HOLOMORPHIC MOTIONS AND STRUCTURAL STABILITY FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbf{C}^2$

GREGERY T. BUZZARD AND ADRIAN JENKINS

Abstract. Combining ideas from real dynamics on compact manifolds and complex dynamics in one variable, we prove the structural stability of hyperbolic polynomial automorphisms in $\mathbf{C}^2$. We consider families of hyperbolic polynomial automorphisms depending holomorphically on the parameter λ. This is done over a series of steps - given a family $\{f_\lambda\}$, where $|\lambda|$ is sufficiently small, we construct mappings defined on a neighborhood U of J_0 which conjugate f_0 and f_λ. Moreover, it is shown that J moves holomorphically. This conjugacy is then used to construct a conjugacy between f_0 and f_λ defined on a neighborhood M of $J_0^+ \cup J_0^-$. Finally, we extend such a mapping to construct a conjugacy on all of $\mathbf{C}^2$.

1. Introduction

The goal of this paper is two-fold: one the one hand, given a set of Hénon maps $\{f_\lambda\}$ depending holomorphically on the complex parameter λ, we show that there exists $\rho > 0$ so that for all $|\lambda| < \rho$, there are global homeomorphisms $\{\Phi_\lambda\}$ (also depending holomorphically on the parameter λ) so that $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$. We consider the question in steps. First, we construct a conjugacy on a compact neighborhood of a hyperbolic invariant set J. Here we make use of the work of Robbin and Robinson, explained below, and also make use of holomorphic motions.

When extending this conjugacy globally, however, the ideas on compact manifolds fail, and thus in constructing the extensions Φ_λ, we make heavy use of holomorphic motions in two complex dimensions. Thus, a secondary goal of this paper is the utilization of holomorphic motions to answer dynamics questions in two complex variables (generally, holomorphic motions have been of limited use in several variables, but they prove very convenient in this context).

In the study of dynamics of polynomial automorphisms of $\mathbf{C}^2$, there are at least two natural sources of inspiration. The first is the well-developed study of polynomial maps of the plane and rational maps of the sphere. In particular, the potential theoretic approach to questions of dynamics in one complex variable has been modified and used very successfully to study dynamics in two complex variables by Bedford and Smillie [BS], Fornaess and Sibony [FS], [FS2], Diller [D], and others. However, one of the most fruitful associations in one complex variable has been the link between dynamics and quasiconformal maps. One of the earliest connections between these two areas was in the paper of Mãné, Sad, and Sullivan, [MSS] in which they study the stability properties of rational maps using holomorphic motions, showing (modulo a few missing cases filled in by [MS]) that structural stability is dense in the space of rational maps of the sphere. In several complex variables, there is no clear link between holomorphic maps and quasiconformal maps, and the direct generalization of holomorphic motions to higher dimensions fails to have many of the nice properties of holomorphic motions in one variable, such as automatic continuity, limiting their utility in studies of dynamics in several variables.

A second source of inspiration for dynamics in $\mathbf{C}^2$ is the study of diffeomorphisms on compact manifolds. Again, influence from this work can be found in the papers of Bedford and Smillie (see again, e.g. [BS]). As a concrete example, Hubbard, Papadopol and Veselov [HPV] have constructed a compactification of $\mathbf{C}^2$ (of the form $\mathbf{C}^2 \sqcup S^3$), which is homeomorphic to a (real) 4-ball, so that a given Hénon map f extends continuously (and Buzzard has shown that the extension can actually be made to be C^∞, [B1]). Given this extension, we might hope to apply smooth results to show structural stability. However, in [HPV], it is shown that these extensions cannot be globally stable on S^3. In fact, the extensions to S^3 display two invariant solenoids, one which is attracting and the other repelling, and the relationship between these solenoids leads to a space of conjugacy invariant moduli. For more details, see Hubbard and Oberste-Vorth, [HO].

For C^2 diffeomorphisms of a compact manifold, the Newhouse phenomenon of persistent homoclinic tangencies and the Kupka-Smale theorem combine to show that stability is not dense (a result in contrast with the case of rational maps on the sphere). In the complex case, Buzzard has proven a version of the Newhouse phenomenon [B2], and shown the existence of moduli of stability for homoclinic and heteroclinic tangencies for polynomial automorphisms of $\mathbf{C}^2$ [B4]. Recently, Buzzard, Hruska and Il'yashenko have proven the Kupka-Smale theorem for polynomial automorphisms of $\mathbf{C}^2$ [BHI]. Combining these results implies that structural stability is not dense in the space of polynomial automorphisms of degree d for d sufficiently large, a result which is analogous to the situation for diffeomorphisms of a compact surface but different from the situation for rational maps of the sphere.

