + yt_2 + zt_3.$$

We have

$$\mathcal{R}(f)_p = R(x^p t_1^p + y^p t_2^p + z^p t_3^p) = Rz^p t_3^p = \mathcal{R}(M)_p,$$

as implied by Proposition 1.3.

Next we show that our definition of the Rees algebra agrees with the classical notion for ideals, which is thus independent of the embedding of the ideal in R .

Theorem 1.4. *Let R be a ring, let M be a finitely generated ideal of R , and let $g : M \rightarrow R$ be the inclusion map. The natural map from $\mathcal{R}(M)$ to the classical Rees algebra $\mathcal{R} = \bigoplus_{n=0}^{\infty} I^n = \mathcal{R}(g)$ is an isomorphism.*

Proof. Let $f : M \rightarrow F$ be a versal map to a free module F , of rank n , say. Since f is versal we may find a map $h : F \rightarrow R$ so that $hf = g$. We must show that $\phi := \text{Sym}(h)$ is a monomorphism on the subring of $\text{Sym}(F)$ generated by $f(M)$.

Write $\text{Sym}(F) = R[t_1, \dots, t_n]$, and $\text{Sym}(R) = R[z]$. Let m_1, \dots, m_s be generators of M , and write $a_i = g(m_i) \in R$, so that $h(f(m_i)) = a_i z$. Let $S = R[x_1, \dots, x_s]$ be a polynomial ring, and consider the map $\psi : S \rightarrow \text{Sym}(F)$ sending x_i to $f(m_i)$. We must show that the kernel of ψ is the same as the kernel of $\phi\psi$. Giving each x_i degree 1, the kernel of $\phi\psi$ is homogeneous, so it suffices to show that if $u \in S$ is a form of degree d such that $\phi\psi(u) = 0$ then $\psi(u) = 0$. We do induction on d , the case $d = 0$ being obvious.

We may write $u = \sum_{i=1}^s u_i x_i$ where the u_i are forms of degree $d - 1$. We see that

$$0 = \phi\psi(u) = \phi\psi\left(\sum_i (x_i u_i)\right) = \phi\psi\left(\sum_i (a_i u_i)\right)z,$$

so $\phi\psi(\sum_i (a_i u_i)) = 0$. By our induction hypothesis, $\psi(\sum_i (a_i u_i)) = 0$ too.

We may expand each $\psi(u_i)$ in the form $\psi(u_i) = \sum_{\alpha} r_{i,\alpha} t^{\alpha}$, where the sum runs over all multi-indices α of degree $d - 1$, and thus $\sum_i \sum_{\alpha} a_i r_{i,\alpha} t^{\alpha} = 0$. Since the distinct monomials t^{α} are linearly independent, we have $\sum_i a_i r_{i,\alpha} = 0$ for each α . By our hypothesis that f is an embedding, the $f(m_i)$ satisfy the same linear relations as the a_i , so we get $\sum_i f(m_i) r_{i,\alpha} = 0$ for each α , and finally $\psi(u) = \sum_i \sum_{\alpha} f(m_i) r_{i,\alpha} t^{\alpha} = 0$, as required. \square

Note that a versal map from M to a free module has the same image as the natural map from M to its double dual. This image is called the *torsionless quotient* of M . Any map from M to a free module factors uniquely through the torsionless quotient of M . The following result gives conditions under which the Rees algebra of a torsionless module can be deduced from any inclusion into a free module. For convenience in applications we state it without the torsionless hypothesis.

Theorem 1.5. *Let R be a Noetherian ring, let M be a finitely generated R -module, and let $g : M \rightarrow G$ be a map to a free R -module G inducing an inclusion on the torsionless quotient of M . If for each associated prime Q of R either R_Q is Gorenstein, or M_Q is free, or R_Q is \mathbb{Z} -torsion free, then the natural epimorphism $\mathcal{R}(M) \rightarrow \mathcal{R}(g)$ is an isomorphism.*

Proof. Replacing M by its torsionless quotient, we may assume that $g : M \rightarrow G$ is an inclusion.

Let $f : M \rightarrow F$ be a versal map from M to a free module, and let $\mathcal{R} = \mathcal{R}(M) = \mathcal{R}(f)$ be the Rees algebra of M . Let $h : F \rightarrow G$ be a map with $g = hf$ and let $\phi : \mathcal{R} \rightarrow \mathcal{R}(g)$ be the induced epimorphism. To prove the injectivity of ϕ we may replace R by R_Q , where Q is an associated prime of R .

If R is Gorenstein (and hence Artinian), then free modules are injective, and therefore any monomorphism from a module to a free module is versal. Similarly, if M is free, then since R has depth 0 any monomorphism from M to a free module splits, and thus again is versal. In either case we see that f and g are both versal, and the injectivity of ϕ follows from the functoriality of the Rees algebra.

Finally, we treat the case where R is \mathbb{Z} -torsion free. It suffices to consider the case where $F = G \oplus H$ with H free, and h is the natural projection. Set $J = H \cdot \text{Sym}(F)$, the kernel of $\text{Sym}(h)$. Since g is an inclusion, ϕ is an injection in degree 1, and we must show that ϕ is an injection in every degree, or equivalently $J \cap \mathcal{R} = 0$. This follows from the next Lemma:

Lemma 1.6. *Let R be a ring and let F be a free R -module. Let M be a submodule of F , and let \mathcal{R} be the subalgebra of $\text{Sym}(F)$ generated by M . Let H be a summand of F with $H \cap M = 0$. If $d!$ is a nonzerodivisor in R then $(H \cdot \text{Sym}(F)) \cap \mathcal{R}_d = 0$.*

Proof. It is enough to prove that the Lemma holds after localizing at each maximal ideal of R , so we may assume that R is local. Let t_1, \dots, t_n be a basis of F . Since R is local we may suppose that H is generated by t_{m+1}, \dots, t_n . Set $J = H \cdot \text{Sym}(F)$. We may assume that $d > 1$. Since $(d-1)!$ is a non zerodivisor as well, we know by induction that $J \cap \mathcal{R}_{d-1} = 0$. Writing $\partial_i = \frac{\partial}{\partial t_i}$, one has $\partial_i(\mathcal{R}) \subset \mathcal{R}$ for every i because the R -algebra \mathcal{R} is generated by linear forms, $\partial_i(J) \subset J$ for every $i \leq m$, and $\partial_i(J^2) \subset J$ for every i .

Now if $u \in J \cap \mathcal{R}_d$ then $\partial_i(u) \in J \cap \mathcal{R}_{d-1} = 0$ for every $i \leq m$. Since $d!$ is a non zerodivisor on R it follows that $u \in R[t_{m+1}, \dots, t_n]_d \subset J^2$. Thus $\partial_i(u) \in J \cap \mathcal{R}_{d-1} = 0$ for every i , and hence $u = 0$ because $d!$ is a nonzerodivisor. \square

To connect our definition of the Rees algebra with the torsion in the symmetric algebra, suppose that R be a Noetherian ring, M is a finitely generated R -module, and \mathcal{A} is the R -torsion of $\text{Sym}(M)$. If M_Q is free for every associated prime Q of R and if $f : M \rightarrow F$ is a versal map to a free module, then $\text{Sym}(f)$ induces an isomorphism $\text{Sym}(M)/\mathcal{A} \xrightarrow{\sim} \mathcal{R}(M)$. This is because $\text{Sym}(f)_Q$ is injective for every associated prime Q of R .

If we are only interested in the reduced structure of the Rees algebra of M , then we can compute it from any embedding of the torsionless quotient of M . To prove this we need to identify the minimal primes of $\mathcal{R}(M)$.

Proposition 1.7. *Let R be a Noetherian ring and let M be a finitely generated R -module. There is a one-to-one correspondence between the minimal primes of $\mathcal{R}(M)$ and the minimal primes of R given by $P \mapsto R \cap P$, and similarly for associated primes.*

Proof. Notice that $\mathcal{R}(M)$ is an R -subalgebra of a polynomial ring on finitely many variables over R . Any minimal prime of $\mathcal{R}(M)$ is thus a contraction of a minimal prime of the polynomial ring, and the distinct minimal primes of the polynomial ring contract to the distinct minimal primes of R . The proof for associated primes is similar. \square

Proposition 1.8. *Let R be a Noetherian ring, let M be a finitely generated R -module, and let $g : M \rightarrow G$ be a map to a free R -module G inducing an inclusion on the torsionless quotient of M . The kernel of the natural epimorphism $\mathcal{R}(M) \rightarrow \mathcal{R}(g)$ is nilpotent.*

Proof. By Proposition 1.7 every minimal prime of $\mathcal{R}(M)$ contracts to a minimal prime of R , so it is enough to prove the result after localizing at a minimal prime of R . Thus we may assume that (R, \mathfrak{m}) is Artinian and local. We may also replace M by its torsionless quotient and assume that g is a monomorphism.

Let $f : M \rightarrow F$ be a versal map from M to a free module, and let K be the kernel of the natural epimorphism $\mathcal{R}(M) = \mathcal{R}(f) \rightarrow \mathcal{R}(g)$. Suppose first that M has no free summand. Because R is local, it follows that $\text{Im} f \subset \mathfrak{m}F$. By the versality of f , the map $g : M \rightarrow G$ factors as $g = hf$. The kernel K must be contained in the positive degree part of $\mathcal{R}(M)$, which is contained in $\mathfrak{m}\text{Sym}(F)$. As \mathfrak{m} is nilpotent, K is nilpotent as well.

In the general case, let H be a maximal free submodule of M . Since any inclusion of finitely generated free modules over an Artinian ring splits, we may write $G = G' \oplus H$ and $M = M' \oplus H$ in such a way that $g = g' \oplus 1_H$. The map $\mathcal{R}(M) \rightarrow \text{Sym}(G)$ is obtained from the map $\mathcal{R}(M') \rightarrow \text{Sym}(G')$ induced by g' by adjoining polynomial variables. As the kernel of the latter map is nilpotent, the desired result follows. \square

2. Integral Dependence

In this section we introduce general definitions of integral dependence and of analytic spread for modules that we will apply elsewhere.

Definitions 2.1. Let R be a ring, let M be a finitely generated R -module, and $U \subset L$ submodules of M .

- (1) Let U', L' be the images of U, L in $\mathcal{R}(M)$ and consider the subalgebras $R[U'] \subset R[L'] \subset \mathcal{R}(M)$. We say L is *integral* over U in M if the ring extension $R[U'] \subset R[L']$ is integral.
- (2) We say M is *integral* over U or U is a *reduction* of M , if M is integral over U in M .

In the situation of (1) the Rees algebra of L maps to the Rees algebra of M , so if L is integral over U , then L is integral over U in M .

Theorem 2.2. Let R be a Noetherian ring, let M be a finitely generated R -module, $U \subset L$ submodules of M , let $f : M \rightarrow F$ be a versal map from M to a free R -module. The following are equivalent:

- (1) L is integral over U in M .
- (2) For every minimal prime Q of R , the module L' is integral over U' in M' , where $'$ denotes images in F/QF .
- (3) For every map $M \rightarrow G$ to a free R -module and for every homomorphism $R \rightarrow S$ to a domain S , the module L' is integral over U' in M' , where $'$ denotes tensoring with S and taking images in $S \otimes_R G$.
- (4) For every homomorphism $R \rightarrow V$ to a rank one discrete valuation ring V whose kernel is a minimal prime of R , we have $U' = L'$, where $'$ denotes tensoring with V and taking images in $V \otimes_R F$.

- (5) (*Valuative Criterion of Integrality*) For every map $M \rightarrow G$ to a free R -module and every homomorphism $R \rightarrow V$ to a rank one discrete valuation ring V , we have $U' = L'$, where $'$ denotes tensoring with V and taking images in $V \otimes_R G$.

Proof. By the functoriality of the Rees algebra, we may assume $G = F$ in parts (3) and (5). As $\mathcal{R}(M)$ embeds into $\mathcal{R}(F) = \text{Sym}(F)$, we may replace M by F in (1). In Items (2)–(5), the rings $R/Q, S, V$ are domains, and hence by Theorem 1.5 the embedding of M' into the free modules $F/QF, F \otimes_R S, F \otimes_R V$, respectively, can be used to define $\mathcal{R}(M')$. Thus we may replace M by F in these items as well.

Now it is obvious that (1) implies (3). Part (1) follows from (2) since $\mathcal{R}(F)/\sqrt{0} \subset \prod_Q \mathcal{R}(F/QF)$, where Q ranges over all minimal primes of R and F/QF is considered as a module over R/Q . Finally, the equivalence of (2) and (4) and of (3) and (5) has been shown in Rees [1987, 1.5(ii)]. \square

We see from Theorem 2.2 that our definition of integrality differs from that of Rees [1987, p. 435] when R is not a domain. Rees' definition amounts to saying that for every minimal prime Q of R , the module L' is integral (in our sense or his) over U' in M' , where now $'$ denotes images in M/QM . If, for example, k is a field, $R = k[x]/(x^2)$, and $U = 0 \subset L = M = (x)/(x^2)$, then M is integral over U in our sense but not in the sense of Rees.

Definition 2.3. Let R be a Noetherian local ring with residue field k and let M be a finitely generated R -module. The *analytic spread* $\ell(M)$ of M is the Krull dimension of $k \otimes_R \mathcal{R}(M)$.

By way of illustration, we remark that the analytic spread of a finitely generated module M over an Artinian local ring (R, \mathfrak{m}, k) is equal to the rank r of a maximal free summand H of M (and is also equal to the dimension of the Rees algebra of the module). This is because any homomorphism of M/H to a free module F carries M/H into $\mathfrak{m}F$, which generates a nilpotent ideal of $\text{Sym}(F)$. Thus $(\mathcal{R}(M)/\mathfrak{m}\mathcal{R}(M))_{\text{red}} = \mathcal{R}(M)_{\text{red}}$ is a polynomial ring over k on r variables.

If k is infinite one can show as in the case of ideals, using a homogeneous Noether normalization of $k \otimes_R \mathcal{R}(M)$ and Nakayama's Lemma, that

$$\ell(M) = \min\{\mu(U) \mid U \text{ is a reduction of } M\}.$$

Furthermore, $\ell(M) \leq \mu(M)$ and equality holds if and only if M has no proper reduction.

Proposition 2.4. Let R be a Noetherian local ring with residue field k , let M be a finitely generated R -module, and let $g : M \rightarrow G$ be a map to a free R -module G . If g induces an inclusion on the torsionless quotient of M , then $\ell(M) = \dim k \otimes_R \mathcal{R}(g)$.

Proof. Proposition 1.8 shows that $\mathcal{R}(M)$ differs from $\mathcal{R}(g)$ only by a nilpotent ideal, and thus the same holds after tensoring with k . \square

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