

The structure of the core of ideals^{*}

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Dedicated to the memory of Professor Manfred Herrmann

Abstract The core of an R -ideal I is the intersection of all reductions of I . This object was introduced by D. Rees and J. Sally and later studied by C. Huneke and I. Swanson, who showed in particular its connection to J. Lipman's notion of adjoint of an ideal.

Being an a priori infinite intersection of ideals, the core is difficult to describe explicitly. We prove in a broad setting that: $\text{core}(I)$ is a finite intersection of minimal reductions; $\text{core}(I)$ is a finite intersection of general minimal reductions; $\text{core}(I)$ is the contraction to R of a 'universal' ideal; $\text{core}(I)$ behaves well under faithfully flat extensions. The proofs are based on general multiplicity estimates for certain modules.

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1 Introduction

To study properties of an ideal I of a Noetherian local ring R one often passes to a different ideal, either larger or smaller, which carries most of the information about the original ideal, but has better features. The radical \sqrt{I} of I , and the importance the Nullstellensatz assigns to it, is the most notorious example. The integral closure \bar{I} or a reduction J of I are also very familiar instances.

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We recall that the *radical* \sqrt{I} of I consists of all solutions in R of equations of the form $X^m - a = 0$, with $a \in I$ and m a non negative integer. The *integral closure* \bar{I} of I consists instead of all solutions in R of equations of the form $X^n + b_1X^{n-1} + b_2X^{n-2} + \dots + b_{n-1}X + b_n = 0$, with $b_j \in I^j$ and n a non negative integer. We clearly have $I \subset \bar{I} \subset \sqrt{I}$, with, in general, strict inclusions. Finally, a *reduction* J of I is a subideal of I such that $\bar{J} = \bar{I}$. Equivalently, $J \subset I$ is a reduction of I if $I^{r+1} = JI^r$ for some non negative integer r . Minimal reductions are reductions which are minimal with respect to containment. If the residue field of the ring R is infinite minimal reductions have the same number of generators, namely the *analytic spread* ℓ of I .

A more familiar description is the one of \sqrt{I} as the intersection of all prime ideals containing I or, equivalently, as the intersection of all minimal primes over I . It is well known that this intersection is finite. Also, by work of D. Eisenbud, C. Huneke and W.V. Vasconcelos [6], it is now easy to give an algorithmic approach to \sqrt{I} suitable for effective computer calculations.

On the other hand, reductions of an ideal are highly non unique. Their intersection, dubbed *core* of the ideal I , comes from a more recent vintage. It was studied for the first time by D. Rees and J. Sally [15] and later by C. Huneke and I. Swanson [10], who also showed a connection with work of J. Lipman on the adjoint of an ideal [12]. Being the intersection of an *a priori* infinite number of ideals, this object is difficult to describe in terms of explicit data attached to the ideal. It is known though that $\sqrt{\text{core}(I)} = \sqrt{I}$, see [19] for instance.

The core of an ideal appears naturally in the context of the Briançon–Skoda theorem [13]. In one of its simplest formulations, this theorem says that if R is a regular local ring of dimension d and I is an ideal then $\bar{I}^d \subset J$ for every reduction J of I , or equivalently, $\bar{I}^d \subset \text{core}(I)$.

The issues we address in this paper and to which we give fairly general affirmative answers are: Is the core a finite intersection of minimal reductions of I ? Is the core a finite intersection of general minimal reductions of I ? Is the core the contraction to R of a ‘universal ℓ -generated ideal’? Does the core behave well under faithfully flat extensions? The last question has already been raised by C. Huneke and I. Swanson in [10].

Our results are based on general multiplicity estimates for certain modules (Lemmas 4.1, 4.2 and 4.3), and their proofs use techniques coming from the theory of residual intersections. We are required to introduce and base our constructions on the notions of generic, universal and general ideals. To be more specific, let (R, \mathfrak{m}) be a local Cohen–Macaulay ring with infinite residue field and $I = (f_1, \dots, f_n)$ an R -ideal of height g and analytic spread ℓ . Let X_j

$R[\{X_{jl}\}]_{\mathfrak{m}[\{X_{jl}\}]}$, and consider the S -ideal \mathcal{A} generated by $\sum_{j=1}^n X_{jl}f_j$, $1 \leq l \leq \ell$. This ideal, which we dub a *universal ℓ -generated ideal in IS* , is a minimal reduction of IS . In [15], D. Rees and J. Sally prove that if I is \mathfrak{m} -primary, then $\mathcal{A} \cap R \subset \text{core}(I)$. One of the main results of the present paper says that this containment is actually an equality (Theorem 4.7.b), which remains valid if R is merely Buchsbaum (Remark 4.10). Most notably however, we are able to treat ideals that are not necessarily \mathfrak{m} -primary, such as ideals with $\ell = g$ (called *equimultiple* ideals), generically complete intersection Cohen–Macaulay ideals in a Gorenstein ring with $\ell = g + 1$, or two-dimensional Cohen–Macaulay ideals in a Gorenstein ring which are complete intersections locally on the punctured spectrum. All these ideals fall into the class of *universally weakly $(\ell - 1)$ -residually S_2 ideals satisfying G_ℓ* , a class that provides the framework for this article (see Section 2 for the definition). This assumption is fairly general, it essentially requires the vanishing of $\ell - g$ local cohomology modules. For such ideals we are able to prove that

- (1) $\text{core}(I)$ is the intersection of finitely many general minimal reductions of I (Theorem 4.5);
- (2) $\text{core}(I) = \mathcal{A} \cap R$ (Theorem 4.7.b);
- (3) $\text{core}(IR') = \text{core}(I)R'$ for every flat (not necessarily local) homomorphism $R \rightarrow R'$ of local Cohen–Macaulay rings so that IR' is universally weakly $(\ell - 1)$ -residually S_2 (Theorem 4.8).

The main technical result is the fact that $\text{core}(I)$ can be obtained by intersecting *general* minimal reductions of I . It immediately implies the flat ascent asserted in (3), provided the map is local (Lemma 4.6). This yields (2), which in turn leads to the general case of (3). From the equality in (2) we also deduce an expression for $\text{core}(I)$ as a colon ideal in a polynomial ring over R that allows – at least in principle – for an explicit computation of the core (Proposition 5.4 and Remark 5.5).

The assertions (1) and (2) above are no longer true for arbitrary ideals in Cohen–Macaulay rings (Example 4.11). On the other hand, we are able to prove under fairly weak assumptions that $\text{core}(I)$ is still an intersection of *finitely* many minimal reductions of I , which is far from being obvious for non \mathfrak{m} -primary ideals (Theorem 3.1). In fact we do not know of any examples where this finiteness assertion or the flat ascent as in (3) fail to hold. Thus we are led to ask the following questions, where I is an arbitrary R -ideal and $\mathcal{M}(I)$ the set of its minimal reductions:

- (i) Is $\bigcup_{J \in \mathcal{M}(I)} \text{Ass}_R(I/J)$ finite?
- (ii) Is $\text{core}(I)$ an intersection of finitely many minimal reductions of I ?

- (iii) Is $\text{core}(I) \subset \mathcal{A} \cap R$?
 (iv) Is $\text{core}(IR') \supset (\text{core}(I))R'$ for every flat local homomorphism $R \rightarrow R'$ of local Cohen–Macaulay rings?

Notice that an affirmative answer to (iv) would imply that (iii) holds, and that (ii) and (iv), if valid, would yield the equality $\text{core}(IR') = \text{core}(I)R'$ in the setting of (iv).

We end by remarking that effective ‘closed formulas’ for the computation of $\text{core}(I)$ will appear in another article of ours [5], extending earlier work by C. Huneke and I. Swanson [10]. However, the assumptions on I will be more restrictive than the ones used here and the techniques will be different.

2 Definitions and preliminaries

We begin by reviewing some facts from [4] about residually S_2 ideals. Let R be a Noetherian ring, I an R -ideal of height g , and s an integer. Recall that I satisfies the condition G_s if for each prime ideal \mathfrak{p} containing I with $\dim R_{\mathfrak{p}} \leq s-1$, the minimal number of generators $\mu(I_{\mathfrak{p}})$ is at most $\dim R_{\mathfrak{p}}$. A proper R -ideal K is called an *s-residual intersection* of I if there exists an s -generated ideal $J \subset I$ so that $K = J : I$ and $\text{ht} K \geq s \geq g$. If in addition $\text{ht} I + K \geq s+1$ we say that K is a *geometric s-residual intersection* of I . The ideal I is called *s-residually S_2* (*weakly s-residually S_2*) if R/K satisfies Serre’s condition S_2 for every i -residual intersection (geometric i -residual intersection, respectively) K of I and every $i \leq s$. Finally, whenever R is local, we say I is *universally s-residually S_2* (*universally weakly s-residually S_2*) if IS is residually S_2 (weakly s -residually S_2 , respectively) for every ring $S = R(X_1, \dots, X_n)$ with X_1, \dots, X_n variables over R .

If (R, \mathfrak{m}) is a local Cohen–Macaulay ring of dimension d and I an R -ideal satisfying G_s , then I is universally s -residually S_2 in the following cases:

- (1) I has sliding depth, which means that the i^{th} Koszul homology modules H_i of a generating set f_1, \dots, f_n of I satisfy $\text{depth} H_i \geq d - n + i$ for every i (see [7, 3.3]).
- (2) R is Gorenstein, and the local cohomology modules $H_{\mathfrak{m}}^{d-g-j}(R/I^j)$ vanish for $1 \leq j \leq s - g + 1$ or equivalently $\text{Ext}_R^{g+j}(R/I^j, R) = 0$ for $1 \leq j \leq s - g + 1$ (see [4, 4.1 and 4.3]). The latter condition holds whenever $\text{depth} R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq s - g + 1$.

The depth inequalities in (1) and (2) are satisfied by strongly Cohen–Macaulay ideals, i.e., ideals whose Koszul homology modules are Cohen–Macaulay. The latter condition always holds if I is a Cohen–Macaulay almost complete intersection or a Cohen–Macaulay deviation two ideal of a

Gorenstein ring [2, p. 259]. It is also satisfied for any ideal in the linkage class of a complete intersection [9, 1.11]: standard examples include perfect ideals of grade 2 and perfect Gorenstein ideals of grade 3.

Finally, I is universally s -residually S_2 if $s < g$ or if one of the following conditions hold:

- (3) R is Gorenstein, R/I is Cohen–Macaulay, and $s = g$ (see (2)).
- (4) R is Gorenstein, R/I is Cohen–Macaulay, and $\dim R/I \leq 2$ (see (3) and Lemma 2.1.b below).

Lemma 2.1 *Let R be a Cohen–Macaulay ring, s an integer, and I an R -ideal satisfying G_s .*

- (a) *If I is weakly $(s-1)$ -residually S_2 , then for every $\mathfrak{p} \in V(I)$, $I_{\mathfrak{p}}$ is weakly $(s-1)$ -residually S_2 .*
- (b) *If I is weakly $(s-1)$ -residually S_2 , then every s -residual intersection of I is unmixed of height s .*
- (c) *If I is weakly $(s-1)$ -residually S_2 , then for every s -residual intersection $J: I$ of I with $J \subset I$ and $\mu(J) \leq s$, every associated prime of J has height at most s .*
- (d) *If I is weakly $(s-1)$ -residually S_2 , then for every geometric s -residual intersection $K = J: I$ of I with $J \subset I$ and $\mu(J) \leq s$, $I \cap K = J$.*
- (e) *If I is weakly $(s-2)$ -residually S_2 , then for every flat homomorphism of rings $R \rightarrow R'$ and every s -generated reduction J of IR' , $\text{Supp}_{R'}(IR'/J) = V(\text{Fitt}_s(IR'))$.*
- (f) *If I is weakly $(s-1)$ -residually S_2 , then for every s -generated reduction J of I , $\text{Ass}_R(I/J)$ is empty or consists only of primes of height s .*
- (g) *If I is weakly $(s-2)$ -residually S_2 , then $I = J$ for every $(s-1)$ -generated reduction J of I .*

Proof. Part (a) follows as in [17, the proof of 1.10.a], whereas parts (b), (c) and (d) are identical to [4, 3.4.a, b, c].

To prove part (e) it suffices to show that if $\mathfrak{q} \in V(IR')$ and $\mu(IR'_{\mathfrak{q}}) \leq s$, then $IR'_{\mathfrak{q}} = J'_{\mathfrak{q}}$. So let $\mathfrak{p} = \mathfrak{q} \cap R$. Now $\mu(I_{\mathfrak{p}}) \leq s$ and by part (a), $I_{\mathfrak{p}}$ is still weakly $(s-2)$ -residually S_2 . Thus according to [4, 3.6.b], $I_{\mathfrak{p}}$ can be generated by a d -sequence. Therefore $\mu(I_{\mathfrak{p}}) = \ell(I_{\mathfrak{p}})$ by [8, 3.1] or [18, 3.15], and hence $\mu(IR'_{\mathfrak{q}}) = \ell(IR'_{\mathfrak{q}})$. Since $J_{\mathfrak{q}}$ is a reduction of $IR'_{\mathfrak{q}}$, we conclude that $IR'_{\mathfrak{q}} = J_{\mathfrak{q}}$.

As to part (f) we may assume that $I \neq J$. By (e), $\text{codim Supp}_R(I/J) \geq s$. In particular $J: I$ is an s -residual intersection of I , which according to (c) implies that every prime in $\text{Ass}_R(R/J)$ has height at most s . Now our assertion follows since $\text{Ass}_R(I/J) \subset \text{Supp}_R(I/J) \cap \text{Ass}_R(R/J)$.

Finally we show part (g). Applying (e) with s replaced by $s-1$, we deduce that $\text{codim Supp}_R(I/J) > s-1$, which implies $I = J$ by (f). \square

We are now going to introduce the notions of generic, universal, and general subideals, that will play a crucial role in the sequel. To this end, let R be a Noetherian ring, I an R -ideal, f_1, \dots, f_n a generating sequence of I , and t, s integers. Let $\underline{X} = X_{ijl}$, $1 \leq i \leq t$, $1 \leq j \leq n$, $1 \leq l \leq s$, be variables, $T = R[\underline{X}] = R[\{X_{ijl}\}]$, and \mathcal{B}_i , $1 \leq i \leq t$, the T -ideal generated by $\sum_{j=1}^n X_{ijl} f_j$, $1 \leq l \leq s$. We call $\mathcal{B}_1, \dots, \mathcal{B}_t$ *generic s -generated ideals in IT* . Notice that up to adjoining variables and applying an R -automorphism, this definition does not depend on the choice of n and f_1, \dots, f_n .

Now assume (R, \mathfrak{m}) is local with residue field k . We set $S = T_{\mathfrak{m}T}$ and $\mathcal{A}_i = \mathcal{B}_i S$, and call $\mathcal{A}_1, \dots, \mathcal{A}_t$ *universal s -generated ideals in IS* . Furthermore we say that $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ are *general s -generated ideals in I* if $\mathfrak{a}_i \subset I$ are ideals with $\mu(\mathfrak{a}_i) = s$, $\mathfrak{a}_i \otimes_R k \hookrightarrow I \otimes_R k$, and the point $(\mathfrak{a}_1 \otimes_R k, \dots, \mathfrak{a}_t \otimes_R k)$ lies in some dense open subset of the product of Grassmannians $\times^t \mathbb{G}(s, I \otimes_R k)$.

Now let $\underline{\lambda} = \lambda_{ijl}$, $1 \leq i \leq t$, $1 \leq j \leq n$, $1 \leq l \leq s$ be elements of R , and consider the maximal ideal $M = (\mathfrak{m}, \underline{X} - \underline{\lambda}) = (\mathfrak{m}, \{X_{ijl} - \lambda_{ijl}\})$ of T . We will identify the set $\{M = (\mathfrak{m}, \underline{X} - \underline{\lambda}) \mid \underline{\lambda} \in R^{tns}\}$ with the set of k -rational points of the affine space \mathbb{A}_k^{tns} . Write $\pi = \pi_{\underline{\lambda}} : T \rightarrow R$ for the homomorphism of R -algebras with $\pi(X_{ijl}) = \lambda_{ijl}$. The kernel of π is generated by the T -regular sequence $\underline{X} - \underline{\lambda}$. Now $\pi(\mathcal{B}_1), \dots, \pi(\mathcal{B}_t)$ is a sequence of s -generated ideals in I , whose images in $I \otimes_R k$ only depend on M . Conversely, every sequence of s -generated ideals in I is obtained in this way. As $\underline{X} - \underline{\lambda}$ form a regular sequence modulo every power of IT , Nakayama's Lemma shows that $\pi(\mathcal{B}_i)$ is a reduction of I if and only if $(\mathcal{B}_i)_M$ is a reduction of IT_M . Since the latter condition is equivalent to $M \notin \bigcap_{r \geq 0} \text{Supp}_T(I^{r+1}T/\mathcal{B}_i I^r)$,

the set of all M for which $(\mathcal{B}_i)_M$ is a reduction of IT_M is open in \mathbb{A}_k^{tns} .

Finally, we write $\mathcal{M}(I)$ for the set of all minimal reductions of I , and we define

$$\mathcal{P}(I) = \bigcup_{J \in \mathcal{M}(I)} \text{Ass}_R(I/J)$$

and

$$\gamma(I) = \inf \{t \mid \text{core}(I) = \bigcap_{i=1}^t J_i \text{ with } J_i \in \mathcal{M}(I)\}.$$

3 Finiteness

If I is \mathfrak{m} -primary, then $\text{core}(I)$ is \mathfrak{m} -primary and $\gamma(I) \leq \text{type}(R/\text{core}(I))$. In particular $\gamma(I)$ is finite. The next result establishes this finiteness in a much broader setting.

Theorem 3.1 *Let R be a Noetherian local ring with infinite residue field and I an R -ideal. Assume that $\mathcal{P}(I)$ is finite, every element of $\mathcal{P}(I)$ is minimal in this set, and $\mu(I_{\mathfrak{q}}) = \ell(I_{\mathfrak{q}})$ for every $\mathfrak{q} \in \text{Spec}(R) \setminus \overline{\mathcal{P}(I)}$. Then $\gamma(I) < \infty$.*

Proof. Let K be the intersection of the prime ideals in $\mathcal{P} = \mathcal{P}(I)$. We first prove that $K \subset \sqrt{\text{core}(I)} : I$, or equivalently that there exists a fixed integer r so that $K^r I \subset J$ for every $J \in \mathcal{M} = \mathcal{M}(I)$.

Let \mathfrak{m} be the maximal ideal of R , $k = R/\mathfrak{m}$, $\ell = \ell(I)$, \mathcal{B} a generic ℓ -generated ideal in IT , and $U \subset \mathbb{A}_k^{\ell}$ the open set consisting of all $M = (\mathfrak{m}, \underline{X} - \underline{\lambda}) \subset T$ such that \mathcal{B}_M is a reduction of IT_M . Fix $M \in U$ and let Q be any prime ideal of T with $K \not\subset Q \subset M$. Writing $\mathfrak{q} = Q \cap R$ we have $\mathfrak{q} \in \text{Spec}(R) \setminus \overline{\mathcal{P}}$. Thus $\mu(I_{\mathfrak{q}}) = \ell(I_{\mathfrak{q}})$ and hence $\mu(IT_{\mathfrak{q}}) = \ell(IT_{\mathfrak{q}})$. Since \mathcal{B}_Q is a reduction of IT_Q we conclude that $IT_Q = \mathcal{B}_Q$. Therefore $K^n IT_M \subset \mathcal{B}_M$ for some integer n . Now $K^n IT_N \subset \mathcal{B}_N$ for every N in some open neighborhood of M in U . As U is quasicompact there exists an integer r so that $K^r IT_N \subset \mathcal{B}_N$ for every $N \in U$. Specializing we conclude that $K^r I \subset J$ for every $J \in \mathcal{M}$.

Now for every $\mathfrak{p} \in \mathcal{P}$, since \mathfrak{p} is minimal in the finite set \mathcal{P} , $\mathfrak{p}^r I_{\mathfrak{p}} = K^r I_{\mathfrak{p}} \subset (\text{core}(I))_{\mathfrak{p}}$ and hence $\text{length}_{R_{\mathfrak{p}}}((I/\text{core}(I))_{\mathfrak{p}}) < \infty$. Again as \mathcal{P} is finite there exist finitely many minimal reductions J_1, \dots, J_t so that for every

$\mathfrak{p} \in \mathcal{P}$, $\bigcap_{i=1}^t (J_i/\text{core}(I))_{\mathfrak{p}} = \bigcap_{J \in \mathcal{M}} (J/\text{core}(I))_{\mathfrak{p}}$. Thus $(\bigcap_{i=1}^t J_i)_{\mathfrak{p}} \subset J_{\mathfrak{p}}$ for every

$J \in \mathcal{M}$ and every $\mathfrak{p} \in \mathcal{P}$. Hence by the definition of \mathcal{P} , $\bigcap_{i=1}^t J_i \subset J$, which

gives $\text{core}(I) = \bigcap_{i=1}^t J_i$. □

Remark 3.2 The assumptions on $\mathcal{P}(I)$ in Theorem 3.1 are automatically satisfied in any of the following cases, where $g = \text{ht}(I)$ and $\ell = \ell(I)$:

- Locally on the punctured spectrum of R , I is generated by analytically independent elements.
- R satisfies S_{g+1} and I is equimultiple.
- R is Cohen–Macaulay and I is G_{ℓ} and weakly $(\ell - 1)$ -residually S_2 (see Lemma 2.1.e,f,g).

In either case $\mathcal{P}(I) = \text{Min}(\text{Fitt}_\ell(I)) = \text{Ass}_R(I/J)$, where J is any minimal reduction of I .

4 Genericity and the shape of the core

The crucial result of this section is Theorem 4.5, which describes the core as a finite intersection of *general* minimal reductions. To prove it we compare the multiplicities of modules defined by intersecting reduction ideals, universal ideals, and general ideals, respectively. This is done in Lemma 4.3, which in turn is based on Lemmas 4.1 and 4.2. There we deal with modules defined by a single universal ideal (Lemma 4.1) and a single minimal reduction (Lemma 4.2), respectively. For the latter we prove that under suitable assumptions, the multiplicity of I/J is independent of the choice of a minimal reduction J of I , a fact reminiscent of the theme of [4].

If R is a Noetherian local ring and E a finitely generated R -module, we denote by $e(E)$ the multiplicity of E , by $e_I(E)$ the multiplicity with respect to an ideal of definition I of E , and by $e(\underline{y}; E)$ the multiplicity with respect to a system of parameters \underline{y} of E .

Lemma 4.1 *Let R be a Noetherian local ring, I an R -ideal, and s an integer. Let $J \subset I$ be an s -generated ideal and \mathcal{A} a universal s -generated ideal in IS . If $\dim I/J \leq \dim IS/\mathcal{A}$ then $e(I/J) \geq e(IS/\mathcal{A})$.*

Proof. We may assume that the residue field of R is infinite. Let \mathfrak{m} be the maximal ideal of R , and \mathcal{B} a generic s -generated ideal in IT so that $\mathcal{A} = \mathcal{B}S$. Using the notation introduced in Section 2 we have $J = \pi_{\underline{\lambda}}(\mathcal{B})$ for some $\underline{\lambda} \in R^{ms}$. Write $M = (\mathfrak{m}, \underline{X} - \underline{\lambda}) \subset T$. First notice that $I/J \simeq (IT/\mathcal{B} + (\underline{X} - \underline{\lambda})IT)_M$ as $\underline{X} - \underline{\lambda}$ form a regular sequence on $(T/IT)_M$. In particular $\dim I/J \geq \dim (IT/\mathcal{B})_M - ns$. On the other hand, $IS/\mathcal{A} = (IT/\mathcal{B})_{\mathfrak{m}T}$ and we may assume $\mathfrak{m}T \in \text{Supp}(IT/\mathcal{B})$. Thus $\dim (IT/\mathcal{B})_M - ns = \dim (IT/\mathcal{B})_M - \dim (T/\mathfrak{m}T)_M \geq \dim (IT/\mathcal{B})_{\mathfrak{m}T}$, and our assumption implies $\dim I/J = \dim (IT/\mathcal{B})_M - ns = \dim (IT/\mathcal{B})_{\mathfrak{m}T}$.

Hence there exists a sequence \underline{y} in \mathfrak{m} so that \underline{y} is a system of parameters of I/J and of $(IT/\mathcal{B})_{\mathfrak{m}T}$, \underline{y} generates a minimal reduction of \mathfrak{m}/J : I , and $\underline{y}, \underline{X} - \underline{\lambda}$ form a system of parameters of $(IT/\mathcal{B})_M$. Therefore $e(I/J) = e(\underline{y}; I/J) \geq e(\underline{y}, \underline{X} - \underline{\lambda}; (IT/\mathcal{B})_M) \geq e(\underline{y}; (IT/\mathcal{B})_{\mathfrak{m}T})$ where the last inequality holds by the associativity formula for multiplicities as $\dim (T/\mathfrak{m}T)_M = ns$ (see [14, 24.7]). Finally $e(\underline{y}; (IT/\mathcal{B})_{\mathfrak{m}T}) \geq e(IS/\mathcal{A})$. \square

Lemma 4.2 *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ . Assume that I satisfies G_ℓ , but not $G_{\ell+1}$, and I is weakly $(\ell - 2)$ -residually S_2 . Then $e(I/J)$ and $e(R/J; I)$ are independent of the minimal reduction J of I .*

Proof. Let J be any minimal reduction of I . By Lemma 2.1.e, $\text{Supp}(I/J) = V(\text{Fitt}_\ell(I))$, and by our assumption the latter set has codimension ℓ . By the associativity formula for multiplicities we may localize at any minimal prime in $\text{Supp}(I/J) = V(\text{Fitt}_\ell(I))$ of codimension ℓ to assume that $\dim R = \ell$. By Lemma 2.1.a, I is still weakly $(\ell - 2)$ -residually S_2 . Notice that now $\dim I/J = \dim R/J: I = 0$.

We may suppose $\ell > 0$. Let \mathfrak{a} and \mathfrak{b} be minimal reductions of I . By a general position argument (see, e.g., [17, the proof of 1.4]) there are generating sequences a_1, \dots, a_ℓ of \mathfrak{a} and b_1, \dots, b_ℓ of \mathfrak{b} so that for every $1 \leq i \leq \ell$, $(a_1, \dots, a_i, b_{i+1}, \dots, b_\ell)$ is a minimal reduction of I and

$$K = (a_1, \dots, a_{i-1}, b_{i+1}, \dots, b_\ell): I$$

is a geometric $(\ell - 1)$ -residual intersection of I . Write $\mathfrak{c} = (a_1, \dots, a_{i-1}, b_{i+1}, \dots, b_\ell)$. It suffices to prove that

$$e(I/(\mathfrak{c}, a_i)) = e(I/(\mathfrak{c}, b_i)) \quad \text{and} \quad e(R/(\mathfrak{c}, a_i): I) = e(R/(\mathfrak{c}, b_i): I).$$

To this end let ‘ $\bar{}$ ’ denote images in $\bar{R} = R/K$. Since $I \cap K = \mathfrak{c}$ by Lemma 2.1.d, it follows that $\bar{I}/(\bar{a}_i) = I/(I \cap K, a_i) = I/(\mathfrak{c}, a_i)$ and hence $(\bar{a}_i): \bar{I} = (\mathfrak{c}, a_i): I$. The same holds for b_i in place of a_i . Thus it suffices to show $e(\bar{I}/(\bar{a}_i)) = e(\bar{I}/(\bar{b}_i))$ and $e(\bar{R}/(\bar{a}_i): \bar{I}) = e(\bar{R}/(\bar{b}_i): \bar{I})$. Obviously, (\bar{a}_i) and (\bar{b}_i) are reductions of \bar{I} , and by Lemma 2.1.b, \bar{a}_i and \bar{b}_i are non zerodivisors on \bar{R} and $\dim \bar{R} = 1$.

After changing notation, we are reduced to proving that if (R, \mathfrak{m}) is a one-dimensional local Cohen–Macaulay ring with infinite residue field and I is an \mathfrak{m} -primary ideal, then $\text{length}(I/J)$ and $\text{length}(R/J: I)$ do not depend on the minimal reduction J of I . Notice that $\text{length}(R/J) = e_J(R) = e_I(R)$. This gives $\text{length}(I/J) = e_I(R) - \text{length}(R/I)$. If a is an R -regular element generating J , we have $I^{-1}/R \simeq aI^{-1}/(a)$ and $aI^{-1} = J: I$. This yields an exact sequence

$$0 \rightarrow I^{-1}/R \rightarrow R/J \rightarrow R/J: I \rightarrow 0,$$

which shows that $\text{length}(R/J: I) = e_I(R) - \text{length}(I^{-1}/R)$. \square

Lemma 4.3 *Let R be a local Cohen–Macaulay ring with infinite residue field, I an R -ideal of analytic spread ℓ , and J_1, \dots, J_t minimal reductions of I . Assume that I is G_ℓ , but not $G_{\ell+1}$, and I is weakly $(\ell - 2)$ -residually S_2 . Then*

$$e(I/J_1 \cap \dots \cap J_t) \leq e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t) = e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)$$

and $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ are minimal reductions of I , for $\mathcal{A}_1, \dots, \mathcal{A}_t$ universal ℓ -generated ideals in IS and $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ general ℓ -generated ideals in I .

Proof. We may assume that $n = \mu(I) > \ell > 0$. Write $d = \dim(R/\text{Fitt}_\ell(I))$. By Lemma 2.1.e,

$$\text{Supp}_{R'}(IR'/J) = V(\text{Fitt}_\ell(I)R') \quad (4.1)$$

for any flat homomorphism of rings $R \rightarrow R'$ and any ℓ -generated reduction J of IR' . Let $\mathcal{B}_1, \dots, \mathcal{B}_t$ be generic ℓ -generated ideals in IT , defined using a minimal generating sequence of I . We may suppose that $\mathcal{A}_i = \mathcal{B}_i S$.

We first prove the equality $e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t) = e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)$. We may pass from R to \widehat{R} and assume that (R, \mathfrak{m}, k) is complete. Although the residually S_2 assumption may not be preserved by completion, Lemma 2.1.e shows that (4.1) still holds. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal primes of $\text{Fitt}_\ell(I)$ having maximal dimension, namely d . Let C be a coefficient ring of R and write $C_j = C/\mathfrak{p}_j \cap C$, $k_j = \text{Quot}(C_j)$ for $1 \leq j \leq s$.

For $1 \leq i \leq t$ consider the T -modules $E_i = IT/\mathcal{B}_1 \cap \dots \cap \mathcal{B}_i + \mathcal{B}_{i+1}$, where $\mathcal{B}_{t+1} = 0$. By (4.1), $\dim(E_i)_{\mathfrak{p}_j T} = 0$ and hence $(E_i)_{\mathfrak{p}_j T}$ are Cohen–Macaulay. Thus the set of all maximal ideals $(\mathfrak{p}_j, \underline{X} - \underline{\lambda})$ of $T_{\mathfrak{p}_j}$ with $\underline{\lambda} \in C_{\mathfrak{p}_j \cap C}^{tn\ell}$, so that the modules $(E_i)_{(\mathfrak{p}_j, \underline{X} - \underline{\lambda})}$ are Cohen–Macaulay for $1 \leq i \leq t$, form a dense open subset of $\mathbb{A}_k^{tn\ell}$. Let H_j be the largest ideal of $k_j[\underline{X}]$ defining the complement of this subset, and let \overline{H}_j denote the image of $H_j \cap C_j[\underline{X}]$ in $k[\underline{X}]$. Since C_j is either a field or a discrete valuation ring, it follows that $\overline{H}_j \neq 0$ and hence $U_j = D(\overline{H}_j)$ is a dense open subset of $\mathbb{A}_k^{tn\ell}$. Let M be a maximal ideal in T of the form $M = (\mathfrak{m}, \underline{X} - \underline{\lambda})$ with $\underline{\lambda} \in R^{tn\ell}$. As C is a coefficient ring of R we may assume that $\underline{\lambda} \in C^{tn\ell}$. If the image of $\underline{\lambda}$ in $\mathbb{A}_k^{tn\ell}$ lies in $U_j = D(\overline{H}_j)$, then the image of $\underline{\lambda}$ in $\mathbb{A}_{k_j}^{tn\ell}$ belongs to $D(H_j)$ since C_j maps onto k . Thus $(E_i)_{(\mathfrak{p}_j, \underline{X} - \underline{\lambda})}$ are Cohen–Macaulay for $1 \leq i \leq t$, whenever $M = (\mathfrak{m}, \underline{X} - \underline{\lambda})$ lies in U_j . Finally, let U_0 be the dense open subset of $\mathbb{A}_k^{tn\ell}$ so that $\pi_{\underline{\lambda}}(\mathcal{B}_i)$ are reductions of I for $1 \leq i \leq t$. Define U to be the dense open subset $U_0 \cap \dots \cap U_s$ of $\mathbb{A}_k^{tn\ell}$. The natural action of $\times GL_\ell(k)$ on $\mathbb{A}_k^{tn\ell} = \times \text{Hom}_k(k^\ell, k^n)$ induces an action on U , and so the image V of U in the product of Grassmannians $\times \mathbb{G}(\ell, k^n)$ is open and dense. It remains to show that $e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t) = e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)$ whenever $(\mathfrak{a}_1 \otimes_R k, \dots, \mathfrak{a}_t \otimes_R k) \in V$.

So let $(\mathfrak{a}_1 \otimes_R k, \dots, \mathfrak{a}_t \otimes_R k) \in V$. Write $\mathfrak{a}_i = \pi_{\underline{\lambda}}(\mathcal{B}_i)$ where $\underline{\lambda} \in C^{tn\ell}$, $M = (\mathfrak{m}, \underline{X} - \underline{\lambda})T$, and $Q_j = (\mathfrak{p}_j, \underline{X} - \underline{\lambda})T$. By the above \mathfrak{a}_i are reductions of I , $(\mathcal{B}_i)_M$ are reductions of IT_M , and $(E_i)_{Q_j}$ are Cohen–Macaulay modules for $1 \leq i \leq t$, $1 \leq j \leq s$. Since these modules have dimension $tn\ell$ and are annihilated by a power of \mathfrak{p}_j by (4.1), it then follows that $\underline{X} - \underline{\lambda}$ forms a regular sequence on them. Now an induction on i yields $(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)_{\mathfrak{p}_j} \simeq (IT/\mathcal{B}_1 \cap \dots \cap \mathcal{B}_t + (\underline{X} - \underline{\lambda})IT)_{Q_j}$. Furthermore, the latter module

has length equal to $e(\underline{X} - \underline{\lambda}; (IT/\mathcal{B}_1 \cap \dots \cap \mathcal{B}_t)_{Q_j})$. Since $\text{Supp}(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t) = V(\text{Fitt}_\ell(I))$ by (4.1), the associativity formula for multiplicities now yields

$$\begin{aligned} e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t) &= \sum_{j=1}^s \lambda_{R_{\mathfrak{p}_j}}((I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)_{\mathfrak{p}_j}) \cdot e(R/\mathfrak{p}_j) \\ &= \sum_{j=1}^s e(\underline{X} - \underline{\lambda}; (IT/\mathcal{B}_1 \cap \dots \cap \mathcal{B}_t)_{Q_j}) \cdot e((T/Q_j)_M). \end{aligned}$$

To further evaluate this sum, write $E = E_t = IT/\mathcal{B}_1 \cap \dots \cap \mathcal{B}_t$. By (4.1), $\text{Supp}(E_M) = V(\text{Fitt}_\ell(I)T_M)$, which has dimension $d + t n \ell$. For r an integer with $\text{Fitt}_\ell(I)^r \subset \text{ann}(E_M)$, let $\underline{y} = y_1, \dots, y_d$ be a sequence of elements generating a minimal reduction of $\mathfrak{m}/\text{Fitt}_\ell(I)$. Now $\underline{y}, \underline{X} - \underline{\lambda}$ form a system of parameters of E_M . Furthermore in the ring $(T_M/\text{ann}(E))_M$, $(\underline{X} - \underline{\lambda})$ generate an ideal of height $t n \ell$ and dimension d , and \underline{y} generate an ideal of height d and dimension $t n \ell$. The minimal primes of maximal dimension of the first ideal are $Q_1 T_M, \dots, Q_s T_M$, whereas the second ideal has only one minimal prime, $\mathfrak{m} T_M$. Thus the associativity formula (see [14, 24.7]) yields

$$\begin{aligned} \sum_{j=1}^s e(\underline{X} - \underline{\lambda}; E_{Q_j}) \cdot e((T/Q_j)_M) &= \\ &= \sum_{j=1}^s e(\underline{X} - \underline{\lambda}; E_{Q_j}) \cdot e(\underline{y}; (T/Q_j)_M) \\ &= e(\underline{y}, \underline{X} - \underline{\lambda}; E_M) = e(\underline{y}; E_{\mathfrak{m}T}) \cdot e(\underline{X} - \underline{\lambda}; (T/\mathfrak{m}T)_M) \\ &= e(E_{\mathfrak{m}T}) \cdot 1 = e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t). \end{aligned}$$

This completes the proof of the equality $e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t) = e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)$.

We now show the inequality $e(I/J_1 \cap \dots \cap J_t) \leq e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t)$. Writing $\mathcal{J} = J_1 S \cap \dots \cap J_{i-1} S \cap \mathcal{A}_{i+1} \cap \dots \cap \mathcal{A}_t$ with $1 \leq i \leq t$, it suffices to show that $e(IS/\mathcal{J} \cap J_i S) \leq e(IS/\mathcal{J} \cap \mathcal{A}_i)$. Consider the two exact sequences

$$0 \rightarrow IS/\mathcal{J} \cap J_i S \rightarrow IS/\mathcal{J} \oplus IS/J_i S \rightarrow IS/\mathcal{J} + J_i S \rightarrow 0$$

$$0 \rightarrow IS/\mathcal{J} \cap \mathcal{A}_i \rightarrow IS/\mathcal{J} \oplus IS/\mathcal{A}_i \rightarrow IS/\mathcal{J} + \mathcal{A}_i \rightarrow 0.$$

By (4.1), the S -modules occurring in these sequences all have the same dimension, namely d . Applying the equality just proved with $t = 1$, gives $e(IS/\mathcal{A}_i) = e(I/\mathfrak{a}_i)$, whereas Lemma 4.2 yields $e(I/\mathfrak{a}_i) = e(I/J_i)$. Thus $e(IS/\mathcal{A}_i) = e(IS/J_i S)$. Finally, by Lemma 4.1, $e(IS/\mathcal{J} + J_i S) \geq e(IS/\mathcal{J} + \mathcal{A}_i)$, and the asserted inequality follows from the above exact sequences. \square

Remark 4.4 The assumption of I not satisfying $G_{\ell+1}$ in Lemmas 4.2 and 4.3 can be omitted if I is weakly $(\ell - 1)$ -residually S_2 ; for otherwise $J = I$ by Lemma 2.1.g.

Theorem 4.5 *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ . Assume that I is G_ℓ and weakly $(\ell - 1)$ -residually S_2 , and write $t = \gamma(I)$. Then $\text{core}(I) = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t$ for $\mathfrak{a}_1, \dots, \mathfrak{a}_t$ general ℓ -generated ideals in I which are reductions of I .*

Proof. According to Theorem 3.1 and Remark 3.2, $\gamma(I) < \infty$. Hence by Lemma 2.1.f every associated prime of the R -module $I/\text{core}(I)$ has height ℓ , and the same holds for $I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t$. On the other hand Lemma 4.3 and Remark 4.4 show that $e(I/\text{core}(I)) \leq e(I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t)$. Now as $I/\text{core}(I)$ maps onto $I/\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t$, these two modules are equal. \square

Using Theorem 4.5, we can now prove that the formation of the core is compatible with flat local maps:

Lemma 4.6 *Let $R \hookrightarrow R'$ be a flat local extension of local rings with infinite residue fields. Assume R' is Cohen–Macaulay. Let I be an R -ideal of analytic spread ℓ such that IR' is G_ℓ and weakly $(\ell - 1)$ -residually S_2 . Then $\text{core}(IR') = (\text{core}(I))R'$ and $\gamma(IR') = \gamma(I)$.*

Proof. Notice that R is Cohen–Macaulay and I is G_ℓ and weakly $(\ell - 1)$ -residually S_2 . By Theorem 3.1, $\text{core}(I) = \bigcap_{i=1}^s J_i$ for finitely many reductions J_1, \dots, J_s of I . Thus $(\text{core}(I))R' = \bigcap_{i=1}^s (J_i R')$. As $J_i R'$ are reductions of IR' , it follows that $(\text{core}(I))R' \supset \text{core}(IR')$.

To show the other inclusion, let $k \subset K$ be the residue field extension of $R \hookrightarrow R'$. By Theorem 4.5, $\text{core}(IR')$ is the intersection of $t = \gamma(IR')$ general ℓ -generated ideals in IR' which are reductions of IR' . On the other hand every dense open subset of $\times \mathbb{G}(\ell, IR' \otimes_{R'} K) = \times \mathbb{G}(\ell, K^n)$ intersects with $\times \mathbb{G}(\ell, I \otimes_R k) = \times \mathbb{G}(\ell, k^n)$, since k is infinite. Thus $\text{core}(IR') = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t$ where \mathfrak{a}_i are minimal reductions of IR' extended from R -ideals \mathfrak{b}_i . Now \mathfrak{b}_i are minimal reductions of I , which gives $\text{core}(I) \subset \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_t$. Hence $(\text{core}(I))R' \subset \mathfrak{b}_1 R' \cap \dots \cap \mathfrak{b}_t R' = \text{core}(IR')$ and thus $(\text{core}(I))R' = \mathfrak{b}_1 R' \cap \dots \cap \mathfrak{b}_t R' = \text{core}(IR')$. In particular $(\text{core}(I))R' = (\mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_t)R'$, hence $\text{core}(I) = \mathfrak{b}_1 \cap \dots \cap \mathfrak{b}_t$, showing $\gamma(I) \leq \gamma(IR')$. The inequality $\gamma(I) \geq \gamma(IR')$ is obvious since $(\text{core}(I))R' = \text{core}(IR')$. \square

Lemmas 4.3 and 4.6 allow us to prove one of our main results:

Theorem 4.7 *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ . Assume that I is G_ℓ and universally weakly $(\ell - 1)$ -residually S_2 .*

- (a) *Let $\mathcal{A}_1, \dots, \mathcal{A}_t$ be universal ℓ -generated ideals in IS . Then $(\text{core}(I))S = \text{core}(IS) = \mathcal{A}_1 \cap \dots \cap \mathcal{A}_t$ if and only if $t \geq \gamma(I) = \gamma(IS)$.*
(b) *Let \mathcal{A} be a universal ℓ -generated ideal in $IR(X)$. Then $\text{core}(I) = \mathcal{A} \cap R$.*

Proof. (a) If $\text{core}(IS) = \mathcal{A}_1 \cap \dots \cap \mathcal{A}_t$ then $\gamma(IS) \leq t$ and so by Lemma 4.6, $t \geq \gamma(I) = \gamma(IS)$. Conversely, assume $t \geq \gamma(I)$. The equality $(\text{core}(I))S = \text{core}(IS)$ holds by Lemma 4.6. As to the second equality, Lemma 4.3 gives $e(IS/\text{core}(IS)) = e(IS/(\text{core}(I))S) \leq e(IS/\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t)$. Now proceed as in the proof of Theorem 4.5.

(b) In the setting of part (a), $\text{Aut}_R(S)$ acts transitively on $\{\mathcal{A}_1, \dots, \mathcal{A}_t\}$. Hence $\mathcal{A}_1 \cap R = \dots = \mathcal{A}_t \cap R$. Now (a) implies that $\text{core}(I) = \mathcal{A}_1 \cap R$. We may assume that $R(X) \hookrightarrow S$ is a flat local extension and $\mathcal{A}_1 = \mathcal{A}S$, thus $\mathcal{A}_1 \cap R = \mathcal{A} \cap R$. \square

Using Theorem 4.7.b we can in turn generalize Lemma 4.6 to the case when the map is not necessarily local.

Theorem 4.8 *Let $R \rightarrow R'$ be a flat map of local Cohen–Macaulay rings with infinite residue fields. Let I be an R -ideal of analytic spread ℓ such that I and IR' are G_ℓ and universally weakly $(\ell - 1)$ -residually S_2 . Then $\text{core}(IR') = (\text{core}(I))R'$.*

Proof. The containment $\text{core}(IR') \subset (\text{core}(I))R'$ is obvious, because $\gamma(I) < \infty$ by Theorem 3.1. If $\ell(IR') < \ell$ then $\text{core}(IR') = IR' \supset (\text{core}(I))R'$ by Lemma 2.1.g. Thus we may assume that $\ell(IR') = \ell$. Let \mathcal{A} be a universal ℓ -generated ideal in $IR(X)$, $S = R(X)$, and $S' = R'(X)$. Notice that $\mathcal{A}S'$ is a universal ℓ -generated ideal in IS' . Thus by Theorem 4.7.b, $\text{core}(IR') = \mathcal{A}S' \cap R'$. Therefore $(\text{core}(IR')) \cap R = \mathcal{A}S' \cap R' \cap R = \mathcal{A}S' \cap R = \mathcal{A}S' \cap S \cap R \supset \mathcal{A} \cap R = \text{core}(I)$, where the last equality follows again from Theorem 4.7.b. Hence $\text{core}(IR') \supset (\text{core}(I))R'$. \square

Corollary 4.9 *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ . Assume that I is G_ℓ and universally weakly $(\ell - 1)$ -residually S_2 . Then*

$$\gamma(I) = \max(\{\gamma(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Min}(\text{Fitt}_\ell(I))\} \cup \{1\}).$$

Proof. According to Lemma 2.1.a, every localization of I is universally weakly $(\ell - 1)$ -residually S_2 . Write $t = \max(\{\gamma(I_{\mathfrak{p}}) \mid \mathfrak{p} \in \text{Min}(\text{Fitt}_\ell(I))\} \cup \{1\})$, and notice that t and $\gamma(I)$ are finite by Theorem 3.1. For $\mathfrak{p} \in \text{Spec}(R)$, $(\text{core}(I))_{\mathfrak{p}} = \text{core}(I_{\mathfrak{p}})$ by Theorem 4.8 and hence $\gamma(I) \geq \gamma(I_{\mathfrak{p}})$. Thus $\gamma(I) \geq t$.

To prove the reverse inequality, let $\mathcal{A}_1, \dots, \mathcal{A}_t$ be universal ℓ -generated ideals in IS and recall that $\gamma(IS)$ is finite by Theorem 4.7.a. If $\mathfrak{p} \in \text{Min}(\text{Fitt}_\ell(I))$ then $\ell(I_{\mathfrak{p}}) = \ell$ by Lemma 2.1.g, and $(\mathcal{A}_1)_{\mathfrak{p}S}, \dots, (\mathcal{A}_t)_{\mathfrak{p}S}$ are universal ℓ -generated ideals in $I_{\mathfrak{p}}S_{\mathfrak{p}S}$. Thus by Theorem 4.7.a, $\text{core}(I_{\mathfrak{p}}S_{\mathfrak{p}S}) = (\mathcal{A}_1)_{\mathfrak{p}S} \cap \dots \cap (\mathcal{A}_t)_{\mathfrak{p}S}$, and hence by Theorem 4.8, $(\text{core}(IS))_{\mathfrak{p}S} = (\mathcal{A}_1 \cap \dots \cap \mathcal{A}_t)_{\mathfrak{p}S}$ for every $\mathfrak{p} \in \text{Min}(\text{Fitt}_\ell(I))$. Since furthermore $\text{core}(IS) \subset \mathcal{A}_1 \cap \dots \cap \mathcal{A}_t \subset IS$ and $\text{Ass}_S(IS/\text{core}(IS)) \subset \text{Min}(\text{Fitt}_\ell(IS))$ by Lemma 2.1.e,f, we conclude that $\text{core}(IS) = \mathcal{A}_1 \cap \dots \cap \mathcal{A}_t$. Now $\gamma(I) \leq t$ by Theorem 4.7.a. \square

Remark 4.10 Except for the second assertion in Lemma 4.2, the results of this section remain true for ideals I and IR' primary to the maximal ideals if R and R' are merely Buchsbaum instead of Cohen–Macaulay. Notice that the first part of Lemma 4.2 still holds in this case, because $\text{length}(I/J)$ is independent of the minimal reduction of I by definition of the Buchsbaum property.

However, even for an \mathfrak{m} -primary ideal I the inclusion $\mathcal{A} \cap R \subset \text{core}(I)$ of Theorem 4.7.b may fail to hold if (R, \mathfrak{m}) is not Buchsbaum, as can be seen from [15, p. 246]. We observe a similar failure for Cohen–Macaulay rings provided the ideal I does not satisfy the assumptions of Theorem 4.7.

Let R be a Noetherian local ring with infinite residue field and I an R -ideal. If J is a reduction of I we denote by $r_J(I)$ the least integer $r \geq 0$ with $I^{r+1} = JI^r$. Recall that the *reduction number* of I is defined as $r(I) = \min\{r_J(I) \mid J \in \mathcal{M}(I)\}$.

Example 4.11 Let $R = k[U, V, W]_{(U, V, W)} / (U^2 + V^2, VW)$, where k is an infinite field and U, V, W are variables. Denote the images of U, V, W in R by u, v, w . Consider the R -ideal $I = (u, v)$, and let $\mathcal{A} = (Xu + Yv) \subset S = R[X, Y]$ be a universal one-generated ideal in IS .

Notice that R is Gorenstein, $\ell = \ell(I) = 1$, and I does not satisfy G_1 , but is 0-residually S_2 . In this case $\text{core}(I)$ is the intersection of finitely many minimal reductions of I , $\text{core}(IR') = (\text{core}(I))R'$ for every flat map $R \rightarrow R'$ to a local Cohen–Macaulay ring R' , but $\text{core}(I)$ is not an intersection of general one-generated ideals in I which are reductions of I , and $\text{core}(I) \subsetneq \mathcal{A} \cap R$.

Indeed, (u) and (v) are minimal reductions of I , hence $\text{core}(I) \subset (u) \cap (v) = I^2$. On the other hand a computation using *Macaulay* [3] shows that the associated graded ring of I is Cohen–Macaulay; hence $r_J(I) = r_{(u)}(I) = 1$ for every minimal reduction J of I (see [1, 4.11]). Thus $I^2 \subset \text{core}(I)$. Therefore

$$\text{core}(I) = I^2 = (u) \cap (v).$$

If the map $R \rightarrow R'$ is local then the same argument gives $\text{core}(IR') = (IR')^2$. Otherwise either $\text{core}(IR') = 0 = (IR')^2$ or $\text{core}(IR') = R' = (IR')^2$. Hence in any case $\text{core}(IR') = (\text{core}(I))R'$.

Every general one-generated ideal $(\lambda u + \mu v)$ in I contains $uw = (\lambda u + \mu v)\lambda^{-1}w$, whereas $uw \notin I^2$. Thus $\text{core}(I)$ cannot be the intersection of general one-generated ideals in I . Likewise $I^2 + (uw) \subset \mathcal{A}$, hence $\text{core}(I) \subsetneq \mathcal{A} \cap R$.

5 Computational remarks

Individual minimal reductions of homogeneous ideals tend to be inhomogeneous – for instance, the monomial ideal $I = (U^2, UV, V^3) \subset k[U, V]_{(U, V)}$ has no minimal reduction generated by homogeneous polynomials in U and V . Nevertheless the core of this ideal is monomial due to the following general fact:

Remark 5.1 Let R be a Noetherian local ring with infinite residue field. Assume that $R = R'_{\mathfrak{m}'}$ for R' an \mathbb{N}_0^r -graded ring over a local ring and \mathfrak{m}' its homogeneous maximal ideal. Let I be an R -ideal. If I is generated by homogeneous elements of R' then so is $\text{core}(I)$.

Proof. (This proof was suggested to us by D. Eisenbud.) Let U be the group of units of $[R']_{(0, \dots, 0)}$ and G the direct product U^n . If $\alpha = (\alpha_1, \dots, \alpha_n) \in G$ and $x \in [R']_{(i_1, \dots, i_n)}$ we define αx to be $\alpha_1^{i_1} \cdots \alpha_n^{i_n} x \in [R']_{(i_1, \dots, i_n)}$. This induces an action of G on the ring R . As is well known, an R -ideal is G -stable if and only if it is extended from a homogeneous R' -ideal. To finish the proof, notice that G acts on the set $\mathcal{M}(I)$, which implies the G -stability of $\text{core}(I)$. \square

The next remark gives a fairly efficient probabilistic algorithm for computing the core. In light of Theorem 4.5 it suffices to bound $\gamma(I)$:

Remark 5.2 Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ . Assume that I is G_ℓ and weakly $(\ell - 1)$ -residually S_2 . If $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_{t+1}$ for $\mathfrak{a}_1, \dots, \mathfrak{a}_{t+1}$ general ℓ -generated ideals in I , then $\gamma(I) \leq t$.

Proof. By Theorem 3.1 and Theorem 4.5 there exists an integer $s > t$ so that $\text{core}(I) = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_s$ with $\mathfrak{a}_1, \dots, \mathfrak{a}_s$ general ℓ -generated ideals in I . After passing to a smaller dense open subset of $\times^s \mathbb{G}(\ell, I \otimes_R k)$, we deduce from our assumption that $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t \cap \mathfrak{a}_i$ for every $t + 1 \leq i \leq s$. Thus $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_t = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_s = \text{core}(I)$. \square

As before, it suffices to assume in Remark 5.2 that R is Buchsbaum if I is primary to the maximal ideal.

Lemma 5.3 *Let R be a local Cohen–Macaulay ring, s an integer, I an R -ideal satisfying G_s , \mathcal{B} a generic s -generated ideal in IT , $K = \mathcal{B} :_T IT$, and $\mathfrak{q} \in V(K)$ with $\dim T_{\mathfrak{q}} \leq s$.*

- (a) *If $I \subset \mathfrak{q}$ then \mathfrak{q} is extended from a minimal prime of $\text{Fitt}_s(I)$ and $\text{Fitt}_s(I) \neq R$.*
(b) *If $I \not\subset \mathfrak{q}$ then $\mathfrak{q} \cap R$ is a minimal prime of R .*

Proof. We write $\mathfrak{p} = \mathfrak{q} \cap R$ and replace R by $R_{\mathfrak{p}}$.

(a) If \mathfrak{q} is not extended from a minimal prime of $\text{Fitt}_s(I)$, then $\text{Fitt}_s(I) = R$. Thus since I satisfies G_s , I is G_{s+1} . Hence by [11, 3.2], $\text{ht}(IT + K) \geq s + 1$. But this is impossible because $IT + K \subset \mathfrak{q}$ and $\dim T_{\mathfrak{q}} \leq s$.

(b) One has $\mathcal{B} \subset \mathfrak{q}$ since $\mathfrak{q} \in V(K)$, and $I = R$ since $I \not\subset \mathfrak{q}$. Now after adjoining variables to T and applying an R -automorphism we may suppose that \mathcal{B} is defined using 1 as a generator of I . Hence \mathcal{B} is generated by s variables X_1, \dots, X_s of T , and thus $(\mathfrak{p}, X_1, \dots, X_s) \subset \mathfrak{q}$. As $\dim T_{\mathfrak{q}} \leq s$, \mathfrak{p} must be a minimal prime of R . \square

Proposition 5.4 *Let R be a local Cohen–Macaulay ring with infinite residue field and I an R -ideal of analytic spread ℓ and reduction number r . Let $f \in I$ and $h \in \sqrt{\text{Fitt}_{\ell}(I)}$ be non zerodivisors on R , and \mathcal{B} a generic ℓ -generated ideal in IT . Assume that I is G_{ℓ} and universally weakly $(\ell - 1)$ -residually S_2 , and IT is weakly $(\ell - 1)$ -residually S_2 . Then*

$$\text{core}(I) = [\mathcal{B} :_T (\mathcal{B} :_T h^{\infty} I)]_0 = [\mathcal{B} :_T (\mathcal{B} :_T f^{r+1})]_0.$$

Proof. Write $S = T_{\mathfrak{m}T}$ and $\mathcal{A} = \mathcal{B}S$. By Theorem 4.7.b, $\text{core}(I) = \mathcal{A} \cap R$. Now $\mathcal{A} \cap R = \mathcal{A} \cap T \cap R = [\mathcal{B}_{\mathfrak{m}T} \cap T]_0$. Write $H = \mathcal{B} : (\mathcal{B} : h^{\infty} I)$ and $F = \mathcal{B} : (\mathcal{B} : f^{r+1})$. It remains to prove that $\mathcal{B}_{\mathfrak{m}T} \cap T = H = F$. This will follow once we have shown that $H_{\mathfrak{m}T} = F_{\mathfrak{m}T} = \mathcal{B}_{\mathfrak{m}T}$ and that every associated prime of H or F is contained in $\mathfrak{m}T$.

First, notice that $\mathcal{B}_{\mathfrak{m}T}$ is an ℓ -generated reduction of $IT_{\mathfrak{m}T}$. Hence by Lemma 2.1.e, $(\mathcal{B} : IT)_{\mathfrak{m}T}$ contains some power of $\text{Fitt}_{\ell}(I)$ and hence of h . This gives $(\mathcal{B} : h^{\infty} I)_{\mathfrak{m}T} = ((\mathcal{B} : IT) : h^{\infty})_{\mathfrak{m}T} = T_{\mathfrak{m}T}$. Therefore $H_{\mathfrak{m}T} = \mathcal{B}_{\mathfrak{m}T}$. Since $\mathcal{B}_{\mathfrak{m}T}$ is a universal ℓ -generated ideal in $IT_{\mathfrak{m}T}$ we have $I^{r+1} \subset \mathcal{B}_{\mathfrak{m}T}$ by [16, 3.4], and hence $f^{r+1} \in \mathcal{B}_{\mathfrak{m}T}$. This gives $(\mathcal{B} : f^{r+1})_{\mathfrak{m}T} = T_{\mathfrak{m}T}$, thus $F_{\mathfrak{m}T} = \mathcal{B}_{\mathfrak{m}T}$.

Finally let \mathfrak{q} be an associated prime of H or F . Notice that \mathfrak{q} is also an associated prime of \mathcal{B} . Since $\text{ht} \mathcal{B} : IT \geq \ell$ by [11, the proof of 3.2], Lemma 2.1.c then gives $\dim T_{\mathfrak{q}} \leq \ell$. We claim that $\mathfrak{q} \subset \mathfrak{m}T$. We may assume that $\mathcal{B} : IT \subset \mathfrak{q}$ since otherwise $IT_{\mathfrak{q}} = \mathcal{B}_{\mathfrak{q}}$, thus \mathfrak{q} is an associated prime of IT and hence contained in $\mathfrak{m}T$. Now if $I \subset \mathfrak{q}$ then $\mathfrak{q} \subset \mathfrak{m}T$ by Lemma 5.3.a. If on the other hand $I \not\subset \mathfrak{q}$ then part (b) of the same lemma gives $h \notin \mathfrak{q}$ and $f \notin \mathfrak{q}$. Therefore $(\mathcal{B} : h^{\infty} I)_{\mathfrak{q}} = \mathcal{B}_{\mathfrak{q}}$ and $(\mathcal{B} : f^{r+1})_{\mathfrak{q}} = \mathcal{B}_{\mathfrak{q}}$. Thus $H_{\mathfrak{q}} = F_{\mathfrak{q}} = T_{\mathfrak{q}}$, which is impossible. \square

Remark 5.5 Let (R, \mathfrak{m}) be a local Cohen–Macaulay ring of dimension d with infinite residue field, I an \mathfrak{m} -primary ideal of multiplicity e , h an R -regular element, and \mathcal{B} a generic d -generated ideal in IT . Then

$$\text{core}(I) = [\mathcal{B} :_T (\mathcal{B} :_T h^e)]_0.$$

Proof. The assertion follows from the proof of Proposition 5.4 and the fact that $\lambda((T/\mathcal{B})_{\mathfrak{m}T}) = e$. \square

The equality of Proposition 5.4 gives a method for computing the core of a broad class of ideals generated by homogeneous elements not necessarily of the same degree: by giving suitable degrees to the variables X_{jl} of T , the ideal \mathcal{B} becomes homogeneous and the computation stays in the graded category. As an illustration, taking $I = (U^3, UV^3, V^4) \subset k[U, V]_{(U, V)}$, we obtain $\text{core}(I) = (U^2, UV, V^2)I$ and taking $I = (U^3, UV^2W^2, V^3W^3) \subset k[U, V, W]_{(U, V, W)}$, we obtain $\text{core}(I) = (U^2, UVW, V^2W^2)I$. The outcome of neither computation could have been predicted by the results of [5] or [10].

References

1. I. Aberbach, C. Huneke and N.V. Trung, Reduction numbers, Briançon-Skoda theorems and the depth of Rees rings, *Compositio Math.* **97** (1995), 403–434.
2. L. Avramov and J. Herzog, The Koszul algebra of a codimension 2 embedding, *Math. Z.* **175** (1980), 249–260.
3. D. Bayer and M. Stillman, *Macaulay*: A system for computation in algebraic geometry and commutative algebra, 1992. Available via anonymous ftp from `math.harvard.edu`.
4. M. Chardin, D. Eisenbud and B. Ulrich, Hilbert functions, residual intersections, and residually S_2 -ideals, to appear in *Compositio Math.*
5. A. Corso, C. Polini and B. Ulrich, The core of an ideal, in preparation.
6. D. Eisenbud, C. Huneke and W.V. Vasconcelos, Direct methods for primary decomposition, *Invent. Math.* **110** (1992), 207–235.
7. J. Herzog, W.V. Vasconcelos and R.H. Villarreal, Ideals with sliding depth, *Nagoya Math. J.* **99** (1985), 159–172.
8. C. Huneke, On the symmetric and Rees algebra of an ideal generated by a d -sequence, *J. Algebra* **62** (1980), 268–275.
9. C. Huneke, Linkage and Koszul homology of ideals, *Amer. J. Math.* **104** (1982), 1043–1062.
10. C. Huneke and I. Swanson, Cores of ideals in 2-dimensional regular local rings, *Michigan Math. J.* **42** (1995), 193–208.
11. C. Huneke and B. Ulrich, Residual intersections, *J. reine angew. Math.* **390** (1988), 1–20.
12. J. Lipman, Adjoints of ideals in regular local rings, *Math. Research Letters* **1** (1994), 1–17.
13. J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon–Skoda, *Michigan Math. J.* **28** (1981), 199–222.

14. M. Nagata, *Local Rings*, Krieger, Huntington, 1975.
15. D. Rees and J.D. Sally, General elements and joint reductions, *Michigan Math. J.* **35** (1988), 241–254.
16. A. Simis, B. Ulrich, and W.V. Vasconcelos, Cohen–Macaulay Rees algebras and degrees of polynomial relations, *Math. Ann.* **301** (1995), 421–444.
17. B. Ulrich, Artin-Nagata properties and reductions of ideals, in *Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra*, W. Heinzer, C. Huneke, J. Sally (eds.), *Contemp. Math.* **159**, Amer. Math. Soc., Providence, 1994, 373–400.
18. G. Valla, On the symmetric and Rees algebras of an ideal, *Manuscripta Math.* **30** (1980), 239–255.
19. W.V. Vasconcelos, Reduction numbers of ideals, *J. Algebra* **216** (1999), 652–664.