Returning to the history of dynamics of diffeomorphisms of a compact manifold, Robinson [R2] adapted some ideas from Robbin [Ro] to show that a C^1 diffeomorphism of a compact manifold satisfying Axiom A and the strong transversality condition is structurally stable. Robinson's approach to this problem was to use families of stable and unstable disks. To illustrate the ideas of this approach, let f be a diffeomorphism of an open set U in $\mathbf{R}^n$ such that U contains a unique basic set $\Lambda = \cap_n f^n(U)$ for f, and let g be a diffeomorphism near f. Let m be the unstable dimension of Λ. To each point x in a small neighborhood of Λ we associate a disk D^u_x of dimension m. Ideally, these disks would fit together to form a foliation extending the unstable manifold of Λ, but in general this is not known to be possible, so we do not impose this condition. However, the disks should be roughly parallel to nearby parts of the unstable manifold, should have some continuity properties, and should have the invariance property that $D^u_{fx} \subset g(D^u_x)$. I.e., we apply the map g to the disks but the map f to the base point. Analogously, we can construct stable disks D^s_x satisfying $g(D^s_x) \subset D^s_{fx}$. Because of the hyperbolic splitting of Λ, we can construct these disks so that D^s_x and D^u_x intersect in a single point $h(x)$. The invariance properties imply that $gh(x) = hf(x)$, and h can also be shown to be continuous and one-to-one, hence a conjugacy of f to g on a neighborhood of Λ. In fact, the stable and unstable disks can be chosen to move continuously with g (or in a smooth fashion with extra smoothness conditions).

In this paper, we combine the stable-unstable disk approach of Robinson with holomorphic motions in two dimensions to prove stability results for holomorphic families $\{f_\lambda\}$ of hyperbolic polynomial automorphisms of $\mathbf{C}^2$. Bedford and Smillie have shown that hyperbolic polynomial automorphisms are J-stable (i.e. if f is restricted to the (Julia) set J, defined below, f is conjugate to nearby polynomial automorphisms) ; see [BS] .We extend this result here, and produce conjugacies in neighborhoods of the Julia sets of f_λ, as follows:

Theorem 1.1. *Consider a one-parameter family $\{f_\lambda\}$ of hyperbolic polynomial automorphisms of $\mathbf{C}^2$ depending holomorphically on the parameter $\lambda \in \Delta$, and denote $f_0 = f$. Then, there are a neighborhood U of the (Julia) set $J_0 = J$, a constant $\rho > 0$ and homeomorphisms Φ_λ defined on the set U for all $|\lambda| < \rho$, such that $\Phi_\lambda(z)$ is holomorphic in λ for fixed z, $\Phi_0 = Id$ and $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$.*

Note that the theorem also implies that the family of sets J_λ move holomorphically, a result which was proven by Jonsson [J] via different methods.

Next, we use this family of mappings constructed in Theorem 1.1 to construct a conjugacy on a neighborhood of the set $J^+ \cup J^-$, where J^+ (resp. J^-) is the boundary of points which have bounded forward orbits (resp. bounded backwards orbits). This improves a result obtained in [BV], in which a conjugacy was constructed on the set $J^+ \cup J^-$ by working on leaves in $J^\pm$ and using the canonical Bers-Royden extension of holomorphic motions in the plane [BR] to extend the motion of J given by Jonsson [J].

Theorem 1.2. *Let f_λ be a one-parameter family of hyperbolic polynomial automorphisms of $\mathbf{C}^2$ depending holomorphically on $\lambda \in \Delta$. Then there are a neighborhood M of $J^+ \cup J^-$, $\rho > 0$, and homeomorphisms Φ_λ defined on M for $|\lambda| < \rho$ such that $\Phi_\lambda(z)$ is holomorphic in λ for each fixed z, $\Phi_0 = Id$, and $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$. Moreover, Φ_λ is C^∞ on any open set not intersecting $J_0^+ \cup J_0^-$.*

Finally, we establish a global structural stability theorem for polynomial automorphisms of $\mathbf{C}^2$ of fixed degree d. For the proof of theorem 1.1, we use ideas similar to those in Robinson's proof. However, his proof does not apply directly to the entire space $\mathbf{C}^2$ (since it is not compact). Instead, we combine these techniques with ideas of [B1] and [BV] to get a global conjugacy.

Theorem 1.3. *Let $\{f_\lambda\}$ be a one-parameter family of hyperbolic polynomial automorphisms of $\mathbf{C}^2$ depending holomorphically on $\lambda \in \Delta$. Let the homeomorphisms Φ_λ defined on M satisfy the conclusions of the previous theorem. Then there exists an extension (which we also call) $\Phi_\lambda : \mathbf{C}^2 \to \mathbf{C}^2$ which is a homeomorphism for each fixed λ such that $\Phi_\lambda(z)$ is holomorphic in λ for each fixed z, $\Phi_0 = Id$, and $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$ on $\mathbf{C}^2$.*

There are a couple of interesting questions we do not address in this work. As mentioned in the introduction, Buzzard and Verma [BV] have constructed conjugacies between the mappings f_λ and f_0 on the set $J_0^+ \cup J_0^-$. Can the conjugacy of Buzzard and Verma be extended to all of $\mathbf{C}^2$? We do not address this issue here.

We also can ask the deeper question: is hyperbolicity equivalent to structural stability? This question is closely related to the following result on compact manifolds M: a diffeomorphism f is structurally stable if and only if f satisfies Axiom A and the strong transversality condition (for more information on these two concepts, please see [R1]). As mentioned, the reverse direction has been proven through the work of Robbin (for C^2 diffeomorphisms, in the C^1-topology, [Ro]) and Robinson (for the general case of C^1 diffeomorphisms, [R2]), while the forward direction was proven by Mañé [M]. This question is analogous to the question of the density of hyperbolicity for quadratic polynomials in one variable.

The paper is organized as follows: in section 2, we give a brief recollection of some important results to which we shall refer regarding hyperbolic polynomial automorphisms and holomorphic motions. section 3 develops the machinery of the d_f metric and the graph transform, which are used to prove theorem 1.1. Finally, in section 4, we prove theorems 1.2 and 1.3.

This research of the first author was supported in part by a National Science Foundation Postdoctoral Fellowship. The first author is grateful for the hospitality of the Université Paul Sabatier, Toulouse for its hospitality during parts of the preparation of this paper. The research of the second author was conducted while a Research Assistant Professor at Purdue University; the author is thankful for the support of the department.

2. Hyperbolic Polynomial Automorphisms of $\mathbf{C}^2$

We recall in this section some relevant results on hyperbolic polynomial automorphisms.

Following Bedford and Smillie [BS], given a polynomial diffeomorphism F of $\mathbf{C}^2$, we define the set K^+ (resp. K^-) to be those points p with bounded forward (resp. backward) orbits $F^n(p)$ (resp. $F^{-n}(p)$), $n \in \mathbf{N}$. We define the sets $J^+ = \partial K^+$ and $J^- = \partial K^-$. Finally, we define the (Julia) set $J = J^+ \cap J^-$. We say that a polynomial diffeomorphism f of $\mathbf{C}^2$ is hyperbolic if the set J is a hyperbolic set for f. Recall that a set S is a hyperbolic set (or has a hyperbolic structure) if over each point $p \in S$, the tangent space $T_p\mathbf{C}^2$ splits as the direct sum $A^s_p \oplus A^u_p$, so that the splitting is invariant in the sense that $Df_p(A^{s/u}_p) = A^{s/u}_p$, the splitting varies continuously with the point p and so that there is are constants $C > 0$ and $0 < \lambda < 1$, independent of the point $p \in S$, so that $|Df^n_p v^s| \leq C\lambda^n v^s$ for $v^s \in A^s_p$ and $Df^{-n}_p v^u| \leq C\lambda^n v^u$ for $v^u \in A^u_p$. It is often convenient to change the metric so that such contraction occurs after a *single* iteration, that is, so that given any vector $v^s \in A^s_p$, we have that $|Df^n_p v^s| \leq \lambda^n |\mathbf{v}^s|$, and similarly for $v^u \in A^u_p$ with the roles of f and f^{-1} interchanged. This can always be achieved; see e.g. Robinson [R1]. We will call such a metric an adapted metric.

Our interest here is in non-elementary polynomial automorphisms (those automorphisms with interesting dynamics) with dynamical degree 2 or greater. Elementary polynomial automorphism have very simple dynamics; indeed, Friedland and Milnor [FM] have shown that any such automorphism is polynomially conjugate to an automorphism of the form $(z, w) \to (ax + p(y), cy + d)$. On the other hand, any non-elementary polynomial automorphism of $\mathbf{C}^2$ is polynomially conjugate to a generalized Hénon map, that is, a composition of maps of the form $g(z, w) = (w, p(w) - az)$, where p is a polynomial of degree at least 2 and $a \neq 0$ (we will refer to generalized Hénon maps as simply Hénon maps - this will yield no confusion). Moreover, if $L = \{f_\lambda\}$ is a family of hyperbolic polynomial automorphisms varying holomorphically with the parameter λ, then this family is conjugate to a family of hyperbolic Hénon maps H varying holomorphically with the parameter λ, and the family of conjugating homeomorphisms also varies holomorphically as well. Thus, for most of the paper, we will restrict our attention to the case of Hénon maps.

Given a Hénon map f_0 we say that f_0 is structurally stable if for any nearby mapping f_λ in the C^1-topology, there is a homeomorphism $h_\lambda : \mathbf{C}^2 \to \mathbf{C}^2$ satisfying $f_\lambda h_\lambda = h_\lambda f_0$, for $0 < |\lambda| << 1$. It is convenient that the set $\{h_\lambda\}$ also depend holomorphically on the parameter λ; we shall discuss this in the next paragraph.

Much of the work presented here relies on the idea of a holomorphic motion. This is defined, for a set $E \subseteq \mathbf{C}^n$ and $\mathbb{D} \subseteq \mathbf{C}$ the unit disc, as a mapping $f = f(\lambda, z) : \mathbb{D} \times E \to \mathbf{C}^n$ for which $f(0, z) = z$ for all $z \in E$, $f_\lambda(z) = f(\lambda, z)$ is injective for each fixed λ, and $f(., z)$ is holomorphic for each fixed z. In the case $n = 1$ (in which case we may assume that $E \subseteq \mathbb{P}^1$), this definition forces very strong conditions on each of the maps f_λ. For example, they are automatically continuous (uniformly in z) and each map has an extension to $\overline{E}$ (the so-called λ-lemma of Mané, Sad and Sullivan; see [MSS]). In fact, Bers and Royden [BR] have proved that given a holomorphic motion $f : \mathbb{D} \times E \to \mathbb{P}^1$, this motion admits a canonical extension $\tilde{f} : D(0, \frac{1}{3}) \times \mathbb{P}^1 \to \mathbb{P}^1$. This extension is characterized by a harmonic Beltrami coefficient; that is, given that $\mu(\lambda, z)$ is the Beltrami coefficient of $z \to f(\lambda, z)$, and S is any component of $\mathbb{P}^1 \backslash \hat{E}$, where $\hat{E}$ is the closure of E in $\mathbb{P}^1$, then $\mu(\lambda, z) = (\rho_S(z))^{-2}\overline{\psi(\lambda, z)}$ for $z \in S$, $\lambda \in D(0, 1)$, Here $\rho_S(z)|dz|$ is the hyperbolic (or Poincaré) metric in S, and ψ is holomorphic in z and antiholomorphic in λ.

Unfortunately, in the case $n \geq 2$, many of these facts fail to hold. For example, automatic continuity cannot be achieved, and canonical extensions are generally not possible. However, under certain conditions on the motion, continuity can be achieved, by appealing to the Bers-Royden extension in one complex variable. This follows from a result of Buzzard and Verma [BV], which is quite general. We will give definitions and a precise statement of the theorem in Section 4; for now we content ourselves with a description of the process. We consider holomorphic motions k_i, defined on one-dimensional (complex) subsets $D_i \in \mathbf{C}^2$ (here, $i \in I$ for some index set I). The sets and the motions must converge, in a sense to be described later, to limits which we call k_∞ and D_∞. Let $\tilde{k}_i$ denote the Bers-Royden extension of k_i to the set $S_i \supseteq D_i$. The point of the result is that the extensions $\tