# Infinitely many periodic attractors for holomorphic maps of 2 variables

GREGERY T. BUZZARD*

## Introduction

An important development in the study of discrete dynamical systems was Newhouse's use of persistent homoclinic tangencies to show that a large set of $C^2$ diffeomorphisms of compact surfaces have infinitely many coexisting periodic attractors, or sinks [6], where "large" refers to a residual subset of an open set of diffeomorphisms. In the present paper, we obtain this result for various spaces of holomorphic maps of two variables.

Newhouse later extended his result to show that such residual sets exist near any surface diffeomorphism which has a homoclinic tangency [7]. More recently, Palis and Viana extended this latter result to higher dimensions when the stable manifold has codimension one [10], and Romero obtained an analogous result for higher codimension stable manifolds using saddles in place of sinks [12]. In each case however, the construction reduces to the study of intersecting Cantor sets in the line: under an appropriate projection, the stable and unstable manifolds of a basic set are mapped to Cantor sets in the line, and a tangency between these manifolds corresponds to a point of intersection of the Cantor sets. The generic unfolding of tangencies then gives rise to periodic attractors or saddles. This reduction to linear Cantor sets depends heavily on the fact that there is only one expanding eigenvalue. Even Romero's result for higher codimension stable manifolds involves a reduction to this case.

In the holomorphic setting, eigenvalues come in conjugate pairs (from a real point of view), so this reduction is not possible. Instead, stable and unstable manifolds are Riemann surfaces, and after extending the stable and unstable manifolds of a basic set to foliations, these foliations will be tangent in a (real) 2-dimensional disk, and the stable and unstable manifolds correspond to Cantor sets in this disk. Hence, persistent tangencies between basic sets correspond to the stable intersection of two Cantor sets in the plane.

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Although we have no criterion as general as Newhouse's concept of thickness for Cantor sets in the line, we are able to give reasonably flexible conditions for Cantor sets in the plane to intersect. Using this, we obtain persistent homoclinic tangencies for holomorphic maps of two variables, then we apply a result of Gavosto together with standard arguments to obtain infinitely many sinks. An additional complication here is that since holomorphic maps are quite rigid, we must work a bit to show that we can unfold a tangency generically.

It should be pointed out that Rosay and Rudin constructed a holomorphic automorphism of $\mathbb{C}^2$ having infinitely many attracting fixed points, which of necessity had no limit point [13]. More in the direction of the current result, Fornæss and Gavosto used the results of Newhouse to show that there are quadratic polynomial automorphisms of $\mathbb{C}^2$ having infinitely many sinks contained in a compact set [4], and Gavosto obtained a similar result for holomorphic self-maps of $\mathbf{P}^2$ [5]. However, these results give no indication of the prevalence of this phenomenon.

In this paper we consider the spaces $\mathrm{Aut}(\mathbb{C}^2)$ of holomorphic automorphisms of $\mathbb{C}^2$ and $\mathrm{AutP}_d(\mathbb{C}^2)$ of polynomial automorphisms of degree at most $d$, which is the set of $F \in \mathrm{Aut}(\mathbb{C}^2)$ for which each component function is a polynomial of degree at most $d$. The topology in both cases is that induced by local uniform convergence of the map and its inverse. This topology can also be obtained from a complete metric, so that Baire's theorem applies to both spaces. We also consider the space $\mathcal{H}_d$ of holomorphic self-maps of $\mathbf{P}^2$ of degree $d$ using the natural distance on $\mathbf{P}^2$ to induce the supremum metric on $\mathcal{H}_d$. Again, this makes $\mathcal{H}_d$ a complete metric space.

Main Theorem. *There exists $d > 0$ such that if $X$ is one of*

*(a) the space $\mathrm{Aut}(\mathbb{C}^2)$ of automorphisms of $\mathbb{C}^2$,*

*(b) the space $\mathrm{AutP}_d(\mathbb{C}^2)$ of polynomial automorphisms of $\mathbb{C}^2$ of degree at most $d$,*

*(c) the space $\mathcal{H}_d$ of holomorphic self-maps of $\mathbf{P}^2$ of degree $d$,*

*then there exists $G \in X$ and a neighborhood $\mathcal{N} \subseteq X$ of $G$ such that $\mathcal{N}$ has persistent homoclinic tangencies. More precisely, there is a compact subset $\bar{E}$ of the ambient manifold and a dense subset $\mathcal{S} \subseteq \mathcal{N}$ such that each $H \in \mathcal{S}$ has a homoclinic tangency in $\bar{E}$ between the stable and unstable manifolds of some fixed point of $H$.*

Combining this with a result of Gavosto [5], we obtain the following.

Corollary 1. *With $\mathcal{N}$ and $\bar{E}$ as in the previous theorem, there is a dense $\mathcal{G}_\delta$ subset $\mathcal{R} \subseteq \mathcal{N}$ such that each $H \in \mathcal{R}$ has infinitely many attracting periodic points (sinks) contained in $\bar{E}$.*

At the moment, there is no estimate for the degree of the maps in the main theorem. It is also an interesting open question whether Newhouse's second result is valid in the holomorphic case: that is, given any dissipative holomorphic map with a homoclinic tangency, is there a nearby residual set of holomorphic maps with infinitely many sinks?

## 1. Background and outline of proof

The notion of a basic set plays a key role in the construction of an automorphism with persistent tangencies, so we first review some of the associated ideas. A more complete discussion can be found in many places, e.g. [1, 8, 9].

Let $F : M \to M$ be a $C^k$ diffeomorphism of a Riemannian manifold $M$, and suppose that $\Lambda \subseteq M$ is compact with $F(\Lambda) = \Lambda$. We say that $\Lambda$ is *hyperbolic* if $TM|\Lambda$ has a $DF$-invariant continuous splitting $E^s \oplus E^u$ such that for some $\lambda < 1$, $C > 0$, and all $j \geq 0$,

$$\max\{\, \|DF^j|E^s\|, \, \|DF^{-j}|E^u\|\,\} \leq C\lambda^j.$$

By $DF$-invariant we mean that $(D_pF)E^s_p = E^s_{F(p)}$ and $(D_pF)E^u_p = E^u_{F(p)}$ for each $p \in \Lambda$. Important examples include hyperbolic fixed points and the orbit of a hyperbolic periodic point.

These ideas can be extended to holomorphic self-maps of $\mathbf{P}^2$ which are not invertible, although the resulting stable and unstable manifolds are no longer global manifolds. For purposes of this paper, we will only deal with pieces of the stable and unstable manifolds which are smooth.

Next, let d be the distance function induced by the metric. For a point $p \in \Lambda$ and $\epsilon > 0$, we define the *stable manifold* and *local stable manifold* of $F$ at $p$ by

$$W^s(p) := \{q \in M : \lim_{n\to\infty} \mathrm{d}(F^n(p), F^n(q)) = 0\}$$

and

$$W^s_\epsilon(p) := \{q \in W^s(p) : \mathrm{d}(F^n(p), F^n(q)) < \epsilon, \forall n \geq 0\},$$

respectively. Then $W^s(p)$ is a $C^k$ immersed submanifold containing $p$, and $W^s_\epsilon(p)$ is a $C^k$ disk through $p$ which varies continuously with $p$ in the $C^k$ topology and which is tangent to $E^s_p$ at $p$. There are analogous definitions and results for the unstable versions of these manifolds. We use the notation $W^s_{\mathrm{loc}}(p)$ and $W^u_{\mathrm{loc}}(p)$ to represent compact, simply connected neighborhoods of $p$ in the stable and unstable manifolds, respectively, without specifying the size of these neighborhoods.

We say that a hyperbolic set $\Lambda$ has *local product structure* if there exists $\epsilon > 0$ such that $W^s_\epsilon(x) \cap W^u_\epsilon(y) \subseteq \Lambda$ for all $x, y \in \Lambda$. This condition implies that if $x \in \Lambda$ and $\epsilon$ is small, then there is a neighborhood $U$ of $x$ such that

$U \cap \Lambda$ is homeomorphic to $(W^s_\epsilon(x) \cap \Lambda) \times (W^u_\epsilon(x) \cap \Lambda)$. Finally, we say that $F|\Lambda$ is *transitive* if there exists $x \in \Lambda$ such that $\{F^n(x) : n \in \mathbb{Z}\}$ is dense in $\Lambda$.

DEFINITION 1.1. *A compact, invariant, hyperbolic set $\Lambda$ is a basic set if $\Lambda$ has local product structure and $F|\Lambda$ is transitive.*

The well-known Smale horseshoe gives an example of a nontrivial basic set [15]. One important fact about basic sets is that they persist under perturbations: if $\Lambda(F)$ is a basic set for $F$, then for all $G$ which are $C^k$ near enough to $F$ there is a basic set $\Lambda(G)$ for $G$ and a homeomorphism $h : \Lambda(F) \to \Lambda(G)$ which conjugates $G$ to $F$ in the sense that $F = h^{-1}Gh$ on $\Lambda(F)$. In fact, $h$ may be taken to be $C^0$ close to the identity.

To produce an automorphism of $\mathbb{C}^2$ having persistent homoclinic tangencies, we will use a map of the form $F = F_3F_2F_1$, where $F_1(z,w) = (z+f(w), w)$, $F_2(z,w) = (z, w+g(z))$, $F_3$ is a linear map, and $f$ and $g$ are polynomials. However, we first start with compact sets $K_f, K_g \subseteq \mathbb{C}$ and define $f$ and $g$ to be piecewise linear or quadratic on $K_f$ and $K_g$, respectively. With the correct choice of $K_f$, $K_g$, $f$, and $g$, we obtain a basic set $\Lambda$ which can be analyzed quite easily, as well as a fixed point $p_0$ with a homoclinic tangency.

For $G$ holomorphic and $C^0$ near $F$, we use a generalization of a result of Pixton [11] to construct $C^1$ foliations $\mathcal{F}^u(\Lambda(G))$ and $\mathcal{F}^s(\Lambda(G))$ whose leaves are complex manifolds which are semi-invariant under $G$, which agree with $W^u_\epsilon(\Lambda(G))$ and $W^s_\epsilon(\Lambda(G))$ respectively, and which vary $C^1$ as $G$ varies near $F$. For $G$ near $F$ there is a $C^1$ disk $D_T(G)$ where the leaves of these foliations are tangent, and we can use $\mathcal{F}^u(\Lambda(G))$ to project $W^s(p_0) \cap \Lambda(G)$ to $D_T(G)$, then project from there to the $w$-axis to obtain a Cantor set in the plane. Similarly, we use $\mathcal{F}^s$ to project $W^u(p_0) \cap \Lambda(G)$ to the $w$-axis.

This gives two Cantor sets in the plane, and a careful analysis of $\Lambda(G)$ together with techniques like those in [2] shows that these Cantor sets intersect for all $G$ near $F$. This stable intersection is equivalent to persistent tangencies for the basic set $\Lambda(G)$. For any such $G$, we can compose with an affine linear map near the identity to find a map arbitrarily near $G$ which has a homoclinic tangency. In particular, we can approximate the piecewise linear $f$ and $g$ by polynomials to obtain a polynomial automorphism $G$ which has persistent tangencies.

In the remaining sections, we repeatedly choose parameters $\delta$, $\delta_1$, $\delta_2$ and $\delta_3$. The order of dependence will always be that just given, and at the end we will be able to choose each parameter to satisfy all of the requirements at once. For future reference, we note that $K_0$ is a grid of nine disjoint squares in the plane, $K_1 = K_0 \times K_0 \subseteq \mathbb{C}^2$, $\delta$ is the size of the gaps between squares in the set $K_0$, $\delta_1$ is the $C^2$ distance from pieces of the stable and unstable manifolds to linear manifolds, $\delta_2$ is the $C^1$ distance from the projections along the stable and unstable foliations to certain linear maps, and $\delta_3$ defines a neighborhood of the map $F$ by the condition $\|F - G\|_{C^2} < \delta_3$ on the closure of the domain of $F$.

For notation, $\Delta(z;r) = \{w \in \mathbb{C} : |z-w| < r\}$, $\pi_j$ is projection onto the $j$th coordinate, $\Delta^2(p;r) = \Delta(\pi_1 p;r) \times \Delta(\pi_2 p;r)$, and $\mathrm{B}(p;r) = \{q \in \mathbb{C}^2 : \|p-q\| < r\}$.

## 2. A basic set for the piecewise linear function

In this chapter we construct a biholomorphism of an open set in $\mathbb{C}^2$ which is piecewise linear and which has a basic set which can be easily analyzed. The ideas are quite similar to the construction of the basic set in the horseshoe map [15, 8].

Generalizing the construction of the horseshoe map, we can construct a piecewise linear map as in figure 1. Here we start with nine squares arranged in a regular grid and apply a map of the form $F_1(z,w) = (z + f(w), w)$ so that the top row of squares is moved to the right and the bottom row is moved to the left. Then we apply a map of the form $F_2(z,w) = (z, w + f(z))$ to align the rows vertically. Finally, we apply a linear map which contracts in the horizontal direction and expands in the vertical direction. With the appropriate choice of these maps, the image of the original squares will be nine rectangles stretched over the original squares as in figure 1. Taking the intersection of all forward and backward images under the composition of these maps gives an invariant set which is a basic set for the map.

We can carry out the same procedure in $\mathbb{C}^2$. To be precise, for $z \in \mathbb{C}$, let $S(z;r)$ denote the open square with center $z$ and sides of length $2r$ parallel to the real and imaginary axes. That is,

$$S(z;r) := \{w \in \mathbb{C} : |\mathrm{Im}(w-z)|, |\mathrm{Re}(w-z)| < r\}.$$

Also, let

$$A := \{j + ki : j,k \in \{-1,0,1\}\}.$$

Let $\delta \in (0,1/2)$, and let $c_0 = 1 - \delta$. Set

$$K_0 := \bigcup_{a \in A} S(a; c_0/2).$$

Then $K_0$ is a regular grid of nine disjoint squares in the plane, and we let $K_1 = K_0 \times K_0$.

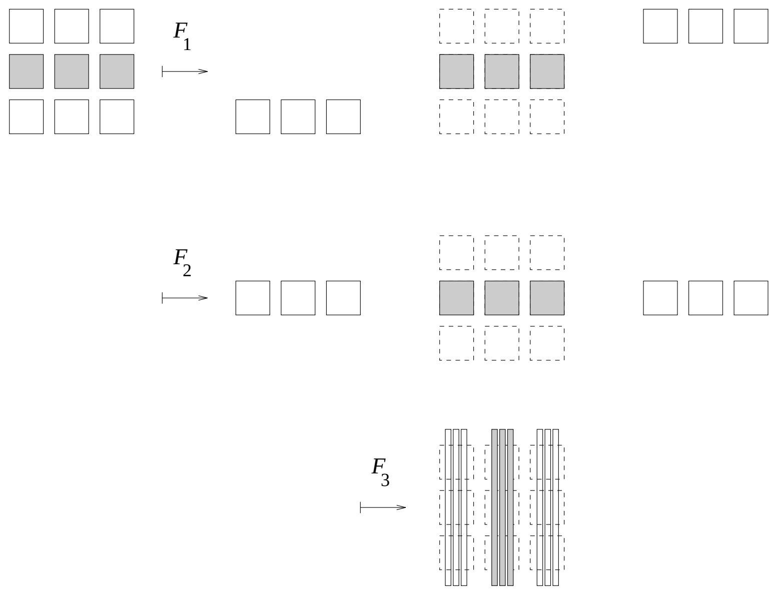


FIGURE 1. A generalized horseshoe formed by the composition of the maps $F_1$, $F_2$, and $F_3$. The original region is shown in dotted lines at each stage.

In order to define a horseshoe map, we want a function $f$ which is piecewise constant on $K_f := K_0$ and which has different values in each component of $K_f$. Explicitly, we choose $c_1 \in (c_0, 3c_0/(2+c_0))$ and set

$$f(w) := \sum_{a\in A} \frac{3a}{c_1} \chi_{\overline{S(a;c_0/2)}}(w), \tag{2.1}$$

where $\chi_E$ is 1 on $E$ and 0 elsewhere. Likewise, $K_g := \cup_{a\in A} S(3a/c_1; 3/2)$ and

$$g(z) := \sum_{a\in A} -a\chi_{\overline{S(3a/c_1;3/2)}}(z). \tag{2.2}$$

Then we define maps

$$\begin{aligned} F_1(z,w) &:= (z + f(w), w), \\ F_2(z,w) &:= (z, w + g(z)), \\ F_3(z,w) &:= \left(\frac{c_1}{3} z, \frac{3}{c_1} w\right). \end{aligned}$$

Taking $F := F_3F_2F_1$, we see that if $(z,w)$ is in the component of $\overline{K_1}$ containing $(a,b) \in A^2$, then

$$F(z,w) = \left(\frac{c_1}{3} z + b, \frac{3}{c_1}(w - b)\right). \tag{2.3}$$

Note that $F^{-1}$ is also defined on $\overline{K_1}$ by a similar formula. Note also that by Runge's theorem, $f$ and $g$ can be approximated by polynomials, uniformly on $K_f$ and $K_g$, respectively, and that if we use these approximations to replace $f$

and $g$ in the definitions of $F_1$ and $F_2$, then the resulting maps are polynomial automorphisms of $\mathbb{C}^2$.

The choice of $c_1$ implies that if we take any component of $K_1$, apply $F$, take the closure, and project to the $z$-axis, the image is contained in $K_0$, and these images are pairwise disjoint. If we follow the same procedure, but instead project to the $w$-axis, the image will contain $K_0$. Thus, each component of $K_1$ is contracted in the $z$-direction, stretched in the $w$-direction, and placed over the original set in analogy with the map in figure 1.

Using an argument like that in [8], we can apply $F$ repeatedly where defined, then take the intersection of all forward images to obtain a forward invariant set which is the product of a Cantor set in the $z$-direction with a square in the $w$-direction. A similar argument applied to $F^{-1}$ implies that there is a backward invariant set which is the product of a square in the $z$-direction and a Cantor set in the $w$-direction. Taking the intersection of all forward and backward images, we obtain a Cantor set $\Lambda$ which is the maximal invariant set in $K_1$. A standard argument shows that $F$ restricted to $\Lambda$ is conjugate to the shift map on 9 symbols, and the splitting of $T\Lambda$ using the standard basis vectors gives an invariant hyperbolic splitting. Thus, $\Lambda$ is a basic set for the map $F$.

## 3. Control of nonlinearity

We will need to analyze the basic set described above more carefully in order to obtain Cantor sets which intersect stably as described in the outline of the proof. Before doing that, we need to establish some results which will allow us to deal with nonlinear maps near $F$.

We will often represent part of some manifold as the graph of a function $g : (D \subseteq \mathbb{C}) \to \mathbb{C}$. For instance, $p_0 = (0,0)$ is a fixed point of the map $F$ above, and $W^u_{\mathrm{loc}}(p_0) = \{(0,w) : w \in \Delta(0;r)\}$ for some $r > 0$.

Definition 3.1. *Let* $\mathrm{graph_j}$ *denote the graph of a function with the* $j$*th variable regarded as the independent variable; e.g.,* $\mathrm{graph}_2(\Phi) = \{(\Phi(w), w) : w \in \mathrm{dom}(\Phi)\}$.

We also need to be able to map between points in $\mathbb{C}^2$ and points in $T_q\mathbb{C}^2$ for $q \in \mathbb{C}^2$. For this we use the standard exponential map $\exp_q : T_q\mathbb{C}^2 \to \mathbb{C}^2$. In $\mathbb{C}^2$ with the usual metric, this is essentially translation by $q$.

Definition 3.2. *For* $p \in \mathbb{C}^2$*, let* $p_q = \exp_q^{-1}(p) \in T_q\mathbb{C}^2$*. For* $M \subseteq \mathbb{C}^2$*, let* $M_q = \exp_q^{-1}(M)$.

The following lemma is a standard result comparing the distance between two points on a complex manifold with the distance between their projections on a tangent plane. We record it for convenience, but omit the proof.

For notation, suppose that $M = \mathrm{graph}_1(\Phi)$ where $\Phi : D \to \mathbb{C}$ is holomorphic on the convex domain $D$ and that $|\Phi''| < \delta$ on $D$. Let $q \in M$ and let $L_q \subseteq \mathbb{C}^2$ be the complex line tangent to $M$ at $q$.

DEFINITION 3.3. *Let $\pi_q$ denote orthogonal projection onto $L_q$ in $T_q\mathbb{C}^2$.*

LEMMA 3.4. *For each $p \in M$,*

$$\|p_q - \pi_q p_q\| \leq \frac{\delta}{2}\|p - q\|^2 \tag{3.1}$$

*and*

$$(1 - \frac{\delta}{2}\|p - q\|)\|p - q\| \leq \|\pi_q p_q\| \leq \|p - q\|. \tag{3.2}$$

Note that for $G$ near $F$, there is a unique saddle fixed point $p_0^G$ contained in $S(0; c_0/2) \times S(0; c_0/2)$. We use the notation $W^s(p_0^G)$ to denote the stable manifold of $p_0^G$ with respect to $G$ and likewise for $W^u(p_0^G)$.

We next show that if $G|K_1$ is close enough to $F$, then each leaf of $W^u(p_0^G) \cap K_1$ which remains in $K_1$ under backward iteration is the graph of a function which is nearly linear in the sense of the preceding lemma. The proof is by induction, using the stable manifold theorem for the basis case, then using the fact that such $G$ expand in the $w$-direction and contract in the $z$-direction to keep further iterates of the unstable manifold flat.

DEFINITION 3.5. *Let $W^u_{-1}$ be the component of $W^u(p_0^G) \cap (S(0; c_0/2) \times S(0; c_0/2))$ containing $p_0^G$. For $m \geq 0$, let $W^u_m = G(W^u_{m-1}) \cap K_1$.*

DEFINITION 3.6. *Let $\lambda_s = c_1/3$ and $\lambda_u = 3/c_1$ be the eigenvalues of $DF$.*

Also, we need to choose an open set $K_0' \subseteq \mathbb{C}$ with $\pi_1\overline{F(K_1)} \subseteq K_0' \subseteq \overline{K_0'} \subseteq K_0$. Finally, let $S_j$, $j = 1, \ldots, 9$ denote the components of $K_0$.

PROPOSITION 3.7. *Let $\delta_1 > 0$. There exists $\delta_3 > 0$ such that if $G : \overline{K_1} \to \mathbb{C}^2$ is holomorphic with $\|F - G\|_{C^2} < \delta_3$ on $\overline{K_1}$, then the following hold.*

*(1) $G$ has a unique saddle fixed point $p_0^G \in S(0; c_0/2) \times S(0; c_0/2)$ and $\pi_1\overline{G(K_1)} \subseteq K_0'$.*

*(2) For all $m \geq 0$, $W^u_m$ is a finite union of graphs* $\mathrm{graph}_2(\Phi|K_0)$, *where $\Phi : S(0; 1 + c_0/2) \to K_0'$ is holomorphic with $|\Phi'(w)|$, $|\Phi''(w)| < \delta_1$ for $w \in S(0; 1 + c_0/2)$.*

*(3) The analog of part 2 holds for $W^s_m$.*

*Proof.* Part 1 is a simple consequence of the implicit function theorem and continuity.

For part 2, we may replace $\delta_1$ by a smaller value. By Rouche's theorem, we see that we can choose $\delta_1 > 0$ followed by $\delta_3 > 0$ small enough that if $G$ and $\Phi$ satisfy the conditions in the statement of the proposition, then for $H(w) := (\Phi(w), w)$, we have $S(0; 1 + c_0/2) \subseteq \pi_2 GH(S_j)$ and $\pi_1 GH(S_j) \subseteq K_0'$ for each $j = 1, \ldots, 9$.

Since the unstable manifold through $(0,0)$ for $F$ contains the set $\{0\} \times \overline{S(0; 1 + c_0/2)}$, part 2 is true for $W_0^u$ by the stable manifold theorem. We show inductively that if $\Phi$ satisfies the conditions in part 2, then $G(\text{graph}_2(\Phi)) \cap K_1$ is also the graph of a function satisfying part 2.

To do this, let $\Phi$ be as in part 2 and let $H(w) := (\Phi(w), w)$. Choosing $\delta_1$ followed by $\delta_3$ small enough, $\pi_2 GH(w)$ will be near $\pi_2 F(0, w)$ and hence will be injective on $S_j$. From the assumptions on the image of $GH$, we can define the graph transform $G_\# \Phi : S(0; 1 + c_0/2) \to K_0'$ by

$$G_\# \Phi(w) = (\pi_1 GH)(\pi_2 GH)^{-1}(w).$$

The definition of $G_\# \Phi$ shows that $\text{graph}_2(G_\# \Phi) = G(\text{graph}_2(\Phi)) \cap (K_0' \times S(0; 1 + c_0/2))$, so all that remains is to check the bounds on the derivatives of $G_\# \Phi$. For the remainder of the proof, fix $\delta_1$ small enough to obtain the above results. We will shrink $\delta_3$ further to obtain the desired result.

For $p \in K_1$, let $A_{j,k}(p)$ be the $(j,k)th$ entry in the matrix $D_p G$, and note that by assumption $|A_{1,1}(p) - \lambda_s|, |A_{2,2}(p) - \lambda_u|, |A_{2,1}(p)|, |A_{1,2}(p)|$, and all first derivatives of each $A_{j,k}(p)$ are bounded by $\delta_3$. Now, if $w \in S(0; 1 + c_0/2)$ and $u := (\pi_2 GH)^{-1}(w)$, a simple calculation shows

$$(G_\# \Phi)'(w) = \frac{A_{1,1}(H(u))\Phi'(u) + A_{1,2}(H(u))}{A_{2,1}(H(u))\Phi'(u) + A_{2,2}(H(u))}.$$

Call the numerator of this last expression $N(u)$ and the denominator $M(u)$. The assumptions on $A_{j,k}$ and $\Phi$ imply that $|N(u)| \leq (\lambda_s + \delta_3)\delta_1 + \delta_3$ and $|M(u)| \geq \lambda_u - \delta_3 - \delta_1\delta_3$. For $\delta_3$ small depending on $\delta_1$, we have $|(G_\# \Phi)'(w)| < \delta_1$.

Differentiating again, we get

$$|(G_\# \Phi)''(w)| = \left| \left( \frac{N'(u)}{M(u)} - \frac{N(u)M'(u)}{M^2(u)} \right) \frac{1}{M(u)} \right|.$$

Calculating $N'(u)$ and $M'(u)$ in terms of the partial derivatives of $A_{j,k}$ and the derivatives of $\Phi$, then using the assumed bounds on these quantities shows that we can choose $\delta_3$ small enough to get $|(G_\# \Phi)''(w)| < \delta_1$.

By induction we obtain part 2, and part 3 is analogous. □

Next, we show how to approximate the behavior of holomorphic maps near $F$ by using linear maps, at least when restricted to the stable or unstable manifold of a fixed point. First, for $p \in K_1$, $D_p F$ is the diagonal matrix

$\operatorname{diag}(\lambda_s, \lambda_u)$, where $\lambda_s = \lambda_u^{-1} = c_1/3$. Hence for $c_1$ near 1 and $G$ near $F$, we have

$$M_0 := \sup\{\|D_pG\| : p \in K_1\} \leq 4,$$

and

$$m_0 := \inf\{\|D_pG^{-1}\|^{-1} : p \in K_1\} \geq \frac{1}{4}.$$

Note that for $v \in T_q\mathbb{C}^2$, $\|(D_qG)v\| \geq m_0\|v\|$.

DEFINITION 3.8. *For $q \in W^s(p_0^G)$, let $L_q \subseteq T_q\mathbb{C}^2$ be the complex line $T_qW^s(p_0^G)$, and let $\pi_q^s : T_q\mathbb{C}^2 \to L_q$ denote orthogonal projection. Define $\pi_q^u$ for $q \in W^u(p_0^G)$ analogously.*

DEFINITION 3.9. *For $p, q \in W_m^s$ and $n \geq 0$, define*

$$J_q^n(p) := \exp_{G^n(q)}(D_qG^n)\pi_q^s(p_q).$$

The following result shows that we can approximate $G|W^s(p_0)$ by the maps $J$. By the linearity of $F$, we may choose $\delta_1$ small enough that if $\Phi : S(0; 1+c_0/2) \to \mathbb{C}$ satisfies part 3 of proposition 3.7, then there exist constants $C_1, C_2, C_3 > 0$ with $C_1^2 < C_2 < C_1 < 1$ such that if $p, q \in \operatorname{graph}_1(\Phi) \cap K_1$, then

$$C_2\|p-q\| < \|F(p) - F(q)\| < C_1\|p-q\|$$

and

$$\|F(p) - F(q) - (D_qF)(p_q)\| < C_3\|p-q\|^2.$$

By taking $\delta_1$ small enough, we can make $C_1$ and $C_2$ arbitrarily close to $\lambda_s$ and $C_3$ arbitrarily close to 0. For $\delta_3$ small and $\|F-G\|_{C^2} < \delta_3$ on $K_1$, these same inequalities will hold with $G$ in place of $F$. We may also assume that analogous inequalities hold for $G^{-1}$ on graphs near the unstable manifolds of $F$ with $C_2' < C_1'$, both near $1/\lambda_u$, in place of $C_2$ and $C_1$. Finally, we require $\delta_1$ and $C_3$ small enough that $(17\delta_1 + 4C_3)\mathrm{diam}(\mathrm{K}_1) \leq \log 2/2$.

LEMMA 3.10. *Using the notation and assumptions on $\delta_1$ and $\delta_3$ from the preceding paragraph, if $n \geq 0$ and $p, q \in \operatorname{graph}_1(\Phi) \cap W_m^s$, $p \neq q$, then*

$$C^{-2\|p-q\|} \leq \frac{\|G^n(p) - G^n(q)\|}{\|J_q^n(p) - J_q^n(q)\|} \leq C^{\|p-q\|}, \tag{3.3}$$

*where $C = (17\delta_1 + 4C_3)/(1-C_1)$. Analogous inequalities are valid for $n \leq 0$, $J_q^n(p) := \exp_{G^n(q)}(D_qG^n)\pi_q^u(p_q)$, and $p, q \in \operatorname{graph}_2(\Phi) \cap W_m^u$, $p \neq q$, with $\Phi$ as in part 2 of proposition 3.7.*

*Proof.* Denote the fraction in equation (3.3) by $A_n$. By induction we prove

$$A_n \leq (1 + \delta_1\|p-q\|)\prod_{j=0}^{n-1}(1 + \|p-q\|(16\delta_1 + 4C_3)C_1^j), \tag{3.4}$$

which implies the upper bound in (3.3).

The case $n = 0$ follows immediately by applying the second part of lemma 3.4, then using the fact that $\delta_1\|p-q\|/2 \le 1/2$ to replace $1/(1-\delta_1\|p-q\|/2)$ by $1+\delta_1\|p-q\|$.

For $n \ge 1$, write $P = G^{n-1}(p)$ and $Q = G^{n-1}(q)$. The triangle inequality and the assumptions on $G$ imply that

$$A_n \le \frac{\|(D_Q G)(P_Q)\| + C_3\|P-Q\|^2}{\|(D_Q G)(D_q G^{n-1})\pi^s_q(p_q)\|}.$$

Let $B$ denote the right hand side. Since $(D_q G^{n-1})(T_q W^s(p_0)) = T_Q W^s(p_0)$ by the invariance of the stable manifold, and since $D_Q G$ is complex linear on $T_Q W^s(p_0)$, we may replace the denominator by $\|(D_Q G)\pi^s_Q\|\|(D_q G^{n-1})\pi^s_q(p_q)\|$. Likewise, $\|(D_Q G)\pi^s_Q(P_Q)\| = \|(D_Q G)\pi^s_Q\|\|\pi^s_Q(P_Q)\|$, so using this together with $\|(D_q G^{n-1})\pi^s_q(p_q)\| = \|J^{n-1}_q(p) - J^{n-1}_q(q)\|$, we obtain

$$B \le \frac{\|P-Q\|}{\|J^{n-1}_q(p) - J^{n-1}_q(q)\|}\left(\frac{\|(D_Q G)(P_Q)\|\|\pi^s_Q(P_Q)\|}{\|(D_Q G)\pi^s_Q(P_Q)\|\|P-Q\|} + \frac{C_3\|P-Q\|}{\|(D_Q G)\pi^s_Q\|}\right).$$

Using the triangle inequality, both parts of lemma 3.4 and the choices of $C_1$ and $\delta_1$ gives

$$\begin{aligned}\frac{\|(D_Q G)(P_Q)\|}{\|(D_Q G)\pi^s_Q(P_Q)\|} &\le \frac{\|(D_Q G)\pi^s_Q(P_Q)\| + \|(D_Q G)(P_Q - \pi^s_Q P_Q)\|}{\|(D_Q G)\pi^s_Q(P_Q)\|} \\ &\le 1 + 16\delta_1\|p-q\|C_1^{n-1}.\end{aligned}$$

Also, note that $\|\pi^s_Q(P_Q)\| \le \|P-Q\|$ since $\pi^s_Q$ is an orthogonal projection, and that

$$\frac{C_3\|P-Q\|}{\|(D_Q G)\pi^s_Q\|} \le \frac{\|p-q\|C_3 C_1^{n-1}}{m_0}.$$

Putting these inequalities together with the previous bound on $B$ gives

$$B \le \frac{\|G^{n-1}(p) - G^{n-1}(q)\|}{\|J^{n-1}_q(p) - J^{n-1}_q(q)\|}\left(1 + (16\delta_1 + 4C_3)\|p-q\|C_1^{n-1}\right),$$

and induction gives (3.4).

The proof of the lower bound in (3.3) is essentially the same using induction to show

$$A_n \ge \prod_{j=0}^{n-1}(1 - \|p-q\|(17\delta_1 + 4C_3)C_1^j).$$

Here we subtract the error terms to get lower bounds and use lemma 3.4 to get $\|\pi^s_Q(P_Q)\|/\|P_Q\| \ge 1 - \|p-q\|\delta_1 C_1^{n-1}$ to finish the induction. The lower bound in equation (3.3) then follows from the assumptions on $\delta_1$ and $C_3$. $\square$

In addition to comparing the magnitude of the error between iterates of $G$ and the appropriate $J^n_q$, we also need estimates on the angle between $G^n(p)$

and $J^n_q(p)$ relative to the base point $G^n(q)$. The following result shows that we can make this angle arbitrarily small independent of $n$ by making $G$ sufficiently close to linear.

Definition 3.11. *For points $q, p_1, p_2 \in \mathbb{C}^2$, $p_1, p_2 \neq q$, let $\mathcal{A}(q;\, p_1,\, p_2)$ be the angle between the vectors $\overrightarrow{qp_1}$ and $\overrightarrow{qp_2}$. More precisely, viewing $p_j$ and $q$ as elements in $\mathbf{R}^4$ and using the real dot product,*

$$\mathcal{A}(q;\, p_1,\, p_2) = \arccos\left(\frac{(p_1 - q)\cdot(p_2 - q)}{\|p_1 - q\|\,\|p_2 - q\|}\right),$$

*where arccos is chosen in the interval $[0, \pi]$.*

Lemma 3.12. *In addition to the assumptions of the preceding proposition, suppose that $(C_3 + \delta_1 M_0/2)\mathrm{diam}(\mathrm{K}_1) < \mathrm{m}_0/2$. If $n \geq 0$ and $p, q$ as in that proposition, then*

$$\mathcal{A}(G^n(q);\, G^n(p),\, J^n_q(p)) \leq C\|p - q\|,$$

*where $C = \delta_1 + (8C_3 + 16\delta_1)/(1 - C_1) + \delta_1/(1 - C_1^2/C_2)$. The analogous inequality holds for backwards iterates.*

*Proof.* Let $\phi_n = \mathcal{A}(G^n(q);\, G^n(p),\, J^n_q(p))$. Again we will induct on $n$ to show

$$\phi_n \leq \delta_1\|p - q\| + \sum_{j=0}^{n-1}(8C_3 + 16\delta_1)\|p - q\|C_1^j + \sum_{j=1}^{n-1}\delta_1\|p - q\|\left(\frac{C_1^2}{C_2}\right)^j.$$

First distinguish both the $n = 0$ and $n = 1$ cases. In these cases, we bound $\phi_n$ by obtaining a lower bound on $\|G^n(q) - J^n_q(p)\|$ and an upper bound on $\|J^n_q(p) - G^n(p)\|$. Viewing these as the lengths of two sides of the triangle formed by the points $G^n(q)$, $G^n(p)$, and $J^n_q(p)$, the angle $\phi_n$ is maximized when the longer of these two sides is the hypotenuse of a right triangle. Using this procedure, a simple calculation together with lemma 3.4 and the choice of $C_3$ gives $\phi_0 = \delta_1\|p - q\|$ and $\phi_1 \leq (2C_3 + \delta_1 M_0)\|p - q\|/m_0$.

Next, let $n \geq 2$, and write $P = G^{n-1}(p)$ and $Q = G^{n-1}(q)$. For the induction, we use the fact that for the fixed base point $F(Q)$, the angle function $\mathcal{A}(F(Q);\, \cdot,\, \cdot)$ satisfies the triangle inequality in the last two slots. Hence, we set

$$A := \mathcal{A}(G(Q);\, G(P),\, \exp_{G(Q)}(D_Q G)\pi^s_Q(P_Q))$$

and

$$B := \mathcal{A}(G(Q);\, \exp_{G(Q)}(D_Q G)\pi^s_Q(P_Q),\, J^n_q(p)),$$

so that $\phi_n \leq A + B$.

From the $n = 1$ case we see that $A \leq C_1^{n-1}\|p - q\|(8C_3 + 16\delta_1)$. Next, applying $\exp^{-1}_{G(Q)}$, which preserves angles, and $(D_Q G)^{-1}$, which preserves angles on the complex line $T_Q W^s_m$, we see that $B = \mathcal{A}(0;\, \pi^s_Q(P_Q),\, (D_q G^{n-1})\pi^s_q(p_q))$. Applying $\exp_Q$ and using the triangle inequality for $\mathcal{A}$ in the last two slots gives

$$B \leq \mathcal{A}(Q;\, \exp_Q \pi^s_Q(P_Q),\, P) + \mathcal{A}(Q;\, P,\, J^n_q(p)).$$

An argument like that in the case $n = 1$ shows that the first term of this sum is bounded by $\delta_1 \|p - q\| (C_1^2/C_2)^{n-1}$. Combining the bounds for $A$ and $B$, we get

$$\phi_n \leq \|p - q\| \left( (8C_3 + 16\delta_1) C_1^{n-1} + \delta_1 \left( \frac{C_1^2}{C_2} \right)^{n-1} \right) + \phi_{n-1},$$

and induction completes the lemma. $\square$

## 4. Dynamically defined Cantor sets

Recall the basic set $\Lambda$ constructed in section 2 for the map $F$ and the corresponding set $\Lambda_G$ for $G$ near $F$. Recall also that $F$ has a saddle fixed point $p_0 = (0, 0)$ and that each $G$ near $F$ has a unique fixed point $p_0^G$ near $p_0$.

In this section we analyze the Cantor sets $\Lambda_G \cap W^s(p_0^G)$ and $\Lambda_G \cap W^u(p_0^G)$ as a prelude to the stable intersection mentioned in the outline of the proof. We give complementary descriptions of these sets in terms of an increasing union of subsets and in terms of a decreasing intersection of neighborhoods.

First we need some notation. Let $L$ denote a complex line in $\mathbb{C}^2$; i.e., $L = \mathbb{C}(a, b)$ for some $(a, b) \in \mathbb{C}^2$. Intuitively, the following set is a pie-shaped wedge in $L$ with the tip removed.

Definition 4.1. *For $q \in L$, $\zeta \in L \backslash \{q\}$, $0 < r_1 < r_2$, and $\delta_\theta \in (0, \pi)$, define*

$$\mathrm{Wedge}^L_\zeta(q;\, r_1,\, r_2,\, \delta_\theta) := \{p \in L : r_1 \leq \|p - q\| \leq r_2 \text{ and } \mathcal{A}(q;\, p,\, \zeta) \leq \delta_\theta\}.$$

Recall the definition of $W^u_j$ from definition 3.5 and define $W^s_j$ analogously as part of the stable manifold for the fixed point $p_0^G$ for $G$.

Definition 4.2. *For $j \geq 0$, let $X_j = W^s_j \cap W^u_1$.*

Note that $G^j(X_j) \subseteq G^{j+1}(X_{j+1}) \subseteq W^s_0$ for all such $j$.

For the first description of the Cantor sets, we construct increasing subsets such that given a point $p$ in the $n$th set and any direction in the plane, there is a point $q$ in the $(n + 1)$st set such that the vector $\overrightarrow{pq}$ has direction which differs from the given direction by no more than $\pi/6$ and such that $\|p - q\|$ has good upper and lower bounds.

For the following proposition, let $V$ be a convex neighborhood of the fixed point $p_0 = (0,0)$, and let $P^u : V \to \mathbb{C}$ be $C^1$-near $\pi_1$. Also, for $j_0 > 0$, $j > 0$, let $Y_j := P^u(G^{j_0+j}(X_j))$. The sets $Y_j$ will form the increasing subsets of a Cantor set in the plane. Note that the set $Y_0$ is a nearly regular grid of 9 points in the plane, and that the set $Y_{j+1}$ is formed from $Y_j$ by using each point of $Y_j$ as the center point of a scaled, distorted copy of $Y_0$. In particular, $Y_j \subseteq Y_{j+1}$, and given $z \in Y_j$ there exist unique $Q_0 \in X_j$, $Q_1 \in G^{-1}(X_j) \subseteq X_{j+1}$ such that $z = P^u G^{j_0+j}(Q_0) = P^u G^{j_0+j+1}(Q_1)$. Finally, recall the definition of $\pi^s_q$ from definition 3.8.

In the proof of this proposition, we choose the parameters $N_0$, $\delta_1$, $\delta_2$ and $\delta_3$ in that order, where $\delta_1$ is as in proposition 3.7. For clarity, we shrink $\delta_1$, $\delta_2$ and $\delta_3$ throughout the proof, but the dependence is the order just given.

Proposition 4.3. *There exist $N_0 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$ such that if $j_0 \geq N_0$, $\|P^u - \pi_1\|_{C^1} < \delta_2$ on $V$ and $\|F - G\|_{C^2} < \delta_3$ on $K_1$, and if $j \geq 1$, $z \in Y_{j-1}$ and $\zeta \in \mathbb{C} - \{z\}$ are arbitrary, then*

$$(Y_j - Y_{j-1}) \cap \mathrm{Wedge}_\zeta(z;\, r_z/2,\, 2r_z,\, \pi/6) \neq \emptyset,$$

*where $r_z = \|(D_Q G^{j_0+j})\pi^s_Q\|$ with $z = P^u G^{j_0+j}(Q)$, $Q \in G^{-1}(X_{j-1}) \subseteq X_j$.*

*Proof.* Since $X_0 \subseteq W^s_0$, we can choose $N_0 \geq 1$ such that $F^{N_0}(X_0) \subseteq V$, and this will remain true for $G$ near $F$. Hence for $\delta_3$ small, $G^{N_0+j}(X_j) \subseteq V$ for all $j \geq 0$.

We use linear maps which approximate $G$ to obtain the desired intersection. Let $j_0 \geq N_0$, $j \geq 1$ and write $m = j_0 + j$. Define the affine linear map $J^m_Q$ as in definition 3.9, and let $\Phi$ be as in part 3 of proposition 3.7 with $\mathrm{graph}_1(\Phi) \subseteq W^s(p_0^G)$ such that $Q \in \mathrm{graph}_1(\Phi) \cap W^s_j$ with $z = P^u G^m(Q)$. Also, let $q = G^m(Q)$, let $M$ be the complex line tangent to $\mathrm{graph}_1(\Phi)$ at $Q$ and let $M_Q \subseteq T_Q\mathbb{C}^2$ be the tangent space of $\mathrm{graph}_1(\Phi)$ at $Q$.

Since $W^s_{j+1}(G)$ varies $C^1$ with $G$, we see from lemma 3.4 that if $\delta_1$ and $\delta_3$ are small enough, then the set

$$\mathrm{graph}_1(\Phi) \cap W^u_1 = \{Q^1, \ldots, Q^9\}$$

is a nearly regular grid of nine points with center point $Q = Q^1$ and distance approximately 1 between adjacent points. See figure 2. Note that each of these points lies in $X_j$. Let $Q^k_Q = \exp_Q^{-1} Q^k \in T_Q\mathbb{C}^2$. Then for any $\zeta \in M - \{Q\}$,

$$\mathrm{Wedge}^M_\zeta(Q;\, 2/3,\, 3/2,\, \pi/7) \cap \exp_Q\{\pi^s_Q Q^2_Q, \ldots, \pi^s_Q Q^9_Q\} \neq \emptyset. \tag{4.1}$$

We map these two intersecting sets forward under $J^m_Q$. Let $L = J^m_q(M)$, $\xi = J^m_Q(\zeta)$ and $r = \|(D_Q G^m)\pi^s_Q\|$. Since $J^m_Q$ is complex linear from $M$ to $L$, we get

$$\mathrm{Wedge}^L_\xi(q;\, 2r/3,\, 3r/2,\, \pi/7) \cap \{J^m_Q(Q^2), \ldots, J^m_Q(Q^9)\} \neq \emptyset. \tag{4.2}$$

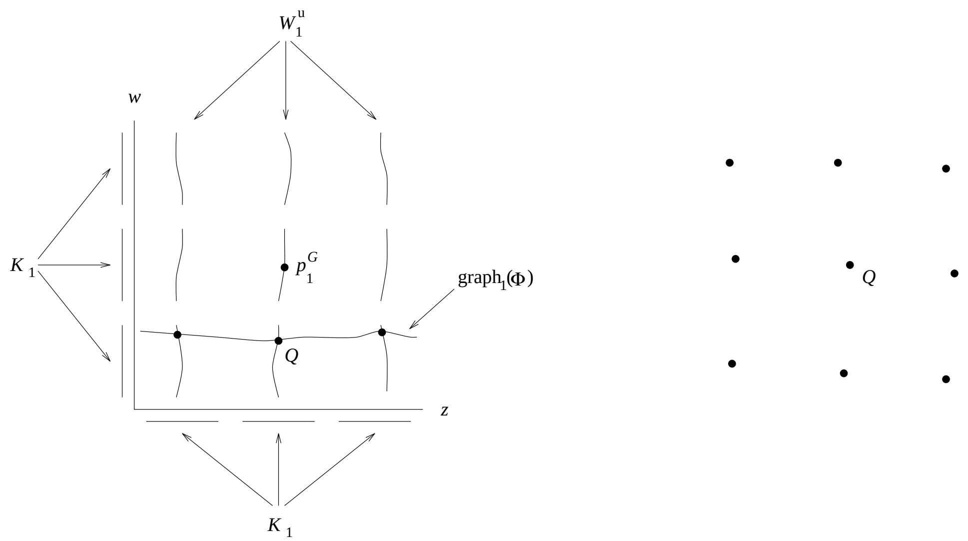


FIGURE 2. On the left is a 2-dimensional representation of the part of $X_j$ given by the intersection of $W_1^u$ with $\mathrm{graph}_1(\Phi)$. On the right is the set $P^u(W_1^u \cap \mathrm{graph}_1(\Phi))$.

Now, define $H_q^u(p) = \pi_1(p-q) + P^u(q)$, and let $a, c \in (0,1)$, $b, d \in (1,2)$, and $\delta_{\theta,1}, \delta_{\theta,2} > 0$ such that $ac > 3/4$, $bd < 4/3$, and $\delta_{\theta,1} + \delta_{\theta,2} < \pi/6 - \pi/7$. If $\delta_1$ and $\delta_2$ are small, then $L$ is nearly parallel to the $z$-axis, so

(4.3)
$$H_q^u(\mathrm{Wedge}_\xi^L(q;\, 2r/3,\, 3r/2,\, \pi/7)) \subseteq \mathrm{Wedge}_\eta(H_q^u(q),\, 2ar/3,\, 3br/2,\, \pi/7 + \delta_{\theta,1}),$$

where $\eta = H_q^u(\xi)$.

From (4.2) and (4.3) we see that for any $\zeta \in \mathbb{C} - \{H_q^u(q)\}$ there exists $k \geq 2$ with

$$H_q^u J_Q^m(Q^k) \in \mathrm{Wedge}_\zeta(H_q^u(q);\, 2ar/3,\, 3br/2,\, \pi/7 + \delta_{\theta,1}). \tag{4.4}$$

Next, it follows from lemmas 3.10 and 3.12 that if $\delta_1$, $\delta_2$, and $\delta_3$ are small enough, then

$$\frac{|H_q^u(J_Q^m Q^k) - H_q^u(J_Q^m Q)|}{|P^u(G^m(Q^k)) - P^u(G^m(Q))|} \in [c, d] \tag{4.5}$$

and

$$\mathcal{A}(P^u(G^m Q);\, P^u(G^m(Q^k)),\, H_q^u(J_Q^m(Q^k))) \leq \delta_{\theta,2}. \tag{4.6}$$

From (4.5) and (4.6) and the fact that $H_q^u(q) = H_q^u(J_Q^m Q) = P^u(G^m Q) = P^u(q)$, we see that

$$P^u(G^m Q^k) \in \mathrm{Wedge}_\zeta(P^u(q);\, 2acr/3,\, 3bdr/2,\, \pi/7 + \delta_{\theta,1} + \delta_{\theta,2}).$$

By choice of $a, b, c, d, \delta_{\theta,1}$, and $\delta_{\theta,2}$, we obtain the proposition. □

We next analyze a Cantor set in $\Lambda_G \cap W^u(p_0^G)$. The proof uses the same outline as that for proposition 4.3, but this time we give a description of the Cantor set as the intersection of a decreasing sequence of sets obtained by taking the intersection of $W^u(p_0^G)$ with images of $K_1$ under iterates of $G^{-1}$.

Here we show that given a point $q$ in the $n$th set, there is a direction in the plane so that the $(n+1)$st set contains a Wedge centered at $q$ in this direction so that the ratio of the inner and outer radii of this Wedge is independent of $n$, as is the angle of opening. This is true because the $n$th set is a collection of disjoints sets which are nearly squares, while the $(n+1)$st set is obtained by subdividing each of these "squares" into 9 sub-"squares" which nearly cover the previous set.

The structure of this proof is much like that of the previous proposition. We first point out that for each $j \geq 0$, the set $G^{-(j+1)}(W_j^u)$ is contained in $W_{-1}^u$ and that the intersection of these sets over all $j \geq 0$ is contained in $\Lambda_G \cap W^u(p_0^G)$.

As in the previous proposition, we let $V$ be a convex neighborhood of the point $p_0 = (0,0)$ and let $P^s : V \to \mathbb{C}$ be $C^1$-near $\pi_2$. Also, for $k, k_0 \geq 0$, let $Z_k = Z_k(G) = P^s(G^{-(k_0+k)}(W_k^u))$, and recall the definition of $\delta$, $c_0 = 1-\delta$, and $c_1 \in (c_0, 3(c_0/(c_0+2)))$ from the definition of $F$ in section 2.

Proposition 4.4. *There exist $\delta > 0$, $N_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ such that if $k_0 \geq N_1$, $\|P^s - \pi_2\|_{C^1} < \delta_2$ on $V$, and $\|F - G\|_{C^2} < \delta_3$ on $K_1$, then for $k \geq 0$ and $z \in Z_k$, there exists $\zeta \in \mathbb{C} - \{z\}$ such that*

$$\mathrm{Wedge}_\zeta(z;\, \beta R_z/16,\, \beta R_z,\, \pi/6) \subseteq Z_{k+1},$$

*where $\beta = 1/9$, $R_z = \|(D_Q G^{-(k_0+k)})\pi_Q^u\|$, $z = P^s G^{-(k_0+k)}(Q)$ and $Q \in W_k^u$.*

*Proof.* Again we choose $N_1$ large enough that $F^{-N_1}(W_0^u) \subseteq V$ independent of $\delta \in (0, 1/2)$. Then $G^{-(N_1+k)}(W_k^u) \subseteq V$ for all $k \geq 0$ and $G$ near $F$. Let $k \geq N_1$, $k \geq 0$, and write $m = k_0 + k$.

First suppose $z \in Z_k(F)$, $Q_F \in W_k^u(F)$ with $z = P^s F^{-m}(Q_F)$, and let $\Phi_F$ be as in part 2 of proposition 3.7 such that $\mathrm{graph}_2(\Phi_F)$ agrees with the component of $W_k^u(F)$ containing $Q_F$, and let $\Psi_F$ be the restriction of $\Phi_F$ such that $\mathrm{graph}_2(\Psi_F) = \mathrm{graph}_2(\Phi_F) \cap F^{-1}(W_{k+1}^u(F))$. See figure 3.

Since $F$ is piecewise linear, $\Psi_F$ is obtained by first restricting $\mathrm{graph}_2(\Phi_F)$ to obtain 9 squares, each with sides of length $1-\delta$, then restricting again so that each of these squares is subdivided into 9 squares, each with sides of length $c_1(1-\delta)/3$. By choosing $\delta$ near 0, we can make the gaps between these squares arbitrarily small. Hence for $\epsilon > 0$ small, we can choose $\delta > 0$ such that if $M = M_F$ is the complex line tangent to $\mathrm{graph}_2(\Phi_F)$ at $Q_F$, then there exists $\zeta \in M - \{Q_F\}$ such that

$$\mathrm{Wedge}_\zeta^M(Q_F;\, \beta/(16+2\epsilon),\, \beta(1+2\epsilon),\, \pi(1+2\epsilon)/6) \subseteq \exp_{Q_F}(\pi_{Q_F}^u(\mathrm{graph}_2(\Psi_F))).$$

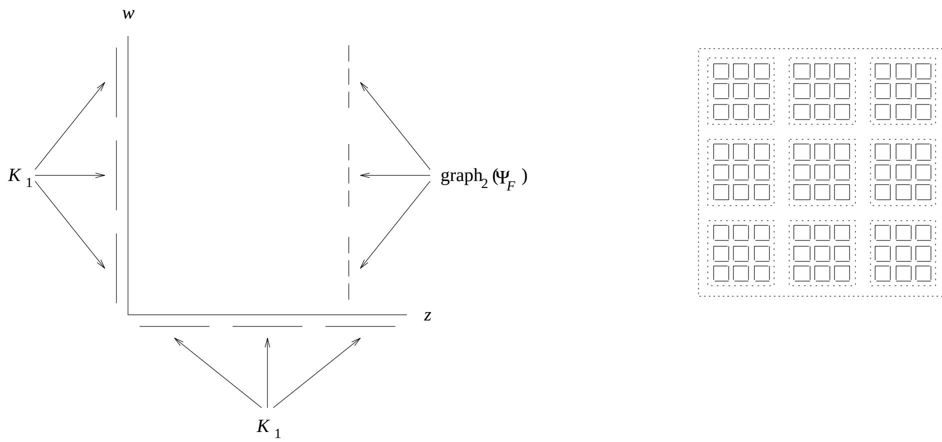


FIGURE 3. On the left is a 2-dimensional representation of $\mathrm{graph}_2(\Psi_F)$. On the right is the set $P^s F^{-m}(\mathrm{graph}_2(\Psi_F)) \subseteq Z_{k+1}(F)$, with previous subdivisions shown in dotted lines.

Using lemma 3.4, proposition 3.7, and Rouche's theorem, it follows that for $\delta_2$ and $\delta_3$ small, $G$ holomorphic with $\|F-G\|_{C^2} < \delta_3$, $z \in Z_k(G)$, and $Q_G$, $\Phi_G$ and $M = M_G$ defined analogously for $G$, there exists $\zeta \in M - \{Q_G\}$ such that

$$\mathrm{Wedge}^M_\zeta(Q_G;\, \beta/(16+\epsilon),\, \beta(1+\epsilon),\, \pi(1+\epsilon)/6) \subseteq \exp_{Q_G}(\pi^u_{Q_G}(\mathrm{graph}_2(\Psi_G))).$$

The remainder of the proof consists of using $J_Q^{-m}$ to approximate $G^{-m}$, verifying relations analogous to those in proposition 4.3, and showing that a similar containment holds after projection. □

With the same proof using extra subdivisions at the beginning, we obtain the following generalization.

PROPOSITION 4.5. *Let $l \in \mathbb{Z}^+$. There exist $N_1 > 0$, $\delta > 0$, $\delta_2 > 0$, $\delta_3 > 0$ such that if $k_0 \geq N_1$, $\|P^s - \pi_2\|_{C^1} < \delta_2$ on $V$, $\|F - G\|_{C^2} < \delta_3$ on $K_1$, then for $k \geq 0$ and $z \in Z_k$, there exists $\zeta \in \mathbb{C} - \{z\}$ such that*

$$\mathrm{Wedge}_\zeta(z;\, \beta_l R_z/16,\, \beta_l R_z,\, \pi/6) \subseteq Z_{k+l},$$

*where $\beta_l = 1/(2(3^l)+3)$, $R_z = \|(D_Q G^{-(k_0+k)})\pi^u_Q\|$, $z = P^s G^{-(k_0+k)}(Q)$, $Q \in W^u_k$.*

*Remark* 4.6. Note that if $Q$ is as in proposition 4.5 and $w$ is a point in the Wedge constructed in that proposition with $P \in W^u_k$ such that $w = P^s G^{-(k_0+k)}(P)$, then $Q$ and $P$ both lie in $\mathrm{graph}_2(\Phi_G)$.

## 5. Tangencies between invariant foliations

In this section we extend the piecewise linear map constructed in section 2 so that it has a homoclinic tangency, then show that for any nearby map, there are semi-invariant foliations extending the stable and unstable manifolds and a $C^1$ disk of points at which the leaves of the two foliations are tangent.

In order to get a homoclinic tangency for $F$, note first that (2.3) implies that $p_0 = (0,0)$ is a fixed point for $F$. Backwards iteration shows that $W^u(p_0)$ contains $\{0\} \times \overline{S(0; 3c_0/2c_1)}$, and likewise, $W^s(p_0)$ contains $\overline{S(0; 3c_0/2c_1)} \times \{0\}$.

Let $\overline{S_0} = \overline{S(a_0; \rho_0)}$ be a small square contained in $S(0; 3c_0/2c_1) - \overline{K_f}$ such that $(3/c_1)\overline{S_0} \cap \overline{K_g} = \emptyset$. In addition to (2.1), we can define $f(w) = \alpha_0 + \alpha_1 w$ on $S_0$, where $\alpha_1 \neq 0$ is arbitrary and $\alpha_0$ is chosen so that $f(a_0) = 3a_0/c_1$ and hence $F_1(0, a_0) \in (3/c_1)S_0 \times \{a_0\}$.

Likewise, in addition to equation (2.2), we can define $g(z) = -a_0 - (z - f(a_0))/\alpha_1 + c_1(z - f(a_0))^2/3\alpha_1^2$ on $(3/c_1)S_0$. Then $F(0, a_0 + w) = ((c_1/3)(f(a_0) + \alpha_1 w), w^2)$ for $|w|$ small. In particular, $F$ is defined in a neighborhood of $\{0\} \times \{a_0 + w : |w| < \rho_1\}$ for some $\rho_1 > 0$, and the image of this disk is tangent to the $z$-axis at $(a_0, 0) \in W^s(p_0)$. Hence $F$ has a homoclinic tangency at $(a_0, 0)$.

Definition 5.1. *$q_0 := F(0, a_0) = (a_0, 0)$ is the homoclinic tangency for $F$.*

Next, we will extend the stable and unstable manifolds of the basic set $\Lambda$ to semi-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ of a neighborhood of $\Lambda$. For the foliation $\mathcal{F}^s$, we let $\mathcal{L}^s(x)$ denote the leaf containing $x$. Each leaf will be a complex 1-dimensional submanifold of $\mathbb{C}^2$, and if $x$ and $F(x)$ are contained in the foliated neighborhood, then $F(\mathcal{L}^s(x)) \subseteq \mathcal{L}^s(F(x))$. Hence we will be able to extend $\mathcal{F}^s$ by applying $F^{-1}$. Analogous results hold for $\mathcal{F}^u$ with $F^{-1}$ in place of $F$. For more background and further details, see [11].

On $K_1$, the form of $F$ implies that the set of complex lines parallel to the $z$-axis is preserved under iteration, so we can take this to be $\mathcal{F}^s$ in $K_1$. Applying $F^{-1}$ we extend this to a neighborhood of $q_0$, so that near $q_0$, the leaves of $\mathcal{F}^s$ are complex lines parallel to the $z$-axis.

Likewise, we can use lines parallel to the $w$-axis to obtain $\mathcal{F}^u$ in $K_1$, then apply $F$ to obtain $\mathcal{F}^u$ in a neighborhood of $q_0$. In this case, for $|z_0|$ small, we can apply $F$ to the disk $\{(z_0, a_0 + w) : |w| < \rho_1\}$. A calculation shows that the point at which the image of this disk is parallel to the $z$-axis is the image of the point $(z_0, a_0 - z_0/\alpha_1)$, and $F(z_0, a_0 - z_0/\alpha_1) = q_0 + (0, -3z_0/\alpha_1 c_1)$. In particular, for $\rho_2$ small and $|w| < \rho_2$, each point of the form $q_0 + (0, w)$ is a point of tangency between a leaf of $\mathcal{F}^s$ and $\mathcal{F}^u$.

Definition 5.2. *Let $D_T$ denote this disk of tangencies: $D_T = \{q_0 + (0, w) : |w| < \rho_2\}$.*

In order to obtain Cantor sets contained in $D_T$, we need to calculate the projection function $P^s$ obtained by projecting along leaves of $\mathcal{F}^s$ from a neighborhood of the origin to $D_T$, and likewise for $P^u$.

Since leaves of $\mathcal{F}^s$ are complex lines parallel to the $z$-axis in a neighborhood of the segment from $(0,0)$ to $q_0$, the projection function along these leaves is simply the projection $(z,w) \mapsto q_0 + (0,w)$. Hence, identifying $D_T$ with a disk in the plane by projecting to the $w$-axis, we see that $P^s = \pi_2$.

On the other hand, leaves of $\mathcal{F}^u$ are complex lines parallel to the $w$-axis in a neighborhood of the segment from $(0,0)$ to $(0,a_0)$. Hence, for a point $p = (z,w)$ near $(0,0)$, we can first project it to $(z, a_0 - z/\alpha_1)$ along a leaf of $\mathcal{F}^u$. Then $(z,w)$ and $(z, a_0 - z/\alpha_1)$ are on the same leaf, so applying $F$, we see that $(c_1 z/3, 3w/c_1)$ and $q_0 + (0, -3z/\alpha_1 c_1)$ lie on the same leaf. Reparametrizing, we see that $(z,w)$ projects to $q_0 + (0, -9z/\alpha_1 c_1^2)$. Hence taking $\alpha_1 = -9/c_1^2$, we obtain $P^u = \pi_1$.

To obtain analogous projection functions for nearby maps, we need the following variant of a result by Pixton [11]. The theorem says that if we are given a biholomorphic map with a basic set and a semi-invariant foliation, we can perturb the map and obtain foliations which vary continuously in the $C^1$ topology. The proof of this version is essentially the same as the original with some extra care taken for the holomorphic objects. The details can be found in the appendix of [3].

THEOREM 5.3 (Pixton). *Let $V \subset \mathbb{C}^2$ be open. Let $\Lambda \subseteq V$ be a basic set of saddle type for the injective holomorphic map $G_0 : V \to \mathbb{C}^2$, with $\Lambda = \cap_{n=-\infty}^{\infty} G_0^n(V)$, and let $E^s \oplus E^u$ be the associated splitting of $T\mathbb{C}^2|\Lambda$. Suppose that*

$$\|DG_0|E^s\| \, \|DG_0^{-1}|E^u\| \, \|DG_0|E^u\| < 1. \tag{5.1}$$

*Then there exists a compact set $L$ and $\delta_3 > 0$ such that if $G$ is holomorphic on $V$ with $\|G - G_0\|_{C^2} < \delta_3$, then there is a basic set $\Lambda_G = \cap_{n=-\infty}^{\infty} G^n(V)$ and a foliation $\mathcal{F}_G^s$ such that $\Lambda_G \subseteq \mathrm{int} L \subseteq L \subseteq \mathcal{F}_G^s$, and such that the assignment $G \mapsto \mathcal{F}_G^s$ is continuous in the $C^1$ topology on the foliations and on their tangent planes. Moreover, each of the following properties hold.*

*(i) Each leaf $\mathcal{L}_G^s(p)$ of $\mathcal{F}_G^s$ is a complex manifold.*
*(ii) If $p \in \Lambda_G$, then $\mathcal{L}_G^s(p)$ agrees with $W^s_{\mathrm{loc}}(p,G)$.*
*(iii) The tangent planes of leaves vary $C^1$ throughout $\mathrm{int} L$.*
*(iv) If $p \in L \cap G(L)$, then $G^{-1}(\mathcal{L}_G^s(p)) \supseteq \mathcal{L}_G^s(G^{-1}(p))$.*

*Finally, if $\mathcal{F}_0^s$ is given satisfying these conditions for $G_0$, then $\mathcal{F}_G^s$ can be chosen so that $\mathcal{F}_{G_0}^s = \mathcal{F}_0^s$.*

*Remark* 5.4. We say that $G_0$ strongly contracts $E^s$ if condition (5.1) holds. Using the piecewise linearity of $F$, we see that $F$ strongly contracts $E^s$

and $F^{-1}$ strongly contracts $E^u$. Hence we can apply the above theorem to both the stable and the unstable foliations.

LEMMA 5.5. *There exists $\delta_3 > 0$ such that if $\|F - G\|_{C^2} < \delta_3$ on the domain of $F$, then $G$ has semi-invariant foliations $\mathcal{F}_G^s$ and $\mathcal{F}_G^u$ as in theorem 5.3 and a $C^1$ disk $D_T^G$ where leaves of $\mathcal{F}_G^s$ and $\mathcal{F}_G^u$ are tangent. Moreover, there are $C^1$ projection functions $P_G^s$ and $P_G^u$ from a neighborhood of the origin to $\mathbb{C}$ defined by projecting to $D_T^G$ along leaves of $\mathcal{F}_G^s$ and $\mathcal{F}_G^u$ respectively, then projecting to the $w$-axis. Finally, the assignments $G \mapsto D_T^G$, $G \mapsto P_G^s$ and $G \mapsto P_G^u$ are continuous in the $C^1$ topology.*

*Proof.* By the preceding remark, we can apply theorem 5.3 to get $\delta_3 > 0$ such that if $\|F - G\|_{C^2} < \delta_3$ on $\mathrm{dom}(F)$, then $G$ has stable and unstable foliations $\mathcal{F}_G^s$ and $\mathcal{F}_G^u$ as in the theorem. By iteration, we can extend these foliations to a neighborhood of $q_0$.

The conclusions of the theorem together with the form of the foliations constructed for $F$ imply that near $q_0$, we can choose $C^1$ parametrizations $\phi_G^s, \phi_G^u : \Delta^2(0;r) \to \mathbb{C}^2$ for some $r$ small such that $\phi_G^s(x,y)$ and $\phi_G^u(x,y)$ are $C^1$ and holomorphic in $x$ for each fixed $y$; such that if $t = s,u$, then $\phi_G^t(\Delta(0;r),y)$ is contained in the leaf of $\mathcal{F}_G^t$ containing $\phi_G^t(0,y)$; such that $\phi_G^s(\Delta^2(0;r))$ and $\phi_G^u(\Delta^2(0;r))$ contain some fixed neighborhood of $q_0$; and such that $\phi_F^s(x,y) = (x,y) + q_0$ and $\phi_F^u(x,y) = F(y, a_0 + x)$. We can do this so that $G \mapsto \phi_G^s$ and $G \mapsto \phi_G^u$ are continuous in the $C^1$ topology.

Define $\Phi(G,x,y) = (\partial/\partial x)\pi_2((\phi_G^s)^{-1}\phi_G^u(x,y))$. A calculation shows that $\Phi(F,x,0) = 2x$. Hence the implicit function theorem gives a unique function $g(G,y)$ defined for $G$ near $F$ and $y$ near 0 such that $\Phi(G, g(G,y), y) = 0$.

Then $D_T^G$ is the image of $g(G,\cdot)$, and $G \mapsto g(G,\cdot)$ is continuous in the $C^1$ topology. We can define the projection functions $P_G^s$ and $P_G^u$ just as we did for $F$, and since the foliations and the disk of tangencies vary continuously in the $C^1$ topology with $G$, so do $P_G^s$ and $P_G^u$. □

## 6. Persistent tangencies between basic sets

In this section we put together some of the previous results to show that for $G$ near $F$, there is a tangency between the stable and unstable manifolds of the basic set $\Lambda_G$. The idea is that the stable manifold of $\Lambda_G$ intersects the disk of tangencies $D_T^G$ in a Cantor set, and likewise for the unstable manifold. Any point of intersection between these two Cantor sets is a point of tangency between the stable and unstable manifolds. To make this precise, we need a technical lemma.

In the following lemma, $\Phi$ satisfies part 3 of proposition 3.7, so that $\mathrm{graph}_1(\Phi)$ is nearly parallel to the $z$-axis and $\mathrm{graph}_1(\Phi) \subseteq W^s(p_0^G)$. For such $\Phi$, $Q \in \mathrm{graph}_1(\Phi)$, and $L_Q := T_Q(\mathrm{graph}_1(\Phi))$, write $\pi_Q^s$ for orthogonal projection in $T_Q\mathbb{C}^2$ onto $L_Q$.

Lemma 6.1. *Let $C > 1$. There exists $\delta_3 > 0$ such that if $\|F - G\|_{C^2} < \delta_3$, $Q_0, Q_1 \in W^s(p_0^G) \cap \mathrm{graph}_1(\Phi)$ and $j \geq 0$, then*

$$\frac{\|(D_{Q_1}G^j)\pi_{Q_1}^s\|}{\|(D_{Q_0}G^j)\pi_{Q_0}^s\|} \leq C.$$

*An analogous distortion result holds for $(D_Q G^{-j})\pi_Q^u$.*

*Proof.* Note that from the piecewise linearity of $F$, given $C' > 0$, we can choose $\delta_3$ such that $\|(D_{Q_1}G)\pi_{Q_1}^s - (D_{Q_0}G)\pi_{Q_0}^s\| < C'\|Q_1 - Q_0\|$. Next, the hypotheses imply that if $j \geq 0$, then $G^j(Q_0), G^j(Q_1) \in \mathrm{graph}_1(\Phi_j)$ for some $\Phi_j$ as in part 3 of proposition 3.7, and hence $\|G^j(Q_1) - G^j(Q_0)\| \leq C_1^j\|Q_1 - Q_2\|$, where $C_1 < 1$ as in the remarks before lemma 3.10. The remainder of the proof is a simple induction using this latter inequality together with

$$\frac{\|(D_{Q_1}G^j)\pi_{Q_1}^s\|}{\|(D_{Q_0}G^j)\pi_{Q_0}^s\|} \leq \left(1 + \frac{C'\|G^{j-1}(Q_1) - G^{j-1}(Q_0)\|}{\|(D_{G^{j-1}Q_0}G)\pi_{G^{j-1}Q_0}^s\|}\right)\frac{\|(D_{Q_1}G^{j-1})\pi_{Q_1}^s\|}{\|(D_{Q_0}G^{j-1})\pi_{Q_0}^s\|}. \ \square$$

The following proposition produces a tangency between the stable and unstable manifolds of a basic set. The idea is the following. Recall that $X_j = W_j^s \cap W_1^u$, $Y_j = P_G^u(G^{j_0+j}(X_{j_0}))$, and $Z_k = P_G^s(G^{-(k_0+k)}(W_k^u))$. Propositions 4.3 implies that for any point $p \in Y_j$, there is a sequence of points in $\cup_{m>j} Y_m$ which converge geometrically to $p$ with scaling factor near 1/3. Proposition 4.5 implies that the ratio of the size of the squares in $Z_k$ to the size of the gaps can be made arbitrarily small, and this ratio is independent of $k$. Hence if $p \in Y_j \cap Z_k$, then one of the points in some $Y_m$ must land in $Z_{k+1}$. Using induction and nested intersection, we obtain a point of intersection between two Cantor sets in the disk of tangencies, and this intersection corresponds to a tangency between the stable and unstable manifolds of the basic set.

Recall that $q_0$ is the point of homoclinic tangency constructed for $F$.

Proposition 6.2. *There exist $r_0 > 0$ and $\delta_3 > 0$ such that if $\|F - G\|_{C^2} < \delta_3$, then $G$ has a basic set $\Lambda_G$ such that $W^s(\Lambda_G)$ is tangent to $W^u(\Lambda_G)$ at some point $q \in \Delta^2(q_0; r_0)$.*

*Proof.* Choose $r_0 > 0$ large enough that $D_T^F \subseteq \Delta^2(q_0; r_0/2)$, and choose $\delta_2 > 0$ small enough for proposition 4.3 and for proposition 4.5 with $l = 3$. By lemma 5.5, we can choose $\delta_3$ small enough that if $\|F - G\|_{C^2} < \delta_3$, then

$D_T^G$, $P_G^s$ and $P_G^u$ are well-defined with $D_T^G \subseteq \Delta^2(q_0; r_0)$, $\|P_G^s - \pi_2\|_{C^1} < \delta_2$ and $\|P_G^u - \pi_1\|_{C^1} < \delta_1$; such that the hypotheses of propositions 4.3 and 4.5 are satisfied for each such $G$ with $l = 3$ in proposition 4.5; and such that the hypotheses of lemma 6.1 are satisfied with $C = 2$.

For $\delta_3$ small, equation (2.3) implies that $1/4 \le \|(D_Q G)\pi_Q^s\| \le 1/2$ and $\|(D_Q G^{-1})\pi_Q^u\| \le 1/2$ for any $Q \in K_1$ and any such $G$. Induction together with the fact that $D_Q G$ preserves the tangent bundle of the stable and unstable manifolds implies that

$$1/4^j \le \|(D_Q G^j)\pi_Q^s\| \le 1/2^j \tag{6.1}$$

if $Q \in W_l^s(G)$ and $j \ge 1$, and that

$$\|(D_Q G^{-k})\pi_Q^u\| \le 1/2^k \tag{6.2}$$

if $Q \in W_l^u(G)$ and $k \ge 1$.

Fix $j_0 \ge N_0$ as in proposition 4.3, then fix $k_0 > N_1$ as in proposition 4.5 and such that $\beta_3/4(2^{k_0}) \le 2/4^{j_0+1}$, where $\beta_3 = 1/(2(3^3)+3)$ as in proposition 4.5.

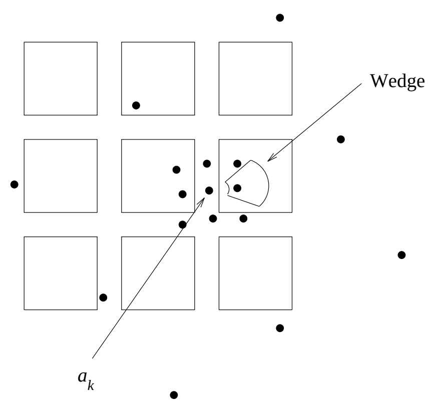


FIGURE 4. The squares in this figure represent 9 components of $Z_{3k+3}$. The large grid is a subset of $Y_{J_k+j_{k+1}-1}$, while the small grid is a subset of $Y_{J_k+j_{k+1}}$ with $a_k$ at the center of both grids. The geometric scaling from $Y_j$ to $Y_{j+1}$ insures that for some $j$, $Y_j$ will intersect $Z_{3k+3}$.

By induction, we construct points $a_k \in Z_{3k} \cap Y_{J_k}$ for some integers $J_k$. For $k = 0$, we shrink $\delta_3$ a final time so that $P_G^u(p_0^G) \in P_G^s(G^{-k_0}(W_0^u))$. This is possible by the $C^1$ dependence of $P_G^u$, $P_G^s$ and $W_0^u$ on $G$ since $\pi_2 q_0 = P_F^u(p_0^F)$ is contained in the interior of $P_F^s(F^{-k_0}(W_0^u))$. Then $a_0 := P_G^u(p_0^G)$ is contained in $Z_0 \cap Y_0$, so we take $J_0 = 0$.

For the induction, suppose $k \ge 0$, $J_k = j_1 + \cdots j_k$ with each $j_l \ge 1$ and $a_k \in Z_{3k} \cap Y_{J_k}$. Define

$$R_k := \|(D_{Q_k} G^{-(k_0+3k)})\pi_{Q_k}^u\|,$$

where $Q_k \in W^u_{3k}$ with $a_k = P^s_G G^{-(k_0+3k)}(Q_k)$. Also, for $j_{k+1} \geq 1$, let

$$r_k(j_{k+1}) := \|(D_{Q'_k} G^{j_0+J_k+j_{k+1}})\pi^s_{Q'_k}\|,$$

where $Q'_k = Q'_k(j_{k+1}) \in G^{-1}(X_{J_k+j_{k+1}-1})$ with $a_k = P^u_G G^{j_0+J_k+j_{k+1}}(Q'_k)$. Suppose also that $\beta_3 R_k/4 \leq 2r_k(1)$. Note that this is true for $k = 0$ by equations (6.1) and (6.2) and choice of $j_0$ and $k_0$.

Proposition 4.5 implies that there exists $\zeta \in \mathbb{C} - \{a_k\}$ such that

$$\mathrm{Wedge}_\zeta(a_k;\, \beta_3 R_k/16,\, \beta_3 R_k,\, \pi/6) \subseteq Z_{3k+3},$$

while proposition 4.3 implies that

$$(Y_{J_k+j_{k+1}} - Y_{J_k+j_{k+1}-1}) \cap \mathrm{Wedge}_\zeta(a_k;\, r_k(j_{k+1})/2,\, 2r_k(j_{k+1}),\, \pi/6) \neq \emptyset.$$

If we can find $j_{k+1} \geq 1$ such that $\beta_3 R_k/4 \leq 2r_k(j_{k+1}) \leq \beta_3 R_k$, then the first Wedge will contain the second, so we can choose $a_{k+1} \in Z_{3k+3} \cap Y_{J_{k+1}}$, where $J_{k+1} = J_k + j_{k+1}$. See figure 4. Hence we verify this inequality, then check the induction hypotheses.

If $2r_k(1) \leq \beta_3 R_k$, then the induction hypotheses imply that $\beta_3 R_k/4 \leq 2r_k(1) \leq \beta_3 R_k$, so we can take $j_{k+1} = 1$.

If $2r_k(1) > \beta_3 R_k$, then (6.1) implies that we can choose $j_{k+1} > 1$ minimal such that $2r_k(j_{k+1}) \leq \beta_3 R_k$. Then $2r_k(j_{k+1} - 1) > \beta_3 R_k$, and fixing $Q'_k \in G^{-1}(X_{J_k+j_{k+1}-1})$ such that $a_k = P^u_G G^{j_0+J_k+j_{k+1}}(Q'_k)$, we have

$$r_k(j_{k+1} - 1) = \|(D_{GQ'_k} G^{j_0+J_k+j_{k+1}-1})\pi^s_{GQ'_k}\|,$$

and hence by the chain rule and invariance of the tangent bundle,

$$r_k(j_{k+1}) = r_k(j_{k+1} - 1)\|(D_{Q'_k} G)\pi^s_{Q'_k}\|.$$

By equation (6.1), we see that $2r_k(j_{k+1}) \geq \beta_3 R_k/4$ as desired.

To complete the induction, we show that $2r_{k+1}(1) \geq \beta_3 R_{k+1}/4$. Using the argument just given, we see that $2r_{k+1}(1) = 2r_k(j_{k+1})\|(D_{Q'_{k+1}} G)\pi^s_{Q'_{k+1}}\| \geq \beta_3 R_k/16$. On the other hand, by the chain rule, equation (6.2), the remark after proposition 4.5 and lemma 6.1 with $C = 2$,

$$\begin{aligned} R_{k+1} &= \|(D_{G^{-3}Q_{k+1}} G^{-(k_0+3k)})\pi^u_{G^{-3}Q_{k+1}}\| \, \|(D_{Q_{k+1}} G^{-3})\pi^u_{Q_{k+1}}\| \\ &\leq 2R_k/8. \end{aligned}$$

Hence $\beta_3 R_{k+1}/4 \leq \beta_3 R_k/16 \leq 2r_{k+1}(1)$ as desired, so the induction is complete.

Thus $Z_{3k} \cap Y_{J_k} \neq \emptyset$ for all $k \geq 0$, so by nested intersection, we see that

$$P^s_G(\Lambda_G \cap W^u(p^G_0)) \cap P^u_G(\Lambda_G \cap W^s(p^G_0)) \neq \emptyset.$$

Let $a$ be a point in this intersection, and let $q = (a', a) \in D^G_T$ be the corresponding point in $D^G_T$ before projection to the plane. Then $q$ is a point of tangency between $W^s(\Lambda_G)$ and $W^u(\Lambda_G)$ as desired. $\square$

## 7. Perturbation to homoclinic tangency

In the previous section, we showed that any map $G$ near $F$ has a tangency between the stable and unstable manifolds of $\Lambda_G$, which means that the stable and unstable manifolds for the fixed point $p_0^G$ are arbitrarily close to a homoclinic tangency. In this section we take any such $G$ and perturb it to get a map with a homoclinic tangency associated to the fixed point near $(0,0)$. We do this by using a perturbation of the form $G_\mu(z,w) = G(z,w) + (0,\mu)$ for $\mu$ near 0. This has the effect of moving the stable and unstable manifolds across one another in order to create a tangency.

The relevant pieces of the stable and unstable manifolds are obtained from a sequence of graph transforms. The next few lemmas consider the behavior of these graphs with respect to the $\mu$ and $z$ variables for the map $F_\mu$ and for nearby maps $G_\mu$. They show that the graphs for $G_\mu$ are $C^2$ near those for $F_\mu$ in $\mu$ and $z$ simultaneously.

Note first that $F_\mu$ has a fixed point $p_0^F(\mu) = (0, \mu/(1-\lambda_u))$, where $\lambda_u$ is the expanding eigenvalue of $F$. For $|\mu|$ small, we see that $W^s_{\mathrm{loc}}(p_0^F(\mu))$ is given by $\mathrm{graph}_1(\phi_F^s(\mu,\cdot))$, where $\phi_F^s(\mu,z) = \mu/(1-\lambda_u)$ for $z \in S(0;1+c_0/2)$. Likewise, $W^u_{\mathrm{loc}}(p_0^F(\mu))$ is given by $\mathrm{graph}_2(\phi_F^u(\mu,\cdot))$, where $\phi_F^u(\mu,w) = 0$.

In the following lemma, we show that the functions giving the local stable and unstable manifolds for $G_\mu$ are $C^2$ near those for $F_\mu$ in the variables $(\mu,z)$ simultaneously.

Lemma 7.1. *Let $\delta_1 > 0$. There exist $r_1 > 0$ and $\delta_3 > 0$ such that if $G$ is holomorphic with $\|F - G\|_{C^2} < \delta_3$ on $\overline{K_1}$, then*

$$\mathrm{graph}_1(\phi_G^s(\mu,\cdot)) = W^s_{\mathrm{loc}}(p_0^G(\mu)), \ \ \mathrm{graph}_2(\phi_G^u(\mu,\cdot)) = W^u_{\mathrm{loc}}(p_0^G(\mu)),$$

*where $\phi_G^s$ and $\phi_G^u$ are defined and holomorphic for $(\mu,z) \in \Delta(0;r_1) \times S(0;1+c_0/2)$ with $\|\phi_G^s - \phi_F^s\|_{C^2}, \|\phi_G^u - \phi_F^u\|_{C^2} < \delta_1$.*

*Proof.* Choose $r_1$ such that $F_\mu$ is in the neighborhood of $F$ given by proposition 3.7 for each $|\mu| < 2r_1$, then choose $\delta_3$ such that any $G$ as in the statement of the current lemma is also in this neighborhood.

Working on a domain slightly larger than $\Delta(0;r_1) \times S(0;1+c_0/2)$, the function $\phi_G^s$ is found as the fixed point of a graph transform just as in a standard proof of the stable manifold theorem [14]. In the case here, this standard proof applies for each fixed $\mu$ to give $\phi_G^s(\mu,\cdot)$, and an examination of the proof of the contraction mapping theorem shows that $\phi_G^s$ can be obtained as a fixed point of a contraction also. If we restrict this contraction to functions holomorphic in $(\mu,z)$, then the resulting $\phi_G^s$ is holomorphic. Similarly, the contraction mapping theorem implies that if $G$ is $C^0$-near $F$, then the corresponding fixed point $\phi_G^s$ is $C^0$-near $\phi_F^s$. But since these latter two functions are holomorphic, this implies that they are close in $C^2$-norm on the desired domain. The same argument applies to $\phi_G^u$. $\square$

If we apply the graph transform induced by $F_\mu$ to $\phi_F^u$ restricted to some component of $K_1$, then restrict so that the new graph is defined on $S(0; 1 + c_0/2)$, we see that the resulting function is also a constant, so that the graph is vertical. We can repeat this process arbitrarily many times to get a new vertical graph. Suppose such a vertical graph intersects the neighborhood of $\{0\} \times \{a_0 + w : |w| < \rho_1\}$ in which $F$ is quadratic as in section 5. We can then obtain a piece of the unstable manifold near the tangent point $q_0$ by applying the graph transform induced by $F_\mu$ in this neighborhood. The formula for $F_\mu$ shows that this gives a piece of the unstable manifold which has the form $\mathrm{graph}_1(\psi_F^u(\mu, \cdot))$, where

$$\psi_F^u(\mu, z) = \lambda_u g(z/\lambda_s) - z + \mu + C,$$

$C$ is constant, $g$ is as in section 5 and $(\mu, z) \in \Delta(0; r_1) \times \Delta(a_0; \rho)$ for some $r_1, \rho > 0$.

Likewise, we can get a piece of the stable manifold near $q_0$ by repeatedly applying graph transforms induced by $F_\mu^{-1}$ to get a sequence of graphs parallel to the $z$-axis. When one of these graphs intersects the central component of $K_1$, we can apply the graph transform induced by $F_\mu^{-1}$ in that component, then restrict to a neighborhood of $q_0$. From the formula for $F_\mu$, this gives a piece of stable manifold of the form $\mathrm{graph}_1(\psi_F^s(\mu, \cdot))$, where

$$\psi_F^s(\mu, \cdot) = \mu/(1 - \lambda_u) + C,$$

$C$ is constant and $(\mu, z) \in \Delta(0; r_1) \times \Delta(a_0; \rho)$.

For $G$ near $F$, we can apply the same sequence of graph transforms to obtain part of the stable and unstable manifolds for $p_0^G(\mu)$. Again we show that the corresponding functions $\psi_G^u$ and $\psi_G^s$ for $G$ are $C^2$ near those for $F$.

Lemma 7.2. *Let $\delta > 0$. There exist $r_1 > 0$ and $\delta_3 > 0$ such that if $G$ is holomorphic with $\|F - G\|_{C^2} < \delta_3$ on the domain of $F$ and $\psi_F^s$ and $\psi_F^u$ are as just described, then*

$$\mathrm{graph}_1(\psi_G^s(\mu, \cdot)) \subseteq W^s(p_0^G(\mu)), \quad \mathrm{graph}_1(\psi_G^u(\mu, \cdot)) \subseteq W^u(p_0^G(\mu)),$$

*where $\psi_G^s$ and $\psi_G^u$ are defined and holomorphic for $(\mu, z) \in \Delta(0; r_1) \times S(a_0; \rho)$ with $\|\psi_G^s - \psi_F^s\|_{C^2}, \|\psi_G^u - \psi_F^u\|_{C^2} < \delta_1$.*

*Proof.* The ideas are similar to those in proposition 3.7 in that we need to control the behavior of the graphs under arbitrarily many graph transforms. However, here we must also consider the $\mu$ parameter.

Choose $r_1$ as in the previous lemma. From that lemma, we know that a piece of the unstable manifold for $G$ near $F$ has the form $\mathrm{graph}_2(\phi_G^u(\mu, \cdot))$

and that all second order partials of $\phi_G^u(\mu, w)$ are bounded by $\delta_1$ for $(\mu, w) \in \Delta(0; r_1) \times S(0; 1+c_0/2)$. As in proposition 3.7, we show that if $\delta_3$ is small, then the graph transforms induced by $G_\mu$ in $K_1$ applied to such a graph preserve these bounds.

For $G$ near $F$, write $G = (G_1, G_2)$ for the component functions of $G$, and let $M_1(\mu, w) = G_1(\phi_G^u(\mu, w), w)$, $M_2(\mu, w) = G_2(\phi_G^u(\mu, w), w) + \mu$. Write $M_j^\mu = M_j(\mu, \cdot)$. Then the graph transform induced by $G_\mu$ is

$$(G_\mu)_\#(\phi_G^u(\mu, \cdot))(w) = M_1^\mu (M_2^\mu)^{-1}(w).$$

Straightforward calculations along the lines of those in the proof of proposition 3.7 imply that if $\delta_3$ is sufficiently small, then all second order partials of $(G_\mu)_\#(\phi_G^u(\mu, \cdot))$ are bounded by $\delta_1$. The analogous results are true for graphs giving the stable manifold.

Hence, given any sequence of graph transforms using $F_\mu$ in $K_1$, we can apply the same sequence using $G_\mu$ to obtain graphs which are $C^2$ near those for $F_\mu$ in both variables. Finally, since the graph transform induced by the quadratic part of $F$ is applied only once to obtain $\psi_F^u$, we see that $\psi_G^u$ will be $C^2$ close to $\psi_F^u$ simply by making $\delta_3$ small. Hence the lemma follows. $\square$

Proposition 7.3. *Let $F$ be the holomorphic map constructed in sections 2 and 5. Then there exists a bounded set $E$ and $\delta_3 > 0$ such that if $\|F - G\|_{C^2} < \delta_3$ on the domain of $F$, then there exists a sequence $\mu_j \to 0$ such that $G_{\mu_j}$ has a point of homoclinic tangency $q_j \in E$ associated with the fixed point $p_0^G(\mu_j)$.*

*Proof.* Let $E = \Delta^2(q_0; r_0)$. Proposition 6.2 implies that if $G$ is $C^2$ near $F$, then there is a point $q = (z_0, w_0) \in E$ and leaves $\mathcal{L}_G^s(q)$ and $\mathcal{L}_G^u(q)$ in the stable and unstable foliations, respectively, which are tangent at $q$. The construction of the Cantor sets in propositions 4.3 and 4.5 imply that pieces of the stable and unstable manifold for $p_0^G$ accumulate on these leaves. These pieces are obtained by applying some sequence of graph transforms induced by $G$ in $K_1$, then applying one graph transform induced by $G$ in the neighborhood where $F$ is quadratic to part of the unstable manifold.

Together with the previous lemma, this is equivalent to saying that there is a sequence of maps $\psi_j^s, \psi_j^u : (\mu, z) \in \Delta(0; r_1) \to K_0$ such that

(a) $\mathrm{graph}_1(\psi_j^s(\mu, \cdot)) \subseteq W^s(p_0^G(\mu))$, $\quad \mathrm{graph}_1(\psi_j^u(\mu, \cdot)) \subseteq W^u(p_0^G(\mu))$,
(b) $|(\frac{\partial}{\partial \mu}\psi_j^s)(\mu, z) - (1 - \lambda_u)^{-1}| < \delta_1$, $\quad |(\frac{\partial}{\partial \mu}\psi_j^u)(\mu, z) - 1| < \delta_1$,
(c) $|(\frac{\partial}{\partial z}\psi_j^s)(\mu, z)| < \delta_1$, $\quad |(\frac{\partial}{\partial z}\psi_j^u)(\mu, z) - 2\lambda_s^2(z - a_0)| < \delta_1$,
(d) $|(\frac{\partial^2}{\partial \mu \partial z}\psi_j^s)(\mu, z)| < \delta_1$, $\quad |(\frac{\partial^2}{\partial \mu \partial z}\psi_j^u)(\mu, z)| < \delta_1$,
(e) $|(\frac{\partial^2}{\partial z^2}\psi_j^s)(\mu, z)| < \delta_1$, $\quad |(\frac{\partial^2}{\partial z^2}\psi_j^u)(\mu, z) - 2\lambda_s^2| < \delta_1$,
(f) $\lim_{j\to\infty} |\psi_j^s(0, z_0) - \psi_j^u(0, z_0)| = 0$, $\quad \lim_{j\to\infty} |(\frac{\partial}{\partial z}(\psi_j^s - \psi_j^u))(0, z_0)| = 0$,

where (c) uses the fact that $a_0 = \lambda_s f(a_0)$, and (f) follows from the fact that $\mathrm{graph}_1(\psi_j^s)$ and $\mathrm{graph}_1(\psi_j^u)$ are parts of leaves in the stable and unstable foliations, respectively, and hence converge to the corresponding pair of tangent leaves, $\mathcal{L}_G^s(q)$ and $\mathcal{L}_G^u(q)$, in a $C^1$ fashion.

Define $\Psi_j : \Delta(0; r_0) \times \Delta(a_0; \rho) \to \mathbb{C}$ by $\Psi_j(\mu, z) = \psi_j^s(\mu, z) - \psi_j^u(\mu, z)$ and $\Gamma_j : \Delta(0; r_0) \times \Delta(a_0; \rho) \to \mathbb{C}^2$ by $\Gamma_j(\mu, z) = (\Psi_j(\mu, z), (\frac{\partial}{\partial z})\Psi_j(\mu, z))$. Then $\Gamma_j(\mu, z) = (0, 0)$ precisely when $\mathrm{graph}_1(\psi_j^s(\mu, \cdot))$ and $\mathrm{graph}_1(\psi_j^u(\mu, \cdot))$ are tangent at $(z, \psi_j^s(\mu, z))$.

Moreover, $\Gamma_j(0, z_0) \to (0, 0)$ as $j \to \infty$ by (f), and by (b)–(e), we see that $D\Gamma_j$ is invertible for $\delta_1$ small depending only on $\lambda_u$, $\lambda_s$ and $\rho$. From the inverse function theorem together with a simple size estimate, it follows that there is a sequence $(\mu_j, z_j) \to (0, z_0)$ such that $\Gamma_j(\mu_j, z_j) = (0, 0)$. This implies that the pieces of the stable and unstable manifold for $G_{\mu_j}$ given by $\mathrm{graph}_1(\psi_j^s(\mu_j, \cdot))$ and $\mathrm{graph}_1(\psi_j^u(\mu_j, \cdot))$ are tangent at $q_j = (z_j, \psi_j^s(\mu_j, z_j))$, which is in $E$ for $j$ large. □

*Proof of Main Theorem.* We first demonstrate parts (a) and (b).

Choose $\delta_3 > 0$ and $E$ as in the previous proposition, and recall the definition of $F$, $f$ and $g$ from sections 2 and 5. Since the domains of definition for $f$ and $g$ are the disjoint union of finitely many simply connected sets, we can apply Runge's theorem and approximate them as closely as desired by polynomials, and in fact, we can do this uniformly in $C^2$ norm on the closures of the domains of $f$ and $g$. Using these polynomials in place of $f$ and $g$, we obtain a polynomial automorphism $G$ of some degree $d$ with $\|F - G\|_{C^2} < \delta_3/2$ on the closure of the domain of $F$.

Then for any automorphism $H$ with $\|H - G\|_{C_2} < \delta_3/2$ on the closure of the domain of $F$, the previous proposition gives a sequence of automorphisms converging to $H$ such that each of these has a homoclinic tangency contained in $E$. Moreover, if $H$ is polynomial, then each polynomial in this sequence is polynomial of the same degree as $H$.

Since $\overline{\mathrm{dom}(F)} \subseteq \mathbb{B}(0; 4)$, we can choose $\epsilon > 0$ small enough that each $H$ in $\mathcal{N} = \{H \in X : \|H - G\| < \epsilon \text{ on } \mathbb{B}(0; 4)\}$ satisfies $\|H - G\|_{C^2} < \delta_3/2$ on $\overline{\mathrm{dom}(F)}$, so that $\mathcal{N}$ is the desired neighborhood.

For case (c), the proof is the same except that first we replace $G$ by its lift to a meromorphic map of degree $d$ on $\mathbf{P}^2$, with homogeneous coordinates $[z : w : t]$. This lift is holomorphic on the set $\{t = 0\}$, which we identify with $\mathbb{C}^2$.

By standard results from algebraic geometry, there are holomorphic self-maps of $\mathbf{P}^2$ of degree $d$ which converge uniformly on compact subsets of $\{t = 0\}$ to the lift of $G$. Hence choosing $H \in \mathcal{P}_d$ near $F$ on the closure of the domain of

$F$, and using the supremum metric to define a neighborhood of $H$, we obtain the theorem. $\square$

*Proof of corollary :.* First we check the conditions necessary to apply a result of Gavosto [5] which implies that a perturbation of a homoclinic tangency leads to the creation of a sink. After that, the corollary is a standard induction.

The first condition is that the tangencies constructed earlier should be "generic:" i.e., that the order of contact is quadratic and that the stable and unstable manifolds cross at a nonzero speed under perturbation. Translating this into the notation from proposition 7.3, we have an automorphism $H$ and a family of perturbations $H_\mu(z,w) = H(z,w) + (0,\mu)$ such that $H_{\mu_j}$ has a homoclinic tangency in $E$ for some $\mu_j \to 0$, and we have corresponding maps $\Psi_j(\mu,z) = \psi_j^s(\mu,z) - \psi_j^u(\mu,z)$ with $\Psi_j(\mu_j,z_j) = 0$ and $(\partial/\partial z)\Psi_j(\mu_j,z_j) = 0$ indicating a homoclinic tangency as in proposition 7.3. Moreover, the inequalities in (e) in that proposition imply that $(\partial^2/\partial z^2)\Psi(\mu_j,z_j) \neq 0$, so that the order of contact is quadratic, and the inequalities in (b) imply that $(\partial/\partial\mu)\Psi(\mu_j,z_j) \neq 0$, so that the speed of crossing is nonzero.

The second condition is that the maps under consideration should be volume decreasing. Although the map $G$ constructed in the proof of the main theorem is volume preserving, we can compose with a linear contraction near the identity to obtain a volume decreasing automorphism with generic homoclinic tangencies as above.

Finally, in the case of noninvertible maps, the relevant parts of the stable and unstable manifolds must be smooth, which is clear from the earlier analysis of these manifolds as graphs.

With these conditions satisfied, [5, theorem 4.1] implies that if $q_j$ is the homoclinic tangency for $H_{\mu_j}$ and $\epsilon > 0$, then there exists $\nu_j$ with $|\nu_j - \mu_j| < \epsilon$ such that $H_{\nu_j}$ has an attracting periodic point contained in $\mathbf{B}(q_j;\epsilon)$. Since attracting periodic points persist under $C^2$ perturbations, $H_\mu$ will have an attracting periodic point contained in $\mathbf{B}(q_j;\epsilon)$ for all $\mu$ in some neighborhood of $\nu_j$.

From this, the existence of a dense $\mathcal{G}_\delta$ set $\mathcal{R}$ as claimed is a standard induction [9]. The idea is to show inductively that the subset $\mathcal{R}(k)$ of maps in $\mathcal{N}$ which have at least $k$ sinks contained in $E$ is open and dense in $\mathcal{N}$. This is certainly true for $k = 0$, and each $R_k$ is clearly open. Moreover, given a map in $R_k$, we can use persistent homoclinic tangencies together with Gavosto's result to make a perturbation small enough to preserve the original $k$ sinks and to create a new sink contained in $E$. Thus $R_{k+1}$ is dense in $R_k$, hence in $\mathcal{N}$. $\square$

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Department of Mathematics, Indiana University, Bloomington, IN 47405



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