Algebraic surfaces holomorphically dominable by $\mathbb{C}^2$

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1 Introduction

An n-dimensional complex manifold M is said to be (holomorphically) *dominable* by $\mathbb{C}^n$ if there is a map $F : \mathbb{C}^n \to M$ which is holomorphic such that the Jacobian determinant $\det(DF)$ is not identically zero. Such a map F is called a *dominating map*. In this paper, we attempt to classify algebraic surfaces X which are dominable by $\mathbb{C}^2$ using a combination of techniques from algebraic topology, complex geometry and analysis. One of the key tools in the study of algebraic surfaces is the notion of Kodaira dimension (defined in Sect. 2). By Kodaira's pioneering work [Ko1] and its extensions (see, for example, [CG] and [KO]), an algebraic surface which is dominable by $\mathbb{C}^2$ must have Kodaira dimension less than two. Using the Kodaira dimension and the fundamental group of X, we succeed in classifying algebraic surfaces which are dominable by $\mathbb{C}^2$ except for certain cases in which X is an algebraic surface of Kodaira dimension zero and the case when X is rational without any logarithmic 1-form. More specifically, in the case when X is compact (namely projective), we need to exclude only the case when X is birationally equivalent to a K3 surface (a simply connected compact complex surface which admits a globally non-vanishing holomorphic 2-form) that is neither elliptic nor Kummer (see Sects. 3 and 4 for the definition of these types of surfaces).

With the exceptions noted above, we show that for any algebraic surface of Kodaira dimension less than 2, dominability by $\mathbb{C}^2$ is equivalent to the apparently weaker requirement of the existence of a holomorphic image of $\mathbb{C}$ which is Zariski dense in the surface. With the same exceptions, we will also show the very interesting and revealing fact that dominability by $\mathbb{C}^2$ is preserved even if a sufficiently small neighborhood of any finite set of points is removed from the surface. In fact, we will provide a complete

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classification in the more general category of (not necessarily algebraic) compact complex surfaces before tackling the problem in the case of non-compact algebraic surfaces.

We remark that both elliptic K3 and Kummer K3 surfaces are dense in the moduli space of K3 surfaces; the former is dense of codimension-one while the latter is dense of codimension sixteen in this moduli space (see [PS,LP]) and intersects the former transversally (these density results hold also in any universal family). Dominability by $\mathbb{C}^2$ holds for both types of K3 surfaces. This suggests that it might hold for all K3 surfaces so that our statements above would be valid without exception for projective (and, more generally, for compact Kähler) surfaces. Indeed, their density plus Brody's Lemma ([Br]) tell us that every K3 surface contains a non-trivial holomorphic image of $\mathbb{C}$ and that the generic K3 surface, which is non-projective but remains Kähler, even contains such an image that is Zariski dense. We mention here that dominability by $\mathbb{C}^2$ can be shown for some non-elliptic K3 surfaces which are close to Kummer surfaces using an argument similar to that of Sect. 6; for length considerations, we omit this non-elliptic case from this paper. However, we note that the statement equating dominability to the weaker condition of having a Zariski dense image of $\mathbb{C}$ is quite false in the non-Kähler category, as is amply demonstrated by Inoue surfaces (see [In0] or [BPV, V.19]).

Observe that if there is a dominating map $F : \mathbb{C}^2 \to X$, then there is also a holomorphic image of $\mathbb{C}$ which is Zariski dense: First we may assume that the Jacobian of F is non-zero at the origin. Defining $h : \mathbb{C} \to \mathbb{C}^2$ by $h(z) = (\sin(2\pi z), \sin(2\pi z^2))$, we see that $h(n) = (0, 0)$ with corresponding tangent direction $(2\pi, 4\pi n)$ for each $n \in \mathbb{Z}$. Taking $F \circ h$, we obtain a holomorphic image of $\mathbb{C}$ with an infinite number of tangent directions at one point, which implies that the image is Zariski dense.

We say that an algebraic variety X satisfies property C if every holomorphic image of $\mathbb{C}$ in X is algebraically degenerate; i.e., is not Zariski dense. Our first main result is that for algebraic surfaces of Kodaira dimension less than 2 and with the exceptions mentioned above, dominability by $\mathbb{C}^2$ is equivalent to the failure of property C. We will state only the main results in the projective category in this introduction for simplicity but will discuss fully the compact non-projective case and much of the quasi-projective case in this paper.

Theorem 1.1 *Let X be a projective surface of Kodaira dimension less than 2 and suppose that X is not birational to a K3 surface which is either elliptic or Kummer. Then X is dominable by $\mathbb{C}^2$ if and only if it does not satisfy property C. Equivalently, there is a dominating holomorphic map $F : \mathbb{C}^2 \to X$ if and only if there is a holomorphic image of $\mathbb{C}$ in X which is Zariski dense.*

By a recent result of the second named author, this theorem is also true for a projective surface of Kodaira dimension 2, which is the maximum Kodaira dimension for surfaces. As previously mentioned, a surface of Kodaira

dimension 2 is not dominable by $\mathbb{C}^2$ [Ko1]; indeed, a surface of Kodaira dimension 2 is precisely a surface which admits a possibly degenerate hyperbolic volume form. Thus in the case of Kodaira dimension 2, Theorem 1.1 can be established by showing that such a surface satisfies property C. The question of whether a variety of maximum Kodaira dimension satisfies property C was first raised explicitly by Serge Lang [Lang].

In the following theorem we give, again modulo the above mentioned exceptions, a classification of projective surfaces which are dominable by $\mathbb{C}^2$ and hence a classification of projective surfaces of Kodaira dimension less than 2 which fail to satisfy property C. We will do this in terms of the Kodaira dimension and the fundamental group, both of which are invariant under birational maps.

Theorem 1.2 *A projective surface X not birationally equivalent to a K3 surface is dominable by $\mathbb{C}^2$ if and only if it has Kodaira dimension less than two and its fundamental group is a finite extension of an abelian group (of even rank four or less). If $\kappa(X) = -\infty$, then the fundamental group condition can be replaced by the simpler condition of non-existence of more than one linearly independent holomorphic one-form. If $\kappa(X) = 0$ and X is not birationally equivalent to a K3 surface, then X is dominable by $\mathbb{C}^2$. If X is birationally equivalent to an elliptic K3 surface or to a Kummer K3 surface, then X is dominable by $\mathbb{C}^2$.*

As with Theorem 1.1, this theorem fails if we include compact non-Kähler surfaces (even after simple minded modification of this theorem). For instance, the Kodaira surfaces are dominable by $\mathbb{C}^2$ but their fundamental groups are not finite extensions of abelian groups ([Ko4]). But this theorem remains valid in the Kähler category, thanks, for example, to Kodaira's result that all Kähler surfaces are deformations of projective surfaces ([Ko2], [Ko3]).

More general versions of Theorem 1.1 and Theorem 1.2 for compact complex surfaces will be given at the end of Sect. 4.

In the quasi-projective category, we also prove the analogue of Theorem 1.1 modulo the same exceptions mentioned in the beginning, following mainly the work of Kawamata [K1] and M. Miyanishi [M]. In this setting, the analogue of the fundamental group characterization requires the study of a new but very natural class of objects of complex dimension one that are related to orbifolds. As for explicit examples, we will work out Theorems 1.1 and an analogue of 1.2 for the complement of a reduced curve C in $\mathbb{P}^2$ in the case when C is normal crossing, where we show that dominability is characterized by $\deg C \leq 3$, and for the overlapping case in which C is either not a rational curve of high degree or has at most one singular point. Here, the most fascinating and revealing example is the case in which C is a non-singular cubic curve, whose complement is a noncompact analogue of a K3 surface. The question of the dominability of the complement of a non-singular cubic was discussed by Bernard Shiffman at MSRI in 1996,

and the positive resolution of this problem served as the first result in and inspiration for this paper.

The key tools we introduce here for constructing dominating maps are the mapping theorems we establish via a combination of complex geometry and analysis. One of these theorems utilizes Kodaira's theory of Jacobian fibrations to deal with general elliptic fibrations (see Sect. 3). Other such theorems construct the required self-maps of $\mathbb{C}^2$ directly via complex analysis to deal with $\mathbb{C}^*$-fibrations, abelian and Kummer surfaces.

In particular, the constructions in Sects. 4 and 6 show that given any complex 2-torus and any finite set of points in this torus, there is an open set containing this finite set and a dominating map from $\mathbb{C}^2$ into the complement of the open set. This should be compared with [Gr] in which it was claimed that the complement of any open set in a simple complex torus is Kobayashi hyperbolic (a complex torus is simple if it has no nontrivial complex subtori). There is no contradiction because it was later realized that the proof given in [Gr] is incorrect since the topological closure of a complex one-parameter group need not be a complex torus. Despite this, the validity of this claim appears to have been an open question until the current paper, which shows the claim to be false in dimension 2. The n-dimensional analogue of our result is given in [Bu].

Many of the tools and results we develop may be of interest to other areas of mathematics besides complex analysis and holomorphic geometry, especially to Diophantine (arithmetic) geometry in view of the connection between the transcendental holomorphic properties and arithmetic properties of algebraic varieties. For example, the important technique of constructing sections of elliptic fibrations, which is very difficult to achieve in the algebro-geometric category but certainly useful in arithmetic and algebraic geometry, turns out to be quite natural and relatively easy to do in the holomorphic category. Also, we undertake a global study, from the viewpoint of holomorphic geometry, of the monodromy action on the fundamental group of an elliptic fibration. Needless to say, without the deep and beautiful contributions of Kodaira on complex analytic surfaces, we would not be able to go much beyond dealing with some special examples, as is the case with much of the scarce literature on the subject. However, we have not avoided, due to the nature of this joint paper, giving elementary lemmas and proofs while avoiding the unnecessary full force of Kodaira's theory on elliptic fibrations, especially as we deal with fibrations over curves that are not necessarily quasiprojective.

The paper is organized as follows. Section 2 introduces some basic birational invariants and general notation and provides a list of the classification of projective surfaces. Section 3 deals with projective surfaces not of zero Kodaira dimension and solves the dominability problem completely for elliptic fibrations, including the non-algebraic ones. Section 4 deals with the remaining projective and compact complex cases while Sect. 5 deals with

the non-compact algebraic surfaces. Section 6 goes beyond these theorems to deal with algebraic surfaces minus small open balls.

2 Classification of algebraic surfaces

In this section we will first introduce some basic invariants in the (logarithmic) classification theory of algebraic varieties (see [Ii] for more details, also compare with [Ue]). Then we will provide a list of the birational classification of projective surfaces and discuss briefly the dominability problem in the quasi-projective category. Finally, we will introduce the more general category of compactifiable complex manifolds and a basic invariant which distinguishes the algebraic case in dimension two.

Let $\bar{X}$ be a complex manifold with a normal crossing divisor D. This means that around any point q of $\bar{X}$, there exists a local coordinate $(z_1, ..., z_n)$ centered at q such that, for some $r \leq n$, D is defined by $z_1 z_2 ... z_r = 0$ in this coordinate neighborhood. If all the components of D are smooth, then D is called a simple normal crossing divisor. Following Iitaka ([Ii]), we define the logarithmic cotangent sheaf $\Omega_{\bar{X}}(\log D)$ as the locally free subsheaf of the sheaf of meromorphic 1-forms, whose restriction to $X = \bar{X} \setminus D$ is identical to Ω_X and whose localization at any point $q \in D$ is given by

$$\Omega_{\bar{X}}(\log D) = \sum_{i=1}^{r} \mathcal{O}_{\bar{X},q} \frac{dz_i}{z_i} + \sum_{j=r+1}^{n} \mathcal{O}_{\bar{X},q} dz_j,$$

where the local coordinates $z_1, ..., z_n$ around q are chosen as before. Its dual, the logarithmic tangent sheaf $T_{\bar{X}}(-\log D)$, is a locally free subsheaf of $T_{\bar{X}}$. We will follow a general abuse of notation and use the same notation to denote both a locally free sheaf and a vector bundle.

By an algebraic variety in this paper, we mean a complex analytic space X_0 such that X_0 has an algebraic structure in the following sense: X_0 is covered by a finite number of neighborhoods, each of which is isomorphic to a closed analytic subspace of a complex vector space defined by polynomial equations and which piece together with rational coordinate transformations. A proper birational map from X_0 to another variety X_1 is, by the graph definition, an algebraic subvariety of $X_0 \times X_1$ which projects generically one-to-one onto each factor as a proper morphism. If such a map exists, we say that the two varieties are properly birational. This notion corresponds to that of a bimeromorphic map in the holomorphic context. Two algebraic varieties are said to be birationally equivalent if they have isomorphic rational function fields; or equivalently, if they have birational compactifications. Hironaka's resolution of singularities theorem [Hi] (an

elementary proof of which can be found in [BM]) implies that given any algebraic variety X_0, there is a smooth projective variety $\bar{X}$ with a simple normal crossing divisor D such that $X = \bar{X} \setminus D$ is properly birational to X_0. If X_0 is smooth, then we can even take X to be X_0 so that X_0 can be compactified by adding a simple normal crossing boundary divisor. In this paper, a surface will mean a complex two dimensional manifold while a curve that is not explicitly a subvariety (or a subscheme) will mean a (not necessarily quasi-projective) complex one-dimensional manifold. All surfaces and curves are assumed to be connected. In particular, every algebraic surface is isomorphic to the complement of a finite set of transversely intersecting smooth curves without triple intersection in some projective surface. We will use the Enriques-Kodaira classification of compact surfaces to simplify our problem for surfaces.

One of the most important invariants under proper birational maps is the (logarithmic) Kodaira dimension. Let X_0, X, $\bar{X}$, and D be as above, and let $K_{\bar{X}} = \det_{\mathbb{C}}(T^{\vee}_{\bar{X}})$ where $T^{\vee}_{\bar{X}}$ is the complex cotangent bundle of $\bar{X}$. The (holomorphic) line bundle $K_{\bar{X}}$ is called the canonical bundle of $\bar{X}$. Identifying a line bundle and its sheaf of holomorphic sections, we define a new line bundle $K = K_{\bar{X}}(D) = K_{\bar{X}} \otimes \mathcal{O}(D)$ corresponding to the sheaf of meromorphic sections of $K_{\bar{X}}$ which are holomorphic except for simple poles along D (see Griffiths and Harris [GH] among many other standard references). In fact,

$$K = \det \Omega_{\bar{X}}(\log D).$$

This line bundle on $\bar{X}$ is called the logarithmic canonical bundle of $X = \bar{X} \setminus D$, or more specifically, of $(\bar{X}, D)$. We will write tensor products of line bundles additively by a standard abuse of notation; for example, $mK = K^{\otimes m}$. Given a projective manifold $\bar{Y}$ and a birational morphism $f : \bar{Y} \to \bar{X}$ such that $f^{-1}(D)$ is the same as a normal crossing divisor E in $\bar{Y}$, then any section of mK as a tensor power of rational 2-form on X pulls back via f to a section of $mK_{\bar{Y}}(E)$. Conversely, any section of $mK_{\bar{Y}}(E)$ pulls back (via f^{-1}) to a section of mK outside a codimension-two subset (the indeterminacy set of f^{-1}), which therefore extends to a section of mK by the classical extension theorem of Riemann. It follows that, for every positive integer m, $h^0(mK) := \dim H^0(mK)$ is independent of the choice of $\bar{X}$ for X_0 and is a proper birational invariant of X_0. This allows us to introduce the following birational invariant of X_0.

Definition 2.1 *The Kodaira dimension of X_0 is defined as*

$$\bar{\kappa}(X_0) = \limsup_{m \to \infty} \frac{\log h^0(mK)}{\log m}.$$

The simpler notation $\kappa(X_0)$ is used when X_0 is projective. The Riemann-Roch formula shows that $\bar{\kappa}(X_0)$ takes values in the set

$$\{-\infty, 0, 1, ..., \dim X_0\}.$$

By the same argument as that for $h^0(mK)$, we see that another proper birational invariant is given by the (logarithmic) irregularity of X_0 defined by

$$\bar{q}(X_0) = h^0(\Omega_{\bar{X}}(\log D)).$$

If $D = 0$, then $\bar{q}(X_0)$ is just the dimension of the space of global holomorphic one-forms $q(X) = h^0(\Omega_X)$ on X.

If $\bar{\kappa}(X_0) = \dim(X_0)$, then X_0 is called a variety of general type. A theorem of Carlson and Griffiths [CG] (see also Kodaira [Ko1]) says that X_0 cannot be dominated (even meromorphically) by $\mathbb{C}^n$ in this case. Hence for both Theorem 1.1 and Theorem 1.2, we need consider only those surfaces with Kodaira dimension less than 2.

A projective surface X whose canonical bundle has non-negative intersection with (or, equivalently, non-negative degree when restricted to) any curve in X is called minimal. We say that K_X is nef (short for numerically effective) in this case. In general, we say that a line bundle L on X is nef if $L \cdot C \geq 0$ for any curve C in X.

Every algebraic surface is either projective or admits a projective compactification by adding a set of smooth curves with at most normal crossing singularities. Moreover, the Enriques-Kodaira classification [BPV, Ch. VI] says that a projective surface admits a birational morphism (as a composition of blowing up smooth points) to one of the following.

(0) A surface of general type: $\kappa = 2$.
(1) $\mathbb{P}^2$ or a ruled surface over a curve C of genus $g = h^0(\Omega_C)$ (that is, a holomorphic $\mathbb{P}^1$ bundle over C). The latter is birationally equivalent to $\mathbb{P}^1 \times C$. Here, $\kappa = -\infty$.
(2) An abelian surface (a projective torus given by $\mathbb{C}^2$/a lattice). Here, $\kappa = 0$.
(3) A K3 surface (a simply connected surface with trivial canonical bundle). $\kappa = 0$.
(4) A minimal surface with the structure of an elliptic fibration (see Sect. 3.2). Here κ can be 0, 1, or $-\infty$.

The characteristic property of the surfaces listed above is the absence of (-1)-curves. A (-1)-curve is a smooth rational curve (image of $\mathbb{P}^1$) in a surface with self-intersection -1, i.e. whose normal bundle has degree -1. From Castelnuovo's criterion [BPV, III4.1], a (-1) curve is always the blow-up of a (smooth) point on a surface. A simple argument (via the linear independence of the total transform of blown up (-1)-curves in H_1) shows that, given any projective surface with $\kappa < 2$, one can always reach one of the surfaces listed above by blowing down (-1)-curves a finite number of times. It is a standard fact that a projective surface with $\kappa \geq 0$ is minimal if and only if it does not have any (-1)-curve, and that such a surface is the unique one in its birational class having this property.

Let X_0 be an algebraic surface having a compactification $\bar{X}_0$ which is birational to one of the model surfaces listed above, say $\bar{X}$. There is

a maximum Zariski open subset U of X_0 that is properly birational to the complement of a reduced divisor C and a finite set T' of points in $\bar{X}$. Now the indeterminacy set of this proper birational map from $X = \bar{X} \setminus \{C \cup T\}$ to $U \subseteq X_0$ must consist of a finite set of points. So to produce a dominating map from $\mathbb{C}^2$ to X_0, it suffices to produce, for each finite set of points T in $\bar{X}$, a dominating map from $\mathbb{C}^2$ into the complement of T in $\bar{X} \setminus C$. Nevertheless, $\bar{X} \setminus C$ may not be dominable by $\mathbb{C}^2$ when X_0 is dominable by $\mathbb{C}^2$; for example, a point on X_0 may correspond to an infinitely near point on $\bar{X}$ over a point of C. However, if we think of X_0 as an open subset of the space of infinitely near points of $\bar{X}$, then we can recover the equivalence in dominability through the above procedure (see Sect. 5).

Although we have chosen to introduce and state our results so far in the algebraic category for simplicity, we will in fact deal with a more general class of surfaces in the next two sections: the class of compactifiable surfaces. These are Zariski open subsets of compact complex surfaces and the invariants $\bar{\kappa}$ and $\bar{q}$ carry over to them verbatim as they are defined by compactifications with normal crossing divisors, which exist by complex surface theory. If a surface X is compact, the transcendency degree $a(X)$ of the field of meromorphic functions on X is, by definition, a bimeromorphic invariant and is called the algebraic dimension.

3 Compact surfaces with $\kappa \neq 0$ and $a \neq 0$

In this section we solve the $\mathbb{C}^2$ dominability problem for compact surfaces whose Kodaira dimension and algebraic dimension are both non-zero. The bulk of this section is devoted to the case of elliptic fibrations, which we treat completely, including all the noncompact cases. In particular, we solve our problem for every projective surface that is birational to a minimal one listed in (1) and (4) above. Cases (2) and (3) will be discussed in Sect. 5.

3.1 Projective surfaces with Kodaira dimension $-\infty$

Since any $\mathbb{P}^1$-bundle over a curve C is birational to the trivial $\mathbb{P}^1$-bundle over C and since $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, any projective surface X with $\bar{\kappa}(X) = -\infty$ is birational to a surface Y which is a trivial $\mathbb{P}^1$-bundle over a curve C of genus $g := h^0(\Omega_C)$. In the case where C is of genus $g > 1$, any holomorphic image of $\mathbb{C}$ in Y must lie in a fiber of the bundle since C is hyperbolic. Hence X satisfies property C and so cannot be dominated by $\mathbb{C}^2$. In the case Y is a $\mathbb{P}^1$ bundle over an elliptic curve or over $\mathbb{P}^1$, one can easily construct a dominating map from $\mathbb{P}^1 \times \mathbb{C}^1$ and hence from $\mathbb{C}^2$ which respects the bundle structure (even algebraically in the latter case). In fact, by composing with the map

$$(\pi^1, h\pi^2) : \mathbb{C}^2 \to \mathbb{C}^2 \tag{3.1}$$

where $h : \mathbb{C} \to \mathbb{C}$ is holomorphic with prescribed zeros (which we can do by Weierstrass' theorem) and π^1, π^2 are the respective projections, we can arrange to have the dominating map miss any finite subset in Y. Choosing this finite subset to be the set of indeterminacies of the birational map from Y to X, this dominating map lifts to give a dominating map into X. Since $\mathbb{P}^1$ admits no holomorphic differentials and is simply connected, we obtain, respectively,

$$q(X) = q(Y) = q(C) = g \ \text{ and } \ \pi_1(X) = \pi_1(Y) = \pi_1(C).$$

Coupling this with the fact that the fundamental group of a curve of genus greater than 1 is not a finite extension of an abelian group gives us the following.

Theorem 3.1 *If X is a projective surface with $\kappa(X) = -\infty$, then the following are equivalent.*

(a) *X is dominable by $\mathbb{C}^2$.*
(b) *$q(X) := h^0(\Omega_X) < 2$.*
(c) *X admits a Zariski dense holomorphic image of $\mathbb{C}$.*
(d) *$\pi_1(X)$ is a finite extension of an abelian group.*

3.2 Elliptic fibrations

If X is any compact non-projective surface with $a(X) \neq 0$, then X is an elliptic surface by [Ko2]. Also, if X is projective and $\kappa(X) = 1$, then X is again an elliptic surface by classification. Hence the only remaining cases of $\kappa \neq 0$ and $a \neq 0$ are elliptic surfaces. In this section we resolve completely the case of elliptic surfaces.

Definition 3.2 *An elliptic fibration is a proper holomorphic map from a surface to a curve whose general fiber is an elliptic curve, i.e., a curve of genus one. Such a surface is called an elliptic surface. An elliptic fibration is called relatively minimal if there are no (-1)-curves on any fiber.*

Note that an elliptic fibration structure on a minimal surface must be relatively minimal.

Let $f : X \to C$ be a fibration (i.e. a proper holomorphic map with connected fibers) between complex manifolds X and C. If $f' : X' \to C'$ is another map where $C' \subseteq C$, then a map $h : X' \to X$ is called fiber-preserving if $f \circ h = f'$. If $\mathrm{rank}(df) = \dim C$ at every point on a fiber $X_s = f^{-1}(s)$, then X_s must be smooth by the implicit function theorem. If rank $(df) < \dim C$ somewhere on X_s, then X_s is called a singular fiber. Outside the singular fibers, all fibers are diffeomorphic by Ehresmann's theorem.

In the case f is a fibration of a surface X over a curve C, then each fiber, as a subscheme via the structure sheaf from C, is naturally an effective

divisor on X as follows. We write $X_s = \sum n_i C_i$ where C_i is the i-th component of the fiber $(X_s)_{red}$ (without the scheme structure) and where $n_i - 1$ is the vanishing order of df for a generic point on C_i. The positive integer coefficient n_i is called the multiplicity of the i-th component. The multiplicity of a fiber $X_s = \sum n_i C_i$ is defined as the greatest common divisor n_s of $\{n_i\}$. A fiber X_s with $n_s > 1$ is called a multiple fiber. A smooth fiber is then a fiber of multiplicity one having only one component. The singular fibers form a discrete set in X by analyticity. We will assume this setup for X and C from now on.

Let $\alpha : \tilde{C} \to C$ be a finite proper morphism. The ramification index at a point $\tilde{s} \in \tilde{C}$ is defined as the vanishing order of $d\alpha$ at $\tilde{s}$ plus one. Suppose α has ramification index n_s at every point above $s \in C$ and suppose that this is true for every $s \in C$. Then, according to [BPV, III, Theorem 9.1], pulling back the fibration via this ramified cover yields an unramified covering $\tilde{X}$ over X. Also, the resulting fibration $\tilde{X} \to \tilde{C}$ no longer has any multiple fiber. Such a ramified covering $\tilde{C}$ is called an orbifold covering of C with the given branched (orbifold) structure on C. More generally we have:

Definition 3.3 *Given a curve C with an assignment of a positive integer n_s for each $s \in C$ such that the set $S = \{s \in C \mid n_s > 1\}$ is discrete in C, define*

$$D = \sum_{n_s > 1} \left(1 - \frac{1}{n_s} \right) s. \text{ Suppose } \alpha : \tilde{C} \to C \text{ is a holomorphic quotient map}$$

such that $\alpha : \tilde{C} \backslash \alpha^{-1}(S) \to C \backslash S$ is an unramified covering and such that, for each point $s \in S$, every point on $\tilde{C}$ above s has ramification index n_s. Then $\tilde{C}$ is called an orbifold covering of the orbifold (C, D). If also $\tilde{C}$ is simply connected, then $\tilde{C}$ is called a uniformizing orbifold covering. A fibration over C defines a natural (branched) orbifold structure D on C by assigning n_s to be the multiplicity of the fiber at s of the fibration.

Therefore, we have the following:

Proposition 3.4 *Let X be a fibration over C. Let n_s denote the multiplicity of the fiber X_s for every point $s \in C$, thus endowing C with an orbifold structure D as above. Let $\tilde{C}$ be an orbifold covering of (C, D). Then the pull back fibration $\tilde{X} \to \tilde{C}$ has no multiple fibers and $\tilde{X} \to X$ is an unramified holomorphic covering map.*

3.2.1 The Jacobian fibration

We first begin with a preliminary discussion in the absolute case, the case where the base is just one point.

Let Z be a one dimensional subscheme (or a curve) in a complex projective surface. The arithmetic genus of Z, defined by $p_a(Z) = h^1(\mathcal{O}_Z) := \dim_{\mathbb{C}} H^1(\mathcal{O}_Z)$, is equal to the geometric genus when Z is smooth. Assume

now that Z is an arbitrary fiber in an elliptic fibration. Since p_a is an invariant in any algebraic family of curves ([Ha, III, cor. 9.13]), we have $p_a(Z) = 1$ and so $H^1(\mathcal{O}_Z) = \mathbb{C}$. From the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$, we construct the cohomology long exact sequence over Z to deduce:

$$0 \to \underset{\substack{\| \\ \text{a } \mathbb{Z}\text{-module}}}{H^1(Z, \mathbb{Z})} \overset{i}{\to} \underset{\substack{\| \\ \mathbb{C}}}{H^1(\mathcal{O}_Z)} \to \underset{\substack{\| \\ \text{Pic}(Z)}}{H^1(\mathcal{O}_Z^*)} \overset{\delta}{\to} H^2(Z, \mathbb{Z}) \to 0$$

Fact: (Let Z be non-singular.) $\text{Pic}(Z)$ is naturally identified with the space of holomorphic line bundles over Z, which, in our case of $p_a = 1$, is a 1-dimensional complex Lie group under tensor product. Every line bundle L can be written as $\mathcal{O}(E)$ for some divisor $E = \sum a_i s_i$ ($a_i \in \mathbb{Z}$, $s_i \in Z$) and $\delta(L) = \deg E := \sum a_i$.

Definition 3.5 $\text{Pic}^0(Z) := \ker \delta$ *is the subgroup of* $\text{Pic}(Z)$ *of line bundles* L *with trivial first Chern class* $c_1(L) := \delta(L)$.

If Z is a smooth elliptic curve with a base point σ, we can construct a group homomorphism from Z to $\text{Pic}^0(Z)$ by the map

$$x \in Z \overset{f}{\mapsto} \mathcal{O}(x - \sigma) \in \text{Pic}^0(Z).$$

Lemma 3.6 *The map f is holomorphic, one-to-one and hence onto.*

Proof: As f is holomorphic by construction, we need to prove only that it is one-to-one. Assume not, so that $\mathcal{O}(x - \sigma) = \mathcal{O}(x' - \sigma)$ where $x \neq x'$. Then $\mathcal{O}(x - x')$ corresponds to the trivial line bundle over Z and so Z has a rational function with a simple pole at x' and a simple zero at x. This gives a 1-1 and hence surjective holomorphic map from Z, which has genus 1, to $\mathbb{P}^1$, which has genus 0. This is a contradiction. $\square$

Note: $\text{Pic}^0(Z) = H^1(\mathcal{O}_Z)/i(H^1(Z, \mathbb{Z}))$.

We now return to the case in which the base is a curve.

Given an elliptic fibration $f : X \to C$ without multiple fibers, one can construct a relative version of Pic^0 as follows (see [BPV, p. 153]). We first form the $\mathcal{O}_C$ module

$$\mathcal{J}ac(f) = f_{*1}(\mathcal{O}_X)/f_{*1}\mathbb{Z}$$

over C. Since $p_a(X_s) = 1$ for every fiber, it follows that $f_{*1}(\mathcal{O}_X)$ is locally free of rank 1 (by a well known theorem of Grauert) and hence is the sheaf of sections of a line bundle L over C. Hence $\mathcal{J}ac(f)$ corresponds to the sheaf of sections of

$$\text{Jac}(f) := L/f_{*1}\mathbb{Z},$$

which is a holomorphic fibration of complex Lie groups with a zero section (see [Ko2], compare also [BPV, V.9]). Note that when X_s is smooth elliptic, $(f_{*1}\mathbb{Z})_s = H^1(X_s, \mathbb{Z})$ which embeds in $L_s = H^1(\mathcal{O}_{X_s}) = \mathbb{C}$. So $\mathrm{Jac}(f)_s = \mathrm{Pic}^0(X_s)$. Note also that $\mathrm{Jac}(f)$ is a holomorphic quotient of a line bundle L over C.

We have the following theorem from Kodaira [Ko2] (see [BPV, V9.1]).

Proposition 3.7 *Let $f : X \to C$ be a relatively minimal elliptic fibration over a curve C with a holomorphic section $\sigma : C \to X$. Let X'_σ consist of all irreducible components of fibers X_s not meeting $\sigma(C)$, and let $X^\sigma = X \setminus X'_\sigma$. Then there is a canonical fiber-preserving isomorphism h from $\mathrm{Jac}(f)$ onto X^σ mapping the zero-section in $\mathrm{Jac}(f)$ onto $\sigma(C)$.*

Hence it is useful to construct holomorphic sections of elliptic fibrations for which we develop the following key lemma.

Lemma 3.8 *Given a relatively minimal elliptic fibration $f : X \to C$ without multiple fibers, assume C is non-compact. Then f has a holomorphic section. Furthermore, given a countable subset T of X whose image $f(T)$ is discrete in C, the section can be chosen to avoid T.*

Proof: From Kodaira's table of non-multiple singular fibers ([Ko2] or [BPV, Table 3 p. 150]), we see that every fiber which is not multiple in a relatively minimal elliptic fibration has a component of multiplicity one. So, every point on C admits a neighborhood with a section. We now choose a locally finite good covering of C by open sets $U_1, U_2, \ldots$, with sections $\tau_1, \tau_2, \ldots$ of $f|_{U_1}, f|_{U_2}, \ldots$, respectively. We may further stipulate that there are no singular fibers on the intersection of any two U_j's.

Let $L = f_{*1}\mathcal{O}_X$, which is a holomorphically trivial line bundle over C since C is Stein. Let $U \subseteq C$ be open and $\tau' \in H^0(U, L)$ a section. If τ is a section of $f|_U$, then we can form the section $\tau + \tau'$ of $f|_U$ by Proposition 3.7. By the same proposition and the fact that all fibers are elliptic curves over $U_i \cap U_j$, there is a section $\tau'_{ij} \in H^0(U_i \cap U_j, L)$ such that $\tau_i + \tau'_{ij} = \tau_j$ on $U_i \cap U_j$.

As $\{\tau'_{ij}\}$ satisfies the cocycle condition, so does $\{-\tau'_{ij}\}$. By the solution to the classical additive Cousin problem (or from the fact that $H^1(\{U_i\}, L) = H^1(C, L) = H^1(C, \mathcal{O}) = 0$ by Leray's theorem, Dolbeault's isomorphism, and the fact that C is Stein), one can find holomorphic sections $\tau'_i \in H^0(U_i, L)$ such that $\tau'_i - \tau'_{ij} = \tau'_j$. Then $\tau_i + \tau'_i = \tau_j + \tau'_j$ on $U_i \cap U_j$ for all i, j. This gives rise to a global section of $f : X \to C$.

Given such a global section, Proposition 3.7 gives a fiber-preserving dominating map $F : L \to X$ where $F^{-1}(x) \subset L$ is at most a countable discrete set for all x in X. Hence $F^{-1}(T)$ is also a countable set and is supported on the fibers of L over $f(T)$. For each $s \in f(T)$, therefore, we may choose a point q_s in $L \setminus T$. As L is isomorphic to the trivial line bundle

(C being non-compact), the classical interpolation theorems of Mittag-Leffler and of Weierstrass give us a holomorphic section σ of L with the prescribed value q_s for all $s \in T$. But then $F \circ \sigma$ is a section of f which avoids T. This completes the proof. $\qquad \square$

3.2.2 Theorem 1.1 in the case of elliptic fibrations

Theorem 3.9 *Let $f : X \to C$ be a relatively minimal elliptic fibration with a finite number of multiple fibers. Assume that C is a Zariski open subset of a projective curve $\bar{C}$. Let n_s be the multiplicity of the fiber X_s. Then the following are equivalent.*

(a) *X is dominable by $\mathbb{C}^2$.*

(b) *$\chi := 2 - 2g(\overline{C}) - \#(\overline{C} \setminus C) - \sum\limits_{n_s \geq 2} \left(1 - \dfrac{1}{n_s}\right) \geq 0.$*

(c) *There exists a holomorphic map of $\mathbb{C}$ to X whose image is Zariski dense.*

Remark 1: $\chi = \chi(C, D)$ is the orbifold Euler characteristic of (C, D). It can be written as $\chi(C, D) = 2 - 2g(\overline{C}) - \sum\limits_{s \in \overline{C}} \left(1 - \dfrac{1}{n_s}\right)$ if we set $n_s = \infty$ for $s \in \overline{C} \setminus C$ (where $\frac{1}{\infty} = 0$). Hence, if we complete the $\mathbb{Q}$-divisor $D = \sum\limits_{s \in C} \left(1 - \dfrac{1}{n_s}\right)s$ to $\overline{D} = \sum\limits_{s \in \overline{C}} \left(1 - \dfrac{1}{n_s}\right)s$ on $\overline{C}$, then

$$\chi(C, D) = 2 - 2g(\overline{C}) - \deg \overline{D}.$$

Proof of theorem: The pair (C, D) defines an orbifold as given in Definition 3.3. We will show that (a) holds if $\chi(C, D) \geq 0$ while property C holds for X (that is, (c) fails to hold) if $\chi(C, D) < 0$. This will conclude the proof.

From the classical uniformization theorem for orbifold Riemann surfaces (see, for example, [FK, IV 9.12]), (C, D) has a uniformizing orbifold covering $\tilde{C}$ which is $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$ according to $\chi(C, D) > 0$, $\chi(C, D) = 0$ or $\chi(C, D) < 0$ respectively, unless $\overline{C} = \mathbb{P}^1$ and $\overline{D}$ has one or two components. In the latter ("unless") case, we simply redefine C to be the complement of the components of $\overline{D}$ in $\mathbb{P}^1$ and reset D to be 0, shrinking X as a result. We can do this because it does not change the fact that $\chi(C, D) \geq 0$ and because once we show that the resulting X is dominable by $\mathbb{C}^2$, the original X is also.

By pulling back the fibration to $\tilde{C}$, we obtain a relatively minimal elliptic fibration Y over $\tilde{C}$. Now, Proposition 3.4 implies that the natural map from Y to X is an unramified covering. Hence any holomorphic map from $\mathbb{C}$ to X must lift to a holomorphic map to Y. It follows that if $\tilde{C} = \mathbb{D}$, then any

such map must lift to a fiber and hence its image in X must lie in a fiber. So, property C holds and X cannot be dominated by $\mathbb{C}^2$ in this case.

It remains to show that X is dominated by $\mathbb{C}^2$ in the case $\tilde{C} = \mathbb{C}$ or $\mathbb{P}^1$ to complete the proof of this theorem. Note that the latter case can be reduced to the former by simply removing a point from $\tilde{C}$. Hence, we may take $\tilde{C}$ to be $\mathbb{C}$ which is non-compact. Lemma 3.8 now applies to give a section of the pullback fibration $\tilde{f} : Y \to \mathbb{C}$. By Proposition 3.7, Y is dominated by $\mathrm{Jac}(\tilde{f})$ which in turn is dominated by a line bundle L over $\mathbb{C}$ by construction. Hence X is dominated by $L = \mathbb{C}^2$ (since any line bundle over $\mathbb{C}$ is holomorphically trivial) as required.   □

Now, let $f' : X' \to C$ be an arbitrary elliptic fibration. By contracting the (-1)-curves on the fiber, we get a bimeromorphic map α from X' to a surface X having a relatively minimal elliptic fibration structure over C. As before, X defines an orbifold structure D on C. If X has an infinite number of multiple fibers or if C is not quasi-projective, then $\mathbb{D}$ is the universal covering of (C, D) and conditions (a) and (c) of this theorem both fail for X. Otherwise the above theorem can be applied to conclude that conditions (a) and (c) are still equivalent for X. Let T be the indeterminacy set of α^{-1}. By examining the last paragraph of the above proof, we see that Lemma 3.8 actually applies to give us a dominating map from the trivial line bundle L over $\tilde{C}$ to X, and the zero-section of L maps to a section of f that avoids T. Composing with a self-map of L given by a section of L with prescribed zeros (just as in Equation 3.1) then gives us a dominating map from L to X which avoids T. Hence, if X is dominable by $\mathbb{C}^2$, then X' is also. It is clear that X' satisfies property C if X does. Hence, we obtain the following, which covers Theorem 1.1 in the case of elliptic surfaces.

Theorem 3.10 *Let $f : X \to C$ be an elliptic fibration. Then conditions (a) and (c) of Theorem 3.9 above are equivalent for X; that is, dominability by $\mathbb{C}^2$ is equivalent to having a Zariski-dense holomorphic image of $\mathbb{C}$.*

Note that we do not require C to be quasi-projective in this theorem.

3.2.3 An algebro-geometric characterization

In this section, we will give, without proof, a characterization of dominability by $\mathbb{C}^2$ for a projective elliptic fibration in terms of familiar quantities in algebraic geometry and not involving the fundamental group. Unfortunately, the condition given is not straightforward nor does it seem very tractable. Hence, we will leave the proof (which is based on the simple fact that the saturation of the cotangent sheaf of the base, pulled back by the fibration map, includes the orbifold cotangent sheaf as a $\mathbb{Q}$ subsheaf) to the reader. We will deal only with the case of $\kappa = 1$ since the other possibility of $\kappa = 0$ contains the, so far, problematic K3 surfaces. However, all surfaces with $\kappa = 0$ other than the K3's are dominable by $\mathbb{C}^2$. We note that from

the classification list in Sect. 2, a surface with $\kappa = 1$ is necessarily elliptic. Before the statement of the following proposition, recall that a vector sheaf is called big if it contains an ample subsheaf. Recall also that a divisor in a surface is nef if its intersection with any effective divisor is non-negative.

Proposition 3.11 *Let X be a projective surface with $\kappa(X) = 1$. Then X is dominable by $\mathbb{C}^2$ if and only if there exists a nef and big divisor H such that, for every nef divisor N with $K_X N = 0$, there exists a positive integer m with $S^m \Omega_X(H - N)$ big.*

It is not difficult to extract a birational invariant out of this from $\mathbb{Q}$-subsheaves of the cotangent sheaf of such an elliptic surface; again we leave this to the interested reader.

In the remainder of this section, we give a more satisfactory and elementary characterization of dominability, now in terms of the fundamental group.

3.3 The fundamental group of an elliptic fibration

We begin with the remark that, except for our narrow focus on holomorphic geometry, most of the results we obtain in this section are not presumed to be new.

Let $f : X \to C$ be an elliptic fibration. Then the fibration determines a branched orbifold structure D on C as given in Definition 3.3. Let C° be the complement of the set of branch points in C. Then $X^\circ = f^{-1}(C)$ is an elliptic fibration defined by $f^\circ = f|_{X^\circ}$, which has no multiple fiber. Let X' be the complement of the singular fibers in $\bar{X}$. Then $f' = f|_{X'}$ defines a smooth fibration over a curve $C' \subseteq C$, and is therefore differentiably locally trivial by Ehresmann's theorem. We have the following commutative diagram.

$$
\begin{array}{ccc}
X' \hookrightarrow & X^\circ \hookrightarrow X \\
f' \downarrow & f^\circ \downarrow \qquad \downarrow f \\
C' \hookrightarrow & C^\circ \hookrightarrow C
\end{array}
\tag{3.2}
$$

We first observe the following trivial lemma for our consideration of $\pi_1(X)$. Throughout this section, all paths are assumed to be continuous.

Lemma 3.12 *Assume that we are given a real codimension two subset W of X and a path $v : [0, 1] \to X$ such that $v(0)$ and $v(1)$ lies outside W. Then v is homotopic to a path that avoids W keeping the end points fixed.*

Proof: We first impose a metric on X. Since $[0, 1]$ is compact, there is an integer n such that $v([(i - 1)/n, i/n])$ is contained in a geodesically convex open ball B_i for all $i \in \{1, 2, ..., n\}$. Then the intersection of these balls are also geodesically convex and, in particular, connected. Now replace $v(i/n)$ by a point in $B_i \cap B_{i+1} \setminus W$ for each integer $i \in [1, n - 1]$. Then replace

$\nu|_{[(i-1)/n, i/n]}$ by a path in $B_i \setminus W$ connecting $\nu((i-1)/n)$ with $\nu(i/n)$, for each integer $i \in [1, n]$. This is possible because the complement of W in each of the open balls is connected as W is of real codimension two in them. Since the balls are contractible and intersect in connected open sets, we see that the new path is homotopic to the original one fixing the end points but now avoids W. $\qquad\square$

If the path ν given above has the same end points, that is $\nu(0) = \nu(1)$, then we call ν a loop. We will often identify ν with its image.

For the next two propositions, we observe from Kodaira's table of singular fibers (see [BPV, V.7]) that, for a fiber X_s of an elliptic fibration (as a topological space or a simplicial complex), $\pi_1(X_s)$ is either $\mathbb{Z} \oplus \mathbb{Z}$ (corresponding to a nonsingular elliptic curve), $\mathbb{Z}$ (corresponding to the (semi-)stable singular fibers), or the trivial group (corresponding to the other singular fibers).

Proposition 3.13 *Let $f : X \to C$ be an elliptic fibration. In the case $C = \mathbb{P}^1$, let X_∞ be a multiple fiber if one exists. Assume f has no multiple fibers except possibly for X_∞ and that C is simply connected. Then $\pi_1(X)$ is a quotient of $\pi_1(X_s)$ for every fiber outside X_∞. In particular, $\pi_1(X)$ is abelian.*

Proof: Since contracting (-1)-curves does not change the fundamental group, we may assume without loss of generality that f is relatively minimal. Let X_s be an arbitrary fiber. Being a CW-subcomplex of X, it is a deformation retract of a small neighborhood U which we may assume to contain a smooth fiber $X_{s'}$ nearby. Since X is path connected, we can choose any base point in considering its fundamental group. Fix then a base point $q \in X_{s'}$ and a loop Q with this base point. We will show that Q is pointed homotopy equivalent in X to a loop in $X_{s'} \subset U$. The theorem then follows as X_s is a deformation retract of U.

Since the singular fibers form a real codimension two subset, we can modify Q to avoid them up to pointed homotopy equivalence by Lemma 3.12 above. In the case $C = \mathbb{P}^1$ but X_∞ is not already given, let X_∞ be a fiber outside U and Q. Since every homotopy (of Q) in $X \setminus X_\infty$ is also one in X, we may safely replace X by $X \setminus X_\infty$ so that C becomes contractible in this case. Hence, we may assume in all cases that C is contractible and that Q is a loop in X', the complement of the singular fibers in X. So Lemma 3.8 and Proposition 3.7 apply to give an isomorphism from $\mathrm{Jac}(f)$ to X with parts of the singular fibers complemented. Hence, we get, by construction of $\mathrm{Jac}(f)$, a map θ from a holomorphically trivial line bundle L over C to X which is an unramified covering above $X' \subset X$. Fixing a point $q_0 \in \theta^{-1}(q) \subset L_{s'}$, we see that Q can be lifted to a path $\tilde{Q}$ in L from q_0 to a point $q_1 \in L_{s'}$ by the theory of covering spaces. As C is contractible, there is a homotopy retraction of L to $L_{s'}$ which provides a pointed homotopy of $\tilde{Q}$ to a path in $L_{s'}$. Pushing down this homotopy (via θ) to X gives a pointed homotopy from Q to a loop in $X_{s'}$ as required. $\qquad\square$

Looking back at the above proof, we see that we can reach the same conclusion by allowing X_s to be a multiple fiber as long as C is contractible and X is free of other multiple fibers. This can be done by contracting the loop Q, as given in the proof, but only to the neighborhood U of X_s before homotoping to X_s via the deformation retraction of U to X_s. Of course, X_s as stated in the theorem is no longer arbitrary in this case as it is a multiple fiber. If $C = \mathbb{P}^1$ and D has two components (corresponding to X having two multiple fibers), we can remove one of the components (corresponding to removing one multiple fiber from X) for the same conclusion. We recall that

$$D = \sum_{s \in C} \left(1 - \frac{1}{n_s}\right) s$$

defines the orbifold structure on X where n_s is the multiplicity of the fiber X_s. Hence, we get a complement to the above proposition.

Proposition 3.14 *Let $f : X \to C$ be an elliptic fibration defining the orbifold structure D on C. If $C = \mathbb{P}^1$ and D has one or two components, or if C is contractible and D has one component, then $\pi_1(X)$ is a quotient of $\pi_1(X_s)$ for every component s of D. Hence, $\pi_1(X)$ is abelian in these cases.*

$\square$

3.3.1 Monodromy action as conjugation in the fundamental group

Although it is not absolutely necessary, some familiarity with the notion of monodromy and vanishing cycles used in geometry may be useful for reading this section.

Let the setup be as in diagram 3.2 and let X_r be a non-singular fiber. Fix a base point q in X_r for all fundamental group considerations from now on. There is an action of $\pi_1(X', q)$ on $\pi_1(X_r, q)$ via the monodromy action which, in the case C' is not $\mathbb{P}^1$, is just the conjugation action in $\pi_1(X', q)$. Indeed, in this case, we have the following exact sequence from the theory of fiber bundles (or from elementary covering space theory)

$$0 \to \pi_1(X_r, q) \to \pi_1(X', q) \to \pi_1(C', r) \to 0, \tag{3.3}$$

from which we deduce that the monodromy action is really an action of $\pi_1(C', r)$ on $\pi_1(X_r, q)$ since the latter is abelian.

In general, we will let H denote the image of $\pi_1(X_r, q)$ in $\pi_1(X, q)$ under the inclusion of X_r in X. It is easy to see that H is a normal subgroup in $\pi_1(X)$ (by the definition of the monodromy action). In this paper, we will be mainly interested in the monodromy action on H. As opposed to the usual case of the monodromy action on the homology level, this action need not be trivial, unless we know, for example, that $\pi_1(X, q)$ is abelian. Hence, it is of interest for us to know how far $\pi_1(X, q)$ is from abelian.

With the same setup, suppose (C, D) has a uniformizing orbifold cover $\tilde{C}$. This is the case unless $\bar{C} = \mathbb{P}^1$ and $\bar{D}$ has one or two components, again

by the uniformization theorem ([FK, IV 9.12]). In the latter cases, Proposition 3.14 and Proposition 3.13 tell us that $\pi_1(X)$ is abelian so that the monodromy action on H is trivial. In all other cases, let $\tilde{f} : \tilde{X} \to \tilde{C}$ be the pullback fibration. Proposition 3.4 implies that $\tilde{X}$ is an unramified cover over X. Let R be the covering group and $G = \pi_1(X)$. From the theory of covering spaces, we know that G is an extension of $\pi_1(\tilde{X})$ by R. Since $\pi_1(X_r)$ surjects to $\pi_1(\tilde{X}) \subset \pi_1(X)$ by Proposition 3.4, we see that

$$H = \pi_1(\tilde{X}).$$

Note that R is a quotient of $\pi_1(C^\circ)$ and hence also of $\pi_1(C')$, allowing us to identify the conjugation action of R on H with the monodromy action. Hence, we have the following exact sequence (which we can regard as a quotient of the exact sequence 3.3)

$$0 \to H \to G \to R \to 0. \tag{3.4}$$

The following proposition tells us that this monodromy action on H via loops in X', which induces the conjugation action of R on H, depends only on the pointed homotopy class of the image of these loops in C. Hence, the monodromy action on H is really an action by the group $\pi_1(C)$, which is a quotient of R. In particular, it tells us that the action is trivial when C is simply connected. This is the closest analogue, on the level of π_1, of the fact that vanishing cycles are vanishing on the level of homology.

Proposition 3.15 *Let $f : X \to C$ be an elliptic fibration. Let X_r be a nonsingular fiber with a base point q. If α, β and γ are loops based at q with α in X_r, and $f \circ \beta$ is pointed homotopic to $f \circ \gamma$ in C, then $\beta^{-1}\alpha\beta$ is pointed homotopic in X to $\gamma^{-1}\alpha\gamma$.*

Proof: We may assume, via Lemma 3.12, that β and γ lie in X'. Let $h : [0, 1] \times [0, 1] \to C$ be a pointed homotopy between $f \circ \beta$ and $f \circ \gamma$, which exists by assumption. Note that

$$(f \circ \beta)(f \circ \gamma)^{-1} = h\big(\partial([0, 1] \times [0, 1])\big)$$

as loops up to pointed homotopy equivalence, where ∂ means the oriented boundary. Our conclusion would follow if we show that the monodromy action of this latter loop, call it μ, on α is trivial in $\pi_1(X)$.

By compactness of $h([0, 1] \times [0, 1])$, there is a partition $\{0 = a_0 < a_1 < \ldots < a_n = 1\}$ of $[0, 1]$ such that $h([a_{i-1}, a_i] \times [a_{j-1}, a_j])$ is contained in an open disk D_{ij} containing at most one branch point and such that the loop

$$\mu_{ij} := h\big(\partial([a_{i-1}, a_i] \times [a_{j-1}, a_j])\big)$$

lies in X', for all $i, j \in \{1, 2, ..., n\}$. Since $\pi_1\big(f^{-1}(D_{ij})\big)$ is abelian by Proposition 3.14, the monodromy action of μ_{ij} on any pointed loop in the fiber is trivial in $\pi_1(f^{-1}(D_{ij}))$, and hence in $\pi_1(X)$ as well, for all $i, j \in \{1, 2, ..., n\}$. Our result now follows from the fact that the monodromy action of μ is just the sum of the monodromy action of the μ_{ij}'s.   □

We can do a bit better when X is compact.

Lemma 3.16 *With the setup as in the above proposition, assume further that either X is compact or X is a holomorphic fiber bundle over C. Then an integer m exists, independent of β, such that β^m commutes with α in $\pi_1(X)$.*

Proof: Let H be the image of $\pi_1(X_r)$ in $\pi_1(X)$. We may assume, as before, that f is relatively minimal.

If X has a singular fiber, then H is cyclic and hence the result follows from the fact that the automorphism group of a cyclic group is finite.

If X is a holomorphic fiber bundle over C, then the monodromy actions can be realized as holomorphic automorphisms of the fiber. The group of such automorphisms is a finite cyclic extension of the group of lattice translations (this can be deduced easily or determined from the table in V.5 of [BPV] listing such groups). Hence every monodromy action up to a power is a translation on the fiber, which therefore leaves every element of $\pi_1(X_r)$ invariant.

If X is compact and has no singular fibers, then it is a holomorphic fiber bundle by Kodaira's theory of Jacobian fibrations. So the result follows by the last paragraph. □

If X is non-compact and f is algebraic without singular fibers, then the conclusion of this lemma may no longer hold. Nevertheless, we can embed X in a projective surface $\bar{X}$, which is again elliptic. Deligne's Invariant Subspace Theorem [Del] implies that elements in $\pi_1(X_r)$ which vanish in $\pi_1(X)$ are generated over $\mathbb{Q}$ by commutators of the form given by this lemma. But we can deduce this directly from the fact that the abelianization of $\pi_1(\bar{X})$ must have even rank so that either H lies in the center of $\pi_1(\bar{X})$ (in the case when $\bar{X}$ is birational to an elliptic fiber bundle) or the commutator subgroup of $\pi_1(\bar{X})$ generates H over $\mathbb{Q}$. In fact, Kodaira's theory allows us to deduce a strong version of the Invariant Subspace Theorem (in the case of elliptic fibrations) which is valid even outside the algebraic category:

Proposition 3.17 *Let $f : X \to C$ be an elliptic fibration without singular fibers and such that C is the complement of a discrete set in a quasi-projective curve. Let X_r be a fiber. Then either f is holomorphically locally trivial or $\pi_1(X_r) \otimes \mathbb{Q} = H_1(X_r) \otimes \mathbb{Q}$ is generated by the vanishing cycles — that is, by loops of the form $\alpha^{-1}\beta^{-1}\alpha\beta$ (naturally identified as elements of $\pi_1(X_r)$ via monodromy) in the notation of Proposition 3.15, where α is a loop in X_r.*

We remark that a weaker form of this proposition is in fact due to Kodaira and is disguised in the proof of Theorem 11.7 in [Ko2]. We will follow his method, almost verbatim, in our proof.

Proof: We begin with some preliminaries concerning the period function $z(s)$, which takes values in the upper half plane. Recall that, as far as

monodromy actions are concerned, we can identify $\beta \in \pi_1(X)$ with an element of $\pi_1(C)$, which we will denote again as β by abuse of notation.

By Theorem 7.1 and Theorem 7.2 of [Ko2] (neither of which requires the additional assumption of that section concerning the compactification), we have a multivalued holomorphic period function $z(s)$ on C with positive imaginary part such that, under the monodromy representation $(\beta) \in SL(2, \mathbb{Z})$ of $\beta \in \pi_1(C)$ as an automorphism of the lattice $H_1(X_r)$ with a fixed choice of basis, $z(r)$ transforms as

$$\beta_* : z(r) \longmapsto \frac{az(r) + b}{cz(r) + d}, \quad \text{where} \quad (\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

under our choice of basis. By definition, $(1, z(s))$ is the period defining the elliptic curve X_s via analytic continuation of $(1, z(r))$, which is fixed by our choice of basis on $H_1(X_r)$ (see Equation 7.3 in [Ko2]).

With a choice of basis over the point r fixed, we can regard the period function z as a single valued holomorphic function on the universal cover $\tilde{C}$. Also, we can naturally identify $\pi_1(C)$ with the covering transformation group of $\tilde{C}$ over C. Then we have (see Equation 8.2 in [Ko2])

$$z(\beta(\xi)) = \beta_* z(\xi) = \frac{az(\xi) + b}{cz(\xi) + d}, \quad \text{where} \quad (\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and } \xi \in \tilde{C}.$$

Let M denote the submodule of $\pi_1(X_r) = H_1(X_r) = \mathbb{Z} \oplus \mathbb{Z}$ generated by the vanishing cycles. After a suitable change of basis, we may assume that $M = n\mathbb{Z} \oplus m\mathbb{Z} \subseteq \mathbb{Z} \oplus \mathbb{Z}$, where m and n are integers. If M does not generate $H_1(X_r)$ over $\mathbb{Q}$, then either m or n must vanish. If m vanishes, then we must have

$$(\beta) = \begin{pmatrix} 1 & b_\beta \\ 0 & 1 \end{pmatrix} \quad \text{(for some } b_\beta \in \mathbb{Z}),$$

and therefore $z(\beta(\xi)) = z(\xi) + b_\beta$ for all $\beta \in \pi_1(C)$. Since the imaginary part of $z(s)$ is positive, $\exp[2\pi i z(s)]$ defines a single valued holomorphic function on C with modulus less than 1. Hence, it must extend to a bounded holomorphic function on the compactification $\bar{C}$ of C and therefore must be constant. It follows that $z(s)$ is constant and so the fibration is locally holomorphically trivial. If n vanishes, then

$$(\beta) = \begin{pmatrix} 1 & 0 \\ c_\beta & 1 \end{pmatrix} \quad \text{(for some } c_\beta \in \mathbb{Z}),$$

and therefore

$$1/z(\beta(\xi)) = 1/z(\xi) + c_\beta.$$

Hence, considering $\exp[-2\pi i/z(s)]$ instead of $\exp[2\pi i z(s)]$ gives us the same conclusion. This ends our proof. $\square$

In order to study $G = \pi_1(X)$, we need some information about its quotient R. This is fortunately a classical subject that we now turn to.

3.3.2 Fuchsian groups versus elementary groups

In this section, we will collect some basic definitions and facts that we will need about Kleinian groups. We refer the reader to [FK, IV.5-IV.9] and [Mas, I-V] for more details.

Let (C, D) be an orbifold and let $\tilde{C}$ be its uniformizing orbifold covering with covering group R which acts holomorphically on $\tilde{C}$. Since $\tilde{C} = \mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{D}$, all of which have natural embeddings into $\mathbb{P}^1$, R can be identified as a subgroup of the group $\mathbb{M}$ of holomorphic automorphisms of $\mathbb{P}^1$, the group of Mobius transformations. So identified, R becomes a Kleinian group; that is, a subgroup of $\mathbb{M}$ with a properly discontinuous action at some point, and hence in some maximum open subset Ω, of $\mathbb{P}^1$. The set of points $\Lambda = \mathbb{P}^1 \setminus \Omega$ where R does not act properly discontinuously is called the limit set of R.

An elementary group is a Kleinian group R with no more than two points in its limit set. Such a group acts properly discontinuously on $\Delta \subset \mathbb{P}^1$, where Δ is $\mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$.

By a Fuchsian group, we mean a Kleinian group R with a properly discontinuous action on some disk $\mathbb{D} \subset \mathbb{P}^1$ such that $\mathbb{D}/R$ is quasi-projective; that is, $\mathbb{D}$ is the uniformizing orbifold covering of an orbifold (C, D) where $C = \mathbb{D}/R$ is quasi-projective. If D has finitely many components, then (C, D) is known as a finite marked Riemann surface and R is called basic. The limit set of a Fuchsian group necessarily contains the boundary of $\mathbb{D}$ (which characterizes Fuchsian groups of the first kind in the literature). It follows that a Fuchsian group cannot be an elementary group. We can also see this directly as follows.

Lemma 3.18 *An elementary Kleinian group is not a Fuchsian group.*

Proof: Let R be an elementary Kleinian group, then R acts properly discontinuously on $\Delta = \mathbb{P}^1$, $\mathbb{C}$ or $\mathbb{C}^*$ as a subset of $\mathbb{P}^1$. If R also acts on a disk $\mathbb{D} \subset \mathbb{P}^1$, then the boundary of this disk with at most two points removed is contained in Δ. Since R is properly discontinuous on Δ, and hence on this punctured boundary, $\mathbb{D}/R$ is not quasi-projective. Hence R is not Fuchsian. □

The following is a direct consequence of the uniformization theorem.

Proposition 3.19 *Let (C, D) be a uniformizable orbifold where D has a finite number of components. Let R be the uniformizing orbifold covering group of (C, D) properly regarded as a Kleinian group. Then $\chi(C, D) < 0$ if and only if R is a Fuchsian group while $\chi(C, D) \geq 0$ if and only if R is an elementary group.*

The reader is cautioned that Lemma 3.18 is not a corollary of this since the definition of an elementary group is more general than that given in this proposition.

Concerning R as an abstract group, Proposition 3.19 and the basic theory of elementary Kleinian groups (see [Mas, V.C and V.D] or [FK, IV 9.5]) gives:

Proposition 3.20 *With the same setup as Proposition 3.19, assume that $\chi(C, D) \geq 0$. Then there is a finite orbifold covering $\tilde{C}$ of (C, D) such that $\tilde{C} = \mathbb{P}^1$, $\mathbb{C}^*$ or an elliptic curve. In particular, R is a finite extension of a free abelian group of rank at most two.*

Quoting [Mas, V.G.6], using Lemma 3.18 and Proposition 3.20, we have:

Proposition 3.21 *Let R be a Fuchsian group as defined above. Then R is not a finite extension of an abelian group. Hence, R is not isomorphic to an elementary group as an abstract group.*

3.3.3 The fundamental group characterization in Theorem 1.2

Before stating the main theorem of this section, we need the following proposition from [BPV, V.5]. We first note from the same source that an elliptic fiber bundle over an elliptic curve is called a primary Kodaira surface if it is not Kähler. A non-trivial free quotient of such a surface by a finite group is called a secondary Kodaira surface. The fundamental group of such a surface is unfortunately not a finite extension of an abelian group, even though the surface is $\mathbb{C}^2$-dominable.

Proposition 3.22 *An elliptic fiber bundle over an elliptic curve is either a primary Kodaira surface, or a free and finite quotient of a compact complex 2-dimensional torus.*

Armed with this, we are ready to tackle our second main theorem, Theorem 1.2, in the case of elliptic fibrations. We will state a more general theorem:

Theorem 3.23 *Let $f : X \to C$ be an elliptic fibration with C quasi-projective. Assume that X is not bimeromorphic to a free and finite quotient of a primary Kodaira surface. Then X is dominable by $\mathbb{C}^2$ if and only if $\pi_1(X)$ is a finite extension of an abelian group (of rank at most 4).*

Proof: With the assumptions as in the theorem, we let $G = \pi_1(X)$ as before. By the same argument as that for Theorem 3.10, we may assume, without loss of generality, that X is relatively minimal by contracting the (-1)-curves (as G is unchanged in this process). If X has an infinite number of multiple fibers, then the orbifold (C, D) is uniformized by $\mathbb{D}$ and so X is not dominable by $\mathbb{C}^2$. Proposition 3.21 tells us that R is not isomorphic to a finite extension of an abelian group in this case. Hence, we may also assume that D has only a finite number of components for the rest of

the proof. Theorem 3.9 then applies and so it is sufficient to show that $\chi(C, D) \geq 0$ if and only if G is a finite extension of an abelian group.

Assume that $\chi(C, D) < 0$. If (C, D) projectivizes to $(\mathbb{P}^1, \bar{D})$ (see the first remark after Theorem 3.9 for the definition of $\bar{D}$), then $\bar{D}$ must have more than two components by the definition of χ. Hence (C, D) is uniformizable and we may apply Proposition 3.19 and Proposition 3.21 to conclude that the orbifold uniformizing group R of (C, D) is not a finite extension of an abelian group. But then neither is G as R is a quotient of G.

Conversely, assume $\chi(C, D) \geq 0$. If (C, D) projectivizes to $(\mathbb{P}^1, \bar{D})$ and $\bar{D}$ has no more than two components, then G is abelian by Proposition 3.13 and Proposition 3.14. Otherwise, (C, D) is uniformizable and, with the notation as in Sect. 3.3.1, the exact sequence 3.4 implies that G is an extension of H by R. Proposition 3.20 now applies to give a pull back elliptic fibration $\hat{f} : \hat{X} \to \hat{C}$ without multiple fibers such that $\hat{X}$ is a finite unramified covering of X and such that $\hat{C} = \mathbb{P}^1, \mathbb{C}^*$ or an elliptic curve. We will consider each of these cases for $\tilde{C}$ separately. Note first that $G = \pi_1(X)$ (respectively R) is a finite extension of $\hat{G} = \pi_1(\hat{X})$ (respectively $\hat{R}$) and that $\hat{H} = H$. Replacing $\hat{C}$ by a finite unramified covering of $\hat{C}$, we may assume, thanks to Lemma 3.16 and Proposition 3.17, that H lies in the center of $\hat{G}$ (that is, the conjugation action of $\hat{G}$ on H is trivial).

In the case when $\hat{C} = \mathbb{P}^1$, Proposition 3.13 implies that $\hat{G}$ is a quotient of a free abelian group of rank two. Hence $\hat{G}$ is abelian of rank no greater than two. Since X is Kähler if and only if $\hat{X}$ is, this rank is even if X is Kähler and odd if not.

In the case when $\hat{C} = \mathbb{C}^*$, the triviality of the conjugation action of $\hat{R} = \mathbb{Z}$ implies immediately that $\hat{G}$ is abelian, of rank one greater than that of H.

In the case when $\hat{C}$ is an elliptic curve, Proposition 3.17 implies that $\hat{X}$ must either be a holomorphically locally trivial fibration over $\hat{C}$, or H is finite cyclic. In the former case, Proposition 3.22 tells us that $\hat{G}$ is a finite extension of a free abelian group of rank four. In the latter case, let $m/2$ be the order of H. Since $\hat{R}$ is abelian, the commutator of two elements a and b in $\hat{G}$ must lie in H. Hence, $ab = bac$ for some $c \in H$. Since c commutes with both a and b, we have $a^m b = ba^m$ and $a^m b^m = (ab)^m$. This shows that $\hat{G}^m = \{a^m \mid a \in \hat{G}\}$ is an abelian subgroup of $\hat{G}$ intersecting H at 1. Hence, we can form the internal direct sum $G^m \oplus H$ in G which we can easily identify with the inverse image of $\hat{R}^m$ in G, where $\hat{R}^m$ is a normal subgroup of index m^2 in $\hat{R}$. (We note as an aside that G^m is canonically isomorphic to R^m.) It follows that $\hat{G}$ becomes abelian if we replace $\hat{X}$ by a finite covering of itself and so our theorem is proved. $\square$

4 Other compact complex surfaces

We deal with the remaining cases of compact complex surfaces in this section. These are the case of zero Kodaira dimension and the case of zero algebraic dimension. In fact, by Kodaira's classification, all surfaces with Kodaira dimension zero are elliptic fibrations except for those bimeromorphic to compact complex 2-dimensional tori and K3 surfaces, where the elliptic ones form a dense codimension one family in their respective moduli space. As we have already resolved the case of elliptic fibrations in the previous section, we need to consider only the tori and the K3 surface cases. We first resolve the case of tori, and indeed prove a much stronger result of independent interest, before considering the other cases.

4.1 Compact complex tori

A 2-dimensional compact complex torus is the quotient of $\mathbb{C}^2$ by a lattice Λ of real rank 4. Let X be such a surface, which we call a torus surface. Any compact surface Y bimeromorphic to X admits a dominating holomorphic map from the complement of finitely many points in X. We show in this section that the complement of finitely many points in X is dominable by $\mathbb{C}^2$. This will follow immediately from Proposition 4.1 below. Hence, Y is also dominable by $\mathbb{C}^2$ as a result.

Following Rosay and Rudin [RR1], we say that a discrete set Λ in $\mathbb{C}^2$ is *tame* if there is a holomorphic automorphism, F, of $\mathbb{C}^2$ such that $F(\Lambda)$ is contained in a complex line. Using techniques of [RR1] or [BF], the complement of a tame set is dominable by $\mathbb{C}^2$, and in fact, there exists an injective holomorphic map from $\mathbb{C}^2$ to $\mathbb{C}^2 \setminus \Lambda$.

By a lattice, we mean a discrete $\mathbb{Z}$-module. For the following proposition, let Λ be a lattice in $\mathbb{C}^2$, let $q_1, \ldots, q_m \in \mathbb{C}^2$, and let $\Lambda_0 = \cup_{j=1}^m \Lambda + q_j$, where $\Lambda + q_j$ represents translation by q_j.

Proposition 4.1 *The set Λ_0 is tame. In particular, $\mathbb{C}^2 \setminus \Lambda_0$ is dominable by $\mathbb{C}^2$ using an injective holomorphic map.*

This result will be strengthened considerably in Sect. 6. Before proving this proposition, we need a lemma.

Lemma 4.2 *There exists an invertible, complex linear transformation $A :$ $\mathbb{C}^2 \to \mathbb{C}^2$ such that $\operatorname{Im} \pi^1 A(\Lambda_0)$ is a discrete set in $\mathbb{R}$. Moreover, we may assume that if $p, q \in A(\Lambda_0)$ with $p \neq q$, then $|p - q| \geq 1$ and either $\operatorname{Im} \pi^1 p = \operatorname{Im} \pi^1 q$ or $|\operatorname{Im} \pi^1 p - \operatorname{Im} \pi^1 q| \geq 1$.*

Proof: Let v_1, v_2, v_3, v_4 be a $\mathbb{Z}$-basis for Λ, and let E be the span over $\mathbb{R}$ of v_1, v_2, v_3. Using the real inner product, let $u_0 \neq 0$ be orthogonal to E. Using the complex inner product, let $u_1 \neq 0$ be orthogonal to u_0. Then u_1 and iu_1 are both real orthogonal to u_0, so $\mathbb{C}u_1 \subseteq E$. Choose A_1 complex linear such

that $A_1(u_0) = (1, 0)$ and $A_1(u_1) = (0, 1)$. Then $\pi^1 A_1(E)$ is a one (real) dimensional subspace of $\mathbb{C}$, so by rotating in the first coordinate, we may assume that $\pi^1 A_1(E)$ is the real line in $\mathbb{C}$.

Let $\mu_0 = \operatorname{Im} \pi^1 A_1(v_4)$, and $\mu_j = \operatorname{Im} \pi^1 A_1(p_j)$ for $j = 1, \dots, m$. Then for each $j = 1, \dots, m$ and $k \in \mathbb{Z}$, we have

$$\pi^1 A_1(E + kv_4 + p_j) \subseteq \mathbb{R} + i(k\mu_0 + \mu_j),$$

so that $\operatorname{Im} \pi^1 A_1(\Lambda_0)$ is discrete in $\mathbb{R}$. Applying an appropriate dilation to A_1 gives A as desired. $\qquad\square$

Note that this lemma implies that given a finite set of points in a complex 2-torus, there is an open set, U, containing this finite set and a nonconstant image of $\mathbb{C}$ avoiding U. In particular, the complement of U in this torus is not Kobayashi hyperbolic. As mentioned in the introduction, this result will be strengthened in Sect. 6 to show that there is a dominating map into the complement of such an open set U.

Proof of Proposition 4.1: Lemma 4.2 implies that there is a complex line $L = \mathbb{C}(z_0, w_0)$ with orthogonal projection $\pi_L : \mathbb{C}^2 \to L$ and real numbers $\mu_0, \dots, \mu_m$ such that

$$\pi_L(\Lambda_0) \subseteq \cup_{j=1}^m (\mu_0 \mathbb{Z} + \mu_j + i\mathbb{R})(z_0, w_0). \qquad (4.1)$$

I.e., identifying L with $\mathbb{C}$ in the natural way, the image of Λ_0 under π_L is contained in a union of lines parallel to the imaginary axis, and this union of lines intersects the real axis in a discrete set.

Making a linear change of coordinates, we may assume that $L = \mathbb{C}(0, 1)$, in which case we may identify π_L with projection to the second coordinate, π^2. Let π^1 denote projection to the first coordinate, and let $E = \cup_{j=1}^m (\mu_0 \mathbb{Z} + \mu_j + i\mathbb{R})$.

We next show that there is a continuous, positive function f_0 on E such that if $(z, w) \in \Lambda_0$ with $z \neq 0$, then $f_0(w)|z| \geq 2|w|$. First, define

$$r_1(w) = \begin{cases} \frac{|w|}{\min\{|z| : (z, w) \in \Lambda_0, z \neq 0\}} & \text{if } w \in \pi^2(\Lambda_0); \\ 0 & \text{if } w \in E \setminus \pi^2(\Lambda_0). \end{cases}$$

Then $r_1(w) \geq 0$, and since Λ_0 is discrete, r is upper-semicontinuous.

Let $r_2(w) = 2(r_1(w) + 1)$ for $w \in E$. Since r_2 is also upper-semicontinuous, it is bounded above on compacta, so a standard construction gives a function f_0 which is continuous on E with $f_0(w) \geq r_2(w) > 0$. Then for $(z, w) \in \Lambda_0$ with $z \neq 0$, we have $f_0(w)|z| \geq 2r_1(w)|z| \geq 2|w|$ by definition of r_1.

We next find a non-vanishing entire function f so that $|f(w)z| \geq |w|$ if $(z, w) \in \Lambda_0$ with $z \neq 0$. Since f_0 is positive on E, $\log f_0(w)$ is continuous and real-valued on E, and $\log f_0(w) \geq \log 2 + \log(r_1(w) + 1)$. By Arakelian's theorem (e.g. [RR2]), there exists an entire $g(w)$ with

$|\log f_0(w) - g(w)| < \log 2$ for $w \in E$. Then $f(w) = \exp(g(w))$ is entire and non-vanishing, and if $(z, w) \in \Lambda_0$ with $z \neq 0$, then

$$|f(w)z| = \exp(\mathrm{Re}g(w))|z| \geq r_1(w)|z| \geq |w|.$$

Finally, define $F(z, w) = (f(w)z, w)$. Then F is a biholomorphic map of $\mathbb{C}^2$ onto itself, and for $(z, w) \in \Lambda_0$ with $z \neq 0$, we have $|\pi^1 F(z, w)| \geq |\pi^2 F(z, w)|$. Since $F(\Lambda_0)$ is discrete, we see that $\pi^1 F(\Lambda_0)$ is discrete. Hence $F(\Lambda_0)$ is tame by [RR1, Theorem 3.9]. By definition of tame, Λ_0 is also tame, so as mentioned earlier, $\mathbb{C}^2 \setminus \Lambda_0$ is dominable by $\mathbb{C}^2$. $\qquad\square$

Corollary 4.3 *The complement of a finite set of points in a two dimensional compact complex torus is dominable by $\mathbb{C}^2$. Hence any surface bimeromorphic to such a torus is dominable by $\mathbb{C}^2$.*

We remark that not all tori are elliptic. The elliptic torus surfaces form a 3 dimensional family in the 4 dimensional family of torus surfaces and the generic torus contains no curves. All compact complex tori are Kähler. Also a compact surface bimeromorphic to a torus can be characterized by $\kappa = 0$ and $q = 2$.

4.2 K3 surfaces

A compact complex surface X is called a K3 surface if its fundamental group and canonical bundle are trivial. A useful fact in the compact complex category, due to Siu ([Siu]), is that all K3 surfaces are Kähler. One can show that $H^2(X, \mathbb{Z})$ is isometric to a fixed lattice L of rank 22. If ϕ is such an isometry, then (X, ϕ) is called a marked K3 surface. The set of such surfaces is parametrized by a 20 dimensional non-Hausdorff manifold $\mathcal{M}$ [BPV, VIII] (The fact that $\mathcal{M}$ is smooth follows from S.T. Yau's resolution of the Calabi conjecture in [Yau] (see e.g., [T]) and the fact that $\mathcal{M}$ is not Hausdorff is due to Atiyah ([At]).)

We first observe a few facts from the classical work of Piatetsky-Shapiro and Shafarevich in [PS] (see also [LP],[Shi],[BPV, VIII]), where they obtained a global version of the Torelli theorem for K3 surfaces. Given a marked K3 surface and a point $o \in \mathcal{M}$ corresponding to it, there is a smooth Hausdorff neighborhood $\mathcal{U}$ of o, a smooth complex manifold Z, and a proper holomorphic map $Z \xrightarrow{p} \mathcal{U}$ whose fibers are exactly the marked K3 surfaces parametrized by $\mathcal{U}$. Within this local family, the subset of projective K3 surfaces is parametrized by a topologically dense subset of $\mathcal{U}$ which is a countable union of codimension one subvarieties. The elliptic K3 surfaces (that is, K3 surfaces admitting an elliptic fibration) also form a topologically dense codimension one family in $\mathcal{U}$.

The following proposition follows directly from Theorem 3.23 and the fact that the fundamental group of a K3 surface is trivial.

Proposition 4.4 *A compact complex surface bimeromorphic to an elliptic K3 surface is holomorphically dominable by $\mathbb{C}^2$.*

The previous section on complex tori allows us to deal with another class of K3 surfaces — the Kummer surfaces, which form a 4 dimensional family in the 20 dimensional family of K3 surfaces. Such a surface X is, by definition, obtained by taking the quotient of a torus surface A (given as a complex Lie group $\mathbb{C}^2$/lattice) by the natural involution $g(x) = -x$, then blowing up the 16 orbifold singular points (resulting in 16 (-2) curves). Alternatively, one can describe X as a $\mathbb{Z}_2$ quotient of $\hat{A}$, where $\hat{A}$ is the blowing up of A at the 16 points of order 2 and where the quotient map is branched along the exceptional (-1)-curves of the blowing up. Since the inverse image of any finite set of points in X is finite in $\hat{A}$ and hence also finite in A, any surface bimeromorphic to a Kummer surface is dominable by $\mathbb{C}^2$ according to Corollary 4.3.

Proposition 4.5 *A compact surface bimeromorphic to a Kummer surface is dominable by $\mathbb{C}^2$.*

Before we leave the subject of K3 surfaces, it is worth mentioning that projective K3 surfaces are dominable by $\mathbb{D} \times \mathbb{C}$ by the work of [GG] and [MM]. Clearly, elliptic K3 surfaces and Kummer surfaces are so dominable as well. Such a surface cannot be measure hyperbolic as defined by Kobayashi ([Kob]). However, it is still an unsolved problem whether all K3 surfaces are so dominable. The only other compact complex surfaces for which this problem remains open are the non-elliptic and non-Hopf surfaces of class VII_0 outside the Inoue-Hirzebruch construction.

4.3 Other compact surfaces and our two main theorems

Besides those bimeromorphic to K3 and torus surfaces, the remaining compact complex surfaces with zero Kodaira dimension are all elliptic, and are all dominable by $\mathbb{C}^2$. Such a surface must be bimeromorphic to either a Kodaira surface (defined and characterized in Sect. 3.3.3), a hyperelliptic surface (which is a finite free quotient of a product of elliptic curves, and hence projective), or an Enriques surface (which is a surface admitting an unramified double covering by an elliptic K3 surface). Except for the first among these three types, the fundamental group is always a finite extension of an abelian group.

Finally, the only remaining compact complex surfaces are those with algebraic dimension 0 and $\kappa = -\infty$. This category includes the non-elliptic Hopf surfaces, which are dominable by $\mathbb{C}^2$ by construction (see [Ko4]). This category also includes the Inoue surfaces, which must be excluded from our main theorems since their universal cover is $\mathbb{D} \times \mathbb{C}$, hence are not dominable by $\mathbb{C}^2$, while any nonconstant image of $\mathbb{C}$ must be Zariski dense (see Proposition 19.1 in [BPV, V]). However, it is of interest to note that

the Zariski dense holomorphic images of $\mathbb{C}$ are constrained by higher order equations on an Inoue surface so that if we relax property C in this sense, we can in fact include Inoue surfaces in the next theorem. Unfortunately, aside from the Hopf surfaces and the Inoue surfaces, the detailed structure of surfaces of this type is not yet clear even though we know the existence of projective affine structures for a special subclass of these surfaces.

We now summarize our investigation in the compact category by giving the following extensions of our main theorems stated in the introduction:

Theorem 4.6 *Let X be a compact complex surface of Kodaira dimension less than 2. Assume that either $\kappa(X) \neq -\infty$ or $a(X) \neq 0$. In the case that X is bimeromorphic to a K3 surface that is not Kummer, assume further that X is elliptic. Then X is dominable by $\mathbb{C}^2$ if and only if it does not satisfy property C. Equivalently, there is a dominating holomorphic map $F : \mathbb{C}^2 \to X$ if and only if there is a holomorphic image of $\mathbb{C}$ in X which is Zariski dense.*

Theorem 4.7 *Let X be a compact complex surface not bimeromorphic to a Kodaira surface. Assume that either $\kappa(X) \neq -\infty$ or $a(X) \neq 0$. In the case that X is bimeromorphic to a K3 surface that is not Kummer, assume further that X is elliptic. Then X is dominable by $\mathbb{C}^2$ if and only if it has Kodaira dimension less than two and its fundamental group is a finite extension of an abelian group (of rank 4 or less).*

5 Non-compact algebraic surfaces

We begin with a key example which motivated the general algebraic setting. This is the example of the complement of a smooth cubic curve in $\mathbb{P}^2$, which we will show to be dominable by $\mathbb{C}^2$.

5.1 Complement of a cubic in $\mathbb{P}^2$

Let C be a smooth cubic curve in $\mathbb{P}^2$ and let $X = \mathbb{P}^2 \setminus C$. Then its logarithmic canonical bundle $K_{\mathbb{P}^2}(C)$ is the trivial line bundle as $\deg K_{\mathbb{P}^2} = -3$. Hence, $\bar{\kappa}(X) = 0$ and X is a logarithmic K3 surface; that is, a non-compact 2-dimensional Calabi-Yau manifold.

Proposition 5.1 *The surface $X = \mathbb{P}^2 \setminus C$ is dominable by $\mathbb{C}^2$.*

Proof: A tangent line to C at a non-inflection point meets C at one other point. This gives rise to a holomorphic $\mathbb{P}^1$ bundle with two holomorphic sections. To see that this is actually a bundle (i.e. locally trivial), identify it with the projectivization of the tautological vector bundle of rank two over the dual curve of C with the obvious isomorphism. We may pull back this $\mathbb{P}^1$ bundle and the sections to the universal cover $\mathbb{C}$ of C, with two sections s_∞

and s. Hence one may regard the complement of $s_\infty(\mathbb{C})$ of this bundle as a trivial line bundle on $\mathbb{C}$ with a meromorphic section s (with poles coming from points of inflection of the cubic).

Hence, it suffices to construct a holomorphic map from $\mathbb{C}^2$ onto the complement of the graph of a meromorphic function s to give a dominating map to X. Note that each vertical slice of the complement of the graph is $\mathbb{C}^*$ except at a pole of s, where the vertical slice is $\mathbb{C}$.

To construct such a map, first define

$$\psi(t, w) = \frac{\exp(tw) - 1}{t} \tag{5.1}$$

$$= w + \frac{tw^2}{2!} + \frac{t^2 w^3}{3!} + \cdots \tag{5.2}$$

which is entire on $\mathbb{C}^2$. Note that $(t, \psi(t, w))$ is a fiberwise selfmap of $\mathbb{C}^2$ which misses precisely the graph of $-1/t$, a function with a simple pole at the origin.

Since $\mathbb{C}$ is Stein, there exists an entire function g such that $\frac{1}{g}$ has the same principle parts as s. This is because we may write $s = f/f_1$ where f and f_1 are entire with no common zeros. So $\log f$ is well defined in a neighborhood of each zero of f_1. By Mittag-Leffler and Weierstrass, we can find an entire function g_1 with the same Taylor expansion as $\log f$ to the order of vanishing of f_1 at each zero of f_1. Then $g = f_1/\exp g_1$ is our desired function. In particular, g vanishes precisely when s has a pole. Then $h = s - \frac{1}{g}$ is entire, so

$$\phi(z, w) = h(z) - \psi(g(z), w)$$
$$= s(z) - \frac{\exp(wg(z))}{g(z)}$$

is entire on $\mathbb{C}^2$. For fixed z with $g(z) \neq 0$, we see from the second equality that $\phi(z, w)$ can attain any value in $\mathbb{C} \setminus \{s(z)\}$ by varying w. If $g(z) = 0$, then $\phi(z, w) = h(z) - w$, which can attain any value in $\mathbb{C}$ by varying w.

Hence, the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \text{graph}(s)$ given by

$$\Phi(z, w) = (z, \phi(z, w))$$

is holomorphic and onto. Composing this map with the map into the $\mathbb{P}^1$ bundle over C, we obtain a dominating map into the complement of the cubic C. $\qquad\qquad\square$

Note that an important step here is the construction of an entire function h whose graph does not intersect the graph of s. This is certainly analogous to the situation of elliptic fibrations.

Remark: The complement of a smooth cubic does not admit any algebraic map to $\mathbb{P}^1$ whose generic fiber contains $\mathbb{C}^*$. This is the only example among complements of normal crossing divisors in $\mathbb{P}^2$ with this property. In fact,

this is the only meaningful affine example with this property that is dominable by $\mathbb{C}^2$ (see [M, p. 189]). Since this is a logarithmic K3 surface, this phenomenon is suggestive of the situation for a generic compact K3 surface.

We isolate the following useful theorem from the above proof.

Theorem 5.2 *Let s be a meromorphic function on $\mathbb{C}$. Then the complement of the graph of s admits a dominating fiber-preserving holomorphic map from $\mathbb{C}^2$.*

5.1.1 Complements of normal crossing divisors in $\mathbb{P}^2$

Let X be the complement of a normal crossing divisor D in $\mathbb{P}^2$. If $\deg D > 3$, then $\bar{\kappa}(X) = 2$ and hence X is not dominable by $\mathbb{C}^2$. If $\deg D = 3$, then D consists of at most three components and it is easy to check that X is dominable by $\mathbb{C}^2$ as follows. If D has only one component, then it is either a smooth cubic or a cubic with one node. In the first case, the result follows from Proposition 5.1. In the second case, blowing up that node gives us a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ with two sections, one corresponding to the exceptional curve of the blow-up. These two sections intersect precisely at the two fibers of the bundle corresponding to the two tangent directions of the cubic at the node. Hence, removing these two fibers gives us a surface biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, which is dominable by $\mathbb{C}^2$. If D has two components, then it consists of a line and a conic (that is, a smooth curve of degree two) intersecting at two points. Blowing up one of the points of their intersection (corresponding to projecting from this point of intersection) gives us a $\mathbb{P}^1$ bundle over $\mathbb{C}$ with two sections complemented, one of which is the exceptional curve of the blow-up. If we think of one section as ∞, then the other section can be regarded as a meromorphic function on $\mathbb{C}$ and so Theorem 5.2 applies to give a dominating map from $\mathbb{C}^2$ to X. An easier way is to delete the fiber containing the only point of intersection of these two sections. The resulting X is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and hence dominable by $\mathbb{C}^2$. If D has three components, then each must be a line and X is $\mathbb{C}^* \times \mathbb{C}^*$, which is dominable by $\mathbb{C}^2$.

From the above argument, we see also that if $\deg D < 3$, then X is dominable by $\mathbb{C}^2$. In summary, we have:

Theorem 5.3 *Let D be a normal crossing divisor in $\mathbb{P}^2$. Then $\mathbb{P}^2 \setminus D$ is dominable by $\mathbb{C}^2$ if and only if $\deg D \leq 3$.*

We remark that this theorem is no longer true if D is not normal crossing. The unique counterexample in one direction is when D consists of three lines intersecting at only one point, which is not dominable by $\mathbb{C}^2$. Another counterexample, but in the opposite direction, is given by the complement of the union of a conic and two lines intersecting at a point of the conic (which we discussed in the two component case of $\deg D = 3$ above).

5.2 The general quasi-projective case

Let X be an algebraic surface over $\mathbb{C}$. Then $X = \bar{X} \setminus D$ where $\bar{X}$ is projective and D is a normal crossing divisor in $\bar{X}$. This is the notation set forth in Sect. 2 and we will assume this setup throughout this section. Kawamata ([K1],[K2],[K3]) has considered the structure of X and obtained a classification theory analogous to that in the projective case. Much of this is explained in some detail in Miyanishi ([M]). We will use their results directly to tackle our problem in this section.

   If there is a surjective morphism $f : X \to C$ whose generic fiber is connected, then we say that X is fibered over C. (We remind the reader that morphisms are algebraic holomorphic maps.) More generally, if f is required to be only holomorphic rather than a morphism, then we say that X is holomorphically fibered over C. For example, the complement of the graph of a meromorphic function is holomorphically fibered over C with generic fiber $\mathbb{C}^*$. As before, we let $X_s = f^{-1}(s)$ be the fiber over s. We first quote the subadditivity property of (log-)Kodaira dimension due to Kawamata ([K1]):

Proposition 5.4 *If X is fibered over a curve C, then*

$$\bar{\kappa}(X) \geq \bar{\kappa}(C) + \bar{\kappa}(X_s)$$

for s outside a finite set of points in C; that is, for the generic fiber X_s.

   From the definitions, a curve of positive genus with punctures has positive Kodaira dimension. An elliptic curve has Kodaira dimension zero. A punctured $\mathbb{P}^1$ has $\kappa = -\infty, 0$ or 1 according to the number of punctures being 1, 2 or greater than 2, respectively.

   Given a dominating morphism f between algebraic varieties, it is clear that f^* is injective on the level of logarithmic forms (see [Ii]). Since tensor powers of top dimensional logarithmic forms define the Kodaira dimension, we see that if f is equidimensional, then it must decrease Kodaira dimension.

   If $\bar{q}(X) > 0$, then there is a morphism from X to a semi-abelian variety (a commutative algebraic Lie group that is an extension of a compact torus by $(\mathbb{C}^*)^k$ for some k) of dimension $\bar{q}(X)$, called the quasi-Albanese map and constructed by Iitaka in [Ii1]. One has the simple formula relating $\bar{q}(X)$ to the first Betti numbers of X and $\bar{X}$:

$$\bar{q}(X) - q(\bar{X}) = b_1(X) - b_1(\bar{X}).$$

   Note that $\mathbb{C}$ does not support any logarithmic form by this formula.

5.2.1 Surfaces fibered by open subsets of $\mathbb{P}^1$

Let X be fibered over a curve C by a map f whose generic fiber is $\mathbb{P}^1$ (possibly) with punctures. Then, by a finite number of contractions of

(-1)-curves that remain on the fiber, the compactification $\bar{X}$ of X admits a birational morphism g to a ruled surface $\bar{Y}$ over a projective curve $\bar{C}$, the compactification of C, and g is a composition of blowing ups. Hence $Y = \bar{Y}|_C$ is a $\mathbb{P}^1$ bundle over C, whose bundle map will again be denoted by g. We may write $f = h \circ g$, where

$$\mathbb{C} \xrightarrow{r} X \xrightarrow{g} Y \xrightarrow{h} C. \tag{5.3}$$

If every holomorphic image of $\mathbb{C}$ in X is constant in C (when composed with f), then X satisfies property C. Otherwise, there exists a holomorphic map $r : \mathbb{C} \to X$ such that $f \circ r$ is not constant. By taking the fiber product with $f \circ r$, we can pull back the factorization picture 5.3 to one over $\mathbb{C}$

$$\mathbb{C} \xrightarrow{\tilde{r}} \tilde{X} \xrightarrow{\tilde{g}} \tilde{Y} \xrightarrow{\tilde{h}} \mathbb{C},$$

where $\tilde{f} = \tilde{h} \circ \tilde{g}$ is surjective with a holomorphic section $\tilde{r}$. Here, $\tilde{X}$ may be singular, but we will regard it only as an auxiliary space.

We will first deal with the case where the general fiber has at most one puncture; that is, $X_s = \mathbb{P}^1$ or $\mathbb{C}$ for s in an open subset of C. We can then regard $\tilde{Y}$ as a trivial $\mathbb{P}^1$ bundle with a section D_∞ to which the puncture (if one exists) on the "generic" fiber of $\tilde{f}$ is mapped. Note that $\tilde{Y} \setminus D_\infty = \mathbb{C}^2$ with coordinates (z, w), and so we may regard a section of $\tilde{h}$ as a meromorphic function on $\mathbb{C}$. In particular, $\tilde{g} \circ \tilde{r}$ is a meromorphic section of $\tilde{h}$.

Since $\bar{X}$ is obtained from $\bar{Y}$ by a finite number of blow ups, we can identify points on X as infinitely near points on Y of order 0 or more as in [Ha, p. 392]. Note that the set of fibers in Y which contain infinitely near points of order 1 or more is finite (since the set of such fibers in $\bar{Y}$ is finite). This finite set of fibers in Y pulls back to a discrete set of fibers in $\tilde{Y}$. In Y, such a higher order infinitely near point corresponds to a point in X obtained by finitely many blow-ups, hence to the specification of a finite jet at the point in Y. Under pull-back, this corresponds to a finite jet in $\tilde{Y}$. Additionally, there is a finite set of fibers in Y which may have more than one puncture, and these fibers all pull back to a discrete set of fibers in $\tilde{Y}$. Together, these two types of fibers will be called exceptional fibers.

In order to produce a dominating map into X, it suffices to produce a fiberwise dominating map $F(z, w) = (z, H(z, w))$ into $\tilde{Y}$ which respects these exceptional fibers in the following sense. If $\tilde{Y}_s$ is an exceptional fiber, then $F(s, w)$ is a single point independent of w. Moreover, if $\tilde{Y}_s$ is a fiber having more than one puncture, then the image of the map F should avoid all such punctures. If $\tilde{Y}_s$ is a fiber having an infinitely near point, then $F(s, w)$ should equal $\tilde{g} \circ \tilde{r}(s)$. Additionally, if $\tilde{g} \circ \tilde{r}$ passes through this infinitely near point, and ϕ is holomorphic in a neighborhood of s, then the local curve $z \mapsto (z, H(z, \phi(z)))$ should agree with the jet given by the infinitely near point on $\tilde{Y}_s$.

Fortunately, the section $\tilde{g} \circ \tilde{r}$ has the correct jet whenever it intersects one of these exceptional fibers, so we can use this section to obtain such a map. Let $q(z) = \tilde{g} \circ \tilde{r}(z)$, which is meromorphic. We will define $H(z, w) = p(z)w + q(z)$ for some entire $p(z)$. For each exceptional fiber $\tilde{Y}_s$, there is an integer $n_s \geq 1$ such that if p vanishes to order n_s at s, then F defined with this H respects the exceptional fiber as indicated above. By Weierstrass' theorem, there exists p entire vanishing exactly to order n_s at each s. Then $F(z, w) = (z, H(z, w))$ gives a dominating map from $\mathbb{C}^2$ into $\tilde{Y}$ respecting the exceptional fibers, and this map pushes forward to Y, then lifts to give a dominating map into X, as desired.

We now deal with the case where the generic fiber of f is $\mathbb{C}^*$. In this case, $\tilde{Y}$ can be identified with a $\mathbb{P}^1$ bundle with a double section D_Y, to which the punctures on the "generic" fibers of $\tilde{f}$ maps to. Now, either D_Y consists of two components, both of which are smooth sections of $\tilde{h}$, or D_Y consists of one component. In either case, outside of a discrete set of fibers, D_Y can be written locally as the union of two mermorphic sections. Moreover, we define the set of exceptional fibers exactly as in the previous case.

First, using a fiber-preserving biholomorphic map of $\mathbb{C} \times \mathbb{P}^1$ to itself, we may move $\tilde{g} \circ \tilde{r}$ to become the ∞-section. Then the requirement of agreeing with the jet of $\tilde{g} \circ \tilde{r}$ at a point s is equivalent to having a pole of some given order at s in the new coordinate system. Next, let E_1 be the points in $\mathbb{C}$ at which D_Y intersects this new infinity section. Near a point $s \in E_1$, D_Y can be written as $w = h(z) \pm \sqrt{g(z)} = u^{\pm}(z)$ for some meromorphic g and h. Hence there exists $n_s > 0$ such that $u^{\pm}(z)(z - s)^{n_s}$ converges to 0 as z tends to s. We may assume also that if $s \in E_1$ and s is the base point of an exceptional fiber, then the n_s obtained here is larger than the n_s obtained above for this exceptional fiber.

Let E be the union of E_1 and the set of base points corresponding to exceptional fibers. Let p be entire with a zero of order n_s at each $s \in E$ and no other zeros, and let $\Phi(z, w) = (z, p(z)w)$. Then $\Phi(D_Y)$ is a double section in $\mathbb{C} \times \mathbb{C}$, and a dominating map from $\mathbb{C}^2$ to $(\mathbb{C} \times \mathbb{P}^1) \setminus \Phi(D_Y)$ followed by Φ^{-1} gives a dominating map to the complement of D_Y which respects the exceptional fibers.

Hence it suffices to construct a dominating map into the complement of $\Phi(D_Y)$. Note that $\Phi(D_Y)$ can be written as $w = v^{\pm}(z) = p(z)u^{\pm}(z)$, where $v^{\pm}$ are holomorphic except possibly for square root singularities at branch points.

For complex numbers u and v, define a Mobius transformation $N_{u,v}(w) = (uw - v)/(w - 1)$, which takes 0 to v and ∞ to u, and define $G_{u,v}(w) = \exp(w(u - v))$. Note that $N_{u,v}(w) = N_{v,u}(1/w)$ and that $G_{u,v}(w) = 1/G_{v,u}(w)$. Hence $H_0(u, v, w) = N_{u,v}(G_{u,v}(w))$ satisfies $H_0(u, v, w) = H_0(v, u, w)$. Since symmetric functions of v^+ and v^- are holomorphic, we see that $H(z, w) = H_0(v^+(z), v^-(z), w)$ is well-defined and holomorphic from $\mathbb{C}^2$ to $\mathbb{C} \times \mathbb{P}^1$. Moreover, for fixed s such that $v^{\pm}(s)$ are distinct, $H(s, \cdot)$

is nonconstant from $\mathbb{C}$ to $\mathbb{P}^1 \setminus \{v^{\pm}(s)\}$. If $v^{\pm}(s)$ are equal, then assuming without loss that $s = 0$, we have $v^{\pm}(z) = h(z) \pm \sqrt{g(z)}$ for some holomorphic $g(z) = z^m g_1(z)$ with $g_1(0) \neq 0$, $m \geq 1$. Then $v^+ - v^- = 2\sqrt{g}$, so multiplying the numerator and denominator of H by $\exp(-w(v^+ - v^-)/2)$ and using the Taylor expansion of $\exp$ gives

$$H = \frac{(h + \sqrt{g})(1 + w\sqrt{g}) - (h - \sqrt{g})(1 - w\sqrt{g}) + O(|z|^m)}{(1 + w\sqrt{g}) - (1 - \sqrt{g}) + O(|z|^m)}$$

$$= \frac{2hw\sqrt{g} + 2\sqrt{g} + O(|z|^m)}{2w\sqrt{g} + O(|z|^m)}.$$

As $z \to 0$, this last expression tends to $h(0) + 1/w$, and hence $H(0, \cdot)$ maps $\mathbb{C}$ onto $\mathbb{P}^1 \setminus \{v^{\pm}(z)\}$.

Thus H is a dominating map from $\mathbb{C}^2$ to the complement of $\Phi(D_Y)$, hence as noted before, $\Phi^{-1} \circ H$ is a dominating map from $\mathbb{C}^2$ to the complement of D_Y which respects the exceptional fibers. As before, this map pushes forward to Y and lifts to give a dominating map into X, as desired.

We can now summarize with the following theorem.

Theorem 5.5 *Assume that X is fibered over a curve C and that the generic fiber is $\mathbb{P}^1$ with at most two punctures. Then X is dominable by $\mathbb{C}^2$ if and only if there is a Zariski dense image of $\mathbb{C}$ in X.*

The arguments given in this paper are not sufficient to resolve the question of dominability for open fibered surfaces. As an example, we have the following question.

Question 5.6 Let X be the complement of a double section in a conic bundle over $\mathbb{C}$, $\mathbb{C}^*$, or an elliptic curve. Is X dominable by $\mathbb{C}^2$?

We will consider this and related questions in a forthcoming paper.

5.2.2 The $\bar{\kappa} = -\infty$ case

Let $\bar{\kappa}(X) = -\infty$. Then $\kappa(\bar{X}) = -\infty$ as well. Hence $\bar{X}$ is either rational or birationally ruled over a curve of non-negative genus. In the latter case, Proposition 5.4 says that X is fibered over a curve C with $\kappa(C) \geq 0$ where the generic fiber is $\mathbb{P}^1$ with at most one puncture. Hence Theorem 5.5 applies in this case to give us the equivalence of dominability by $\mathbb{C}^2$ and the failure of property C. Note that property C holds in the case $\kappa(C) > 0$ (which include the case $\bar{q}(X) \geq 2$), corresponding to C being hyperbolic.

In the remaining case when $\bar{X}$ is rational, we can again divide into two cases according to whether $\bar{q}(X)$ is zero or not. In the latter case, we again have a fibering of X over a curve C via the quasi-Albanese map with the generic fiber having at most one puncture by Proposition 5.4, as before.

This is because there are no logarithmic 2-forms on X since $\bar{\kappa}(X) = -\infty$. By the same token, every logarithmic 1-form on X is the pull back of a logarithmic form on C (One can also see this from the fact that $\mathbb{P}^1$ with at most one puncture has no logarithmic forms so that any logarithmic form on X becomes trivial when restricted to the generic fiber. Hence, $\bar{q}(X) = \bar{q}(C)$.) So, C must be $\mathbb{P}^1$ with at least two punctures. If it has more than two punctures, corresponding to $\bar{q}(X) \geq 2$, then C is hyperbolic. So we have degeneracy of holomorphic maps from $\mathbb{C}$ in this case. Otherwise, Theorem 5.5 applies.

We are left with the case where $\bar{q}(X) = 0$ where Proposition 5.4 no longer applies, but where much of the analysis has been done in [M]. We now quote Theorem (1′) of [M], (which follows from Theorem I.3.11 of [M])

Theorem 5.7 *With X and D as before, assume that D is connected. Then $\bar{\kappa}(X) = -\infty$ if and only if X fibers over a curve with generic fiber being $\mathbb{P}^1$ or $\mathbb{C}$.*

Except in the case where $X = \mathbb{P}^2$, there is, of course, some fibering of X to a curve (as is clear from, for example, (1) of the classification list given in Sect. 2) and every such fibering must be to a curve C that is either $\mathbb{P}^1$ or $\mathbb{C}$. In these fibered cases, we would like to show that the generic fiber is $\mathbb{P}^1$ with at most two punctures so that Theorem 5.5 can be applied to show that X is dominable by $\mathbb{C}^2$. However, it remains an open question whether or not the generic fiber has this form, and although this question should be resolved by some case checking, this lack prevents us from giving a complete classification in the case $\bar{\kappa}(X) = -\infty$ and $\bar{q}(X) = 0$.

We can now summarize this section as follows.

Theorem 5.8 *Let X be the complement of a normal crossing divisor D in a projective surface. Assume $\bar{\kappa}(X) = -\infty$. If $\bar{q}(X) \geq 2$, then X satisfies property C and hence is not dominable by $\mathbb{C}^2$. If $\bar{q}(X) = 1$ or if $\bar{q}(X) = 0$ and D is connected, then X is dominable by $\mathbb{C}^2$ if and only if there exists a holomorphic map of $\mathbb{C}$ to X whose image is Zariski dense.*

5.2.3 The $\bar{\kappa} = 1$ case

Here, we can directly apply the basic Iitaka fibration theorem, Theorem 11.8 in [Ii] (see also [Ue]):

Theorem 5.9 *Assume $\bar{\kappa}(X) \geq 0$. Then X is properly birational to a variety X^* which is fibered over a variety of dimension $\bar{\kappa}(X)$ and whose generic fiber has Kodaira dimension zero.*

This theorem holds for X of any dimension. But for our situation at hand, it says that X is properly birational to a surface X^* which is fibered over a curve with generic fiber that is either an elliptic curve, or $\mathbb{P}^1$ with

two punctures. Now, we have already shown that for such a fibered variety, dominability is unchanged for any variety properly birational to it. The latter case is already resolved by Theorem 5.5. The former case can also be resolved to give the same conclusion by the same analysis as that of Theorem 5.5 with the help of the Jacobian fibration as in Sect. 3. Thus, combining with Theorem 5.5, we have:

Theorem 5.10 *Assume X is fibered over a curve with generic fiber that is either an elliptic curve or $\mathbb{P}^1$ with at most two punctures. This is the case, for example, when $\bar{\kappa}(X) = 1$. Then X is dominable by $\mathbb{C}^2$ if and only if there exists a holomorphic map of $\mathbb{C}$ to X whose image is Zariski dense.*

5.2.4 The $\bar{\kappa} = 0$ case

It remains to look at the case where $\bar{\kappa}(X) = 0$. If $\bar{q}(X) \geq 2$, then a well known theorem of Kawamata ([K4]) says that X has a birational morphism to a semi-abelian surface. Hence, X is dominable by $\mathbb{C}^2$. If $\bar{q}(X) = 1$, then X is fibered over a curve and the generic fiber is an elliptic curve or is $\mathbb{P}^1$ with at most two punctures by Proposition 5.4. Hence Theorem 5.10 applies in this case. When $\bar{q}(X) = 0$, our problem remains with some K3 surfaces as explained in Sect. 4.2.

Finally, if X is affine rational and D has a component that is not a rational curve, then Lemma II.5.5 of [M] says that either X is fibered over a curve with generic fiber $\mathbb{P}^1$ with at most two punctures or X is the complement of a smooth cubic in $\mathbb{P}^2$. The former is handled by Theorem 5.5 while the latter is dominated by $\mathbb{C}^2$ as shown in Sect. 5.1. This resolves the case of the complement of a reduced divisor C in $\mathbb{P}^2$ unless C is a rational curve, which one can resolve as well when C has either only one singular point or is of low degree (and it is easy to check all the cases for degree less than 4). This is a good exercise for the case when C is a rational curve of high degree, which we will not attempt here. Note that, if C is normal crossing with dominable complement, then C is again a smooth cubic in $\mathbb{P}^2$, being the unique non-rational component.

Theorem 5.11 *Assume $\bar{\kappa}(X) = 0$. If $\bar{q}(X)$ is positive or if X is affine and D has a component that is not a rational curve, then X is dominable by $\mathbb{C}^2$ if and only if there exists a holomorphic map of $\mathbb{C}$ to X whose image is Zariski dense.*

6 Compact complex surface minus small balls

For the compact complex surfaces which we showed to be dominable by $\mathbb{C}^2$, a surprisingly stronger result can be achieved, thanks to the theory of Fatou-Bieberbach domains. We can show that these surfaces remain dominable after removing any finite number of sufficiently small open balls. In this

section we show how this can be done in the most difficult case, the case of a two dimensional compact complex torus. We show that given any finite set of points in a torus T, it is possible to find some open set, U, containing this finite set, and a holomorphic map $F : \mathbb{C}^2 \to T \setminus U$ with non-vanishing Jacobian determinant. In fact, F lifts to an injective holomorphic map from $\mathbb{C}^2$ to $\mathbb{C}^2$. For the statement of the following theorem, we focus only on this lifted map. For notation, $\Delta^2(p; r)$ is the bidisk with center p and radii r in both coordinate directions and π^j represents projection to the jth coordinate axis.

Theorem 6.1 *Let $\Lambda \subseteq \mathbb{C}^2$ be a discrete lattice, let $p_1, \dots, p_m \in \mathbb{C}^2$, let $\Lambda_0 = \cup_{j=1}^m \Lambda + p_j$, and for $r > 0$, let $\Lambda_{0,r} = \cup_{p \in \Lambda_0} \Delta^2(p; r)$. For some $r > 0$, there exists an injective holomorphic map $F : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \Lambda_{0,r}$.*

In fact, the proof will show that any discrete set contained in $\Lambda_{0,r}$ is a tame set in the sense of Sect. 4.1. As an immediate corollary, we obtain the following result, as mentioned in the introduction. An n-dimensional version of this result is found in [Bu].

Corollary 6.2 *Let T be a complex 2-torus and let $E \subset T$ be finite. Then there exists an open set U containing E and a dominating map from $\mathbb{C}^2$ into the complement of U.*

For the remainder of this section, Λ, Λ_0 and $\Lambda_{0,r}$ will be as in the statement of this theorem.

6.1 Preparatory lemmas

In this subsection we state some necessary lemmas. The proofs are straightforward and perhaps even standard, but they are provided for completeness.

Notation: For $\epsilon > 0$, let $S_\epsilon = \{x + iy : x \in \mathbb{R}, -\epsilon < y < \epsilon\}$.

Lemma 6.3 *Let $C > 0$, let $f : \mathbb{R} \to [0, C]$ be measurable, and let $\epsilon \in (0, 1)$. Then there exists a function g holomorphic on S_ϵ such that if $\delta > 0$ and $z_0 = x_0 + iy_0 \in S_\epsilon$ with $f(x) = c_0$ for $x_0 - \delta < x < x_0 + \delta$, then*

$$|g(z_0) - f(x_0)| \le \frac{2C\epsilon}{\pi\delta}.$$

Moreover, Re $g(z) \ge 0$ for all $z \in S_\epsilon$.

Proof: For $n \in \mathbb{Z}$, let

$$g_n(z) = \frac{1}{2\pi i} \int_{-n}^n \left(\frac{f(x)}{x - i\epsilon - z} - \frac{f(x)}{x + i\epsilon - z} \right) dx$$

$$= \frac{1}{\pi} \int_{-n}^n f(x) \left(\frac{\epsilon}{(x - z)^2 + \epsilon^2} \right) dx.$$

I.e., g_n is obtained via the Cauchy integral using the function f on the two boundary components of S_ϵ and truncating at $x = \pm n$. By [R, Thm 10.7], each g_n is holomorphic in S_ϵ. Moreover, for $z_0 = x_0 + iy_0 \in S_\epsilon$, we have $|y_0| < \epsilon$, so

$$|(x - z)^2 + \epsilon^2| \geq \text{Re } (x - (x_0 + iy_0))^2 + \epsilon^2 \geq (x - x_0)^2. \qquad (6.1)$$

Using this last inequality and the boundedness of f, it follows immediately that g_n converges uniformly on compact subsets of S_ϵ to the holomorphic function

$$g(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \left(\frac{\epsilon}{(x - z)^2 + \epsilon^2} \right) dx. \qquad (6.2)$$

A simple contour integration shows that if f is replaced by the constant c_0, then the integral in (6.2) is c_0 for all $z \in S_\epsilon$. Hence, if $z_0 = x_0 + iy_0 \in S_\epsilon$ with $f(x) = c_0$ for $x_0 - \delta \leq x \leq x_0 + \delta$, then using (6.1),

$$\begin{aligned}
|g(z_0) - f(x_0)| &= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} (f(x) - c_0) \left(\frac{\epsilon}{(x - z)^2 + \epsilon^2} \right) dx \right| \\
&\leq \frac{C}{\pi} \left(\int_{-\infty}^{x_0 - \delta} + \int_{x_0 + \delta}^{\infty} \frac{\epsilon}{(x - x_0)^2} dx \right) \\
&\leq \frac{2\epsilon C}{\pi \delta}.
\end{aligned}$$

To show that $\text{Re } g(z) \geq 0$, note that the second part of (6.1) implies that $\text{Re } (\epsilon/((x - z)^2 + \epsilon^2)) \geq 0$ for all $z \in S_\epsilon$, and since f is real, (6.2) implies $\text{Re } g(z) \geq 0$. $\qquad \square$

Lemma 6.4 *Let* $V = \{(z, w) : |w| < 1 + |z|^2\}$. *Then there exists an injective holomorphic map* $\Phi : \mathbb{C}^2 \to V$.

Proof: Let $H(z, w) = (w, w^2 - z/2)$. Then H is a polynomial automorphism of $\mathbb{C}^2$, and $(0, 0)$ is an attracting fixed point for H. By [RR1, appendix], there is an injective holomorphic map Ψ from $\mathbb{C}^2$ onto the basin of attraction of $(0, 0)$, which is defined as $B = \{p \in \mathbb{C}^2 : \lim_{n \to \infty} H^n(p) = (0, 0)\}$. By [BS], there exists $R > 0$ such that B is contained in

$$V_R = \{|z| \leq R, |w| < R\} \cup \{|z| \geq R, |w| < |z|\}.$$

Hence taking $\Phi = \Psi/R$ gives an injective holomorphic map from $\mathbb{C}^2$ into $V_1 \subseteq V$. $\qquad \square$

6.2 Proof of Theorem 6.1

We will construct an automorphism of $\mathbb{C}^2$ mapping $\Lambda_{0,r}$ into the complement of the set V of Lemma 6.4. This will be sufficient to prove the theorem, and by [RR1] this implies that any discrete set contained in $\Lambda_{0,r}$ is tame.

Choose an invertible, complex linear A as in Lemma 4.2. Without loss of generality, we may replace Λ by $A(\Lambda)$, p_j by $A(p_j)$, and Λ_0 by $A(\Lambda_0)$. Then $\pi^1\Lambda_0$ is contained in $\cup_{k=1}^{\infty} L_k$, where each L_k is a line of the form $\mathbb{R} + i\gamma_k$, γ_k real. Moreover, $\text{dist}(L_j, L_k) \geq 1$ if $j \neq k$, and $|p - q| \geq 1$ if $p, q \in \Lambda_0$ with $p \neq q$.

Let $\{q_j\}_{j=1}^{\infty}$ be an enumeration of the set

$$\{q \in \Lambda_0 : |\pi^2 q| \leq 1/8\} = \{q \in \Lambda_0 : \overline{\Delta^2(q; 1/8)} \cap (\mathbb{C} \times \{0\}) \neq \varnothing\}.$$

Let $C = \log 32$, and define $f_k : \mathbb{R} \to [0, C]$ for each k by

$$f_k(x) = \begin{cases} 0 & \text{if } (x + i\gamma_k, 0) \in \overline{\Delta^2(q_j; 1/8)} \text{ for some } q_j \\ C & \text{otherwise.} \end{cases}$$

Let $\delta = 1/16$, and choose $\epsilon \leq \delta/2$ small enough that $2C\epsilon/\pi\delta \leq \log(3/2)$. Let $r = \epsilon/2$, and recall that $\Lambda_{0,r} = \cup_{p \in \Lambda_0} \Delta^2(p; r)$.

Let $S_{\epsilon}^k = \{x + i(y + \gamma_k) : -\epsilon < y < \epsilon\}$ and $U_{\epsilon} = \cup_{k=1}^{\infty} S_{\epsilon}^k$. Define g holomorphic on U_{ϵ} by applying Lemma 6.3 with $f = f_k$ to define g on S_{ϵ}^k. By Arakelian's Theorem (e.g. [RR2]), there exists h entire such that if $z \in U_{\epsilon/2}$, then $|h(z) - g(z)| \leq \log(4/3)$. Define

$$F_1(z, w) = (z, w\exp(h(z))).$$

Then $F_1 : \mathbb{C}^2 \to \mathbb{C}^2$ is biholomorphic.

We show next that there is a complex line in the complement of $F_1(\Lambda_{0,r})$. To do this, let $p \in \Lambda_{0,r}$, and suppose first that $p \in \Delta^2(q_j; r)$ for some q_j. Choose k so that $\gamma_k = \text{Im } \pi^1 q_j$, and write $\pi^1 p = x_0 + iy_0$.

Note that $|y_0 - \gamma_k| < r = \epsilon/2$. Also, since $|\pi^2 q_j| \leq 1/8$, we see that if $|x - x_0| < (1/8) - r$, then $(x + i\gamma_k, 0) \in \overline{\Delta^2(q_j; 1/8)}$. Since $\delta < (1/8) - r$, we have $f_k(x) = 0$ for $x_0 - \delta \leq x \leq x_0 + \delta$, and hence by Lemma 6.3 and the choice of ϵ and h,

$$|h(\pi^1 p)| \leq |g(\pi^1 p)| + \log(4/3)$$
$$\leq \frac{2C\epsilon}{\pi\delta} + \log(4/3)$$
$$\leq \log 2.$$

Hence

$$|\pi^2 F_1(p)| \leq 2|\pi^2 p| \leq 2(|\pi^2 q_j| + r) \leq \frac{1}{3}. \tag{6.3}$$

In the remaining case, $p \in \Lambda_{0,r}$ but $p \notin \Delta^2(q_j; r)$ for any j, in which case $|\pi^2 p| \geq (1/8) - r$. Let $q \in \Lambda_0$ such that $p \in \Delta^2(q; r)$, and choose k so that $\gamma_k = \operatorname{Im} \pi^1 q$.

Suppose first that $x_0 = \operatorname{Re} \pi^1 p$ satisfies $f_k(x) = C$ for $|x - x_0| \leq \delta$. Since $|y_0 - \gamma_k| < r = \epsilon/2$, we have by Lemma 6.3 and choice of ϵ and h that

$$\operatorname{Re} h(\pi^1 p) \geq \operatorname{Re} g(\pi^1 p) - \log(4/3)$$
$$\geq C - \frac{2C\epsilon}{\pi\delta} - \log(4/3)$$
$$\geq \log 16.$$

Hence

$$|\pi^2 F_1(p)| \geq 16|\pi^2 p| \geq 16((1/8) - r) > 1. \tag{6.4}$$

Otherwise, $f_k(x) = 0$ for some x with $|x - x_0| \leq \delta$, so there exists j such that $|\pi^1 p - \pi^1 q_j| \leq (1/8) + \delta + r$, hence

$$|\pi^1 q - \pi^1 q_j| \leq (1/8) + \delta + 2r \leq 1/4.$$

Since q and q_j are distinct points of Λ_0, we have $|q - q_j| \geq 1$ by assumption, so $|\pi^2 q - \pi^2 q_j|^2 \geq 1 - (1/4)^2$, and hence

$$|\pi^2 q| \geq |\pi^2 q - \pi^2 q_j| - |\pi^2 q_j| \geq \frac{\sqrt{15}}{4} - \frac{1}{8}$$

and

$$|\pi^2 p| \geq |\pi^2 q| - r \geq \frac{3}{4}.$$

Since $\operatorname{Re} g(\pi^1 p) \geq 0$ by Lemma 6.3, we have $\operatorname{Re} h(\pi^1 p) \geq -\log(4/3)$, and hence

$$|\pi^2 F_1(p)| \geq \frac{3}{4}|\pi^2 p| \geq \frac{9}{16}. \tag{6.5}$$

From (6.3), (6.4) and (6.5), we conclude that if $p \in \Lambda_{0,r}$, then either $|\pi^2 F_1(p)| \leq 1/3$ or $|\pi^2 F_1(p)| \geq 9/16$. In particular,

$$\operatorname{dist}\left(F_1(\Lambda_{0,r}), \mathbb{C} \times \left\{\frac{1}{2}\right\}\right) \geq \frac{1}{16}.$$

Note also that $\pi^1 F_1(p) = \pi^1 p$ for all $p \in \mathbb{C}^2$.

To finish the proof, we will construct F_2 similar to F_1 so that $F_2(F_1(\Lambda_{0,r}))$ is contained in $\mathbb{C}^2 \setminus V$, where V is as in Lemma 6.4.

First note that for $z = x + iy \in S_\epsilon^k$, we have $|y - i\gamma_k| < \epsilon$, so

$$\operatorname{Re}\left[(z - i\gamma_k)^2 + (|\gamma_k| + r)^2 + 1 + \epsilon^2\right] \geq x^2 + (|\gamma_k| + \epsilon)^2 + 1 > 0. \tag{6.6}$$

Hence we can choose a branch of log so that

$$g_2(z) = \log((z - i\gamma_k)^2 + (|\gamma_k| + r)^2 + 1 + \epsilon^2) + 1 + \log 16 \qquad (6.7)$$

is holomorphic on $\cup_k S_\epsilon^k$. Again by Arakelian's Theorem, there exists h_2 entire such that if $z \in S_{\epsilon/2}^k$, then $|g_2(z) - h_2(z)| \le 1$, so Re $h_2(z) \ge$ Re $g_2(z) - 1$. Let

$$F_2(z, w) = \left(z, \left(w - \frac{1}{2} \right) \exp(h_2(z)) \right).$$

Again, $F_2 : \mathbb{C}^2 \to \mathbb{C}^2$ is biholomorphic. Moreover, if $p \in F_1(\Lambda_{0,r})$, then $|\pi^2 p - \frac{1}{2}| \ge 1/16$, and $\pi^1 p = z = x + iy$ with $|y - \gamma_k| < r$ for some k, so by (6.6) and (6.7), we have

$$|\pi^2 F_2(p)| \ge \left| \pi^2 p - \frac{1}{2} \right| \exp(\text{Re } h_2(z))$$

$$\ge \frac{1}{16} \exp(\text{Re } g_2(z) - 1)$$

$$\ge x^2 + (|\gamma_k| + r)^2 + 1$$

$$\ge 1 + |\pi^1 p|^2$$

$$\ge 1 + |\pi^1 F_2(p)|^2.$$

Hence $F_2 F_1(\Lambda_{0,r}) \cap V = \emptyset$, where $V \supseteq \Phi(\mathbb{C}^2)$ is as in Lemma 6.4, so taking $F = F_1^{-1} F_2^{-1} \Phi$ gives an injective holomorphic map $F : \mathbb{C}^2 \to F_1^{-1} F_2^{-1}(V) \subseteq \mathbb{C}^2 \setminus \Lambda_{0,r}$ as desired. $\qquad \square$

6.3 The general case of complements of small open balls

It is now easy to deduce the following corollary from Theorem 6.1.

Corollary 6.5 *Let X be bimeromorphic to a compact complex torus or to a Kummer K3 surface. Then, given any finite set of points in X, the complement of a neighborhood of this set is dominable by $\mathbb{C}^2$. In particular, such a complement is not measure hyperbolic.*

The case of elliptic fibrations over $\mathbb{P}^1$ or over an elliptic curve can be handled in the same way as that of Theorem 6.1. This is because removing a finite number of small open balls (plus a smooth fiber away from them if the base is $\mathbb{P}^1$) is tantamount to removing via the Jacobian fibration a discrete set of contractible open sets in $\mathbb{C}^2$ bounded away from the axis by fixed constants and whose projection to the first factor $\mathbb{C}$ is also a discrete set of contractible open subsets of $\mathbb{C}$. See also Theorem 2.3 in [Bu].

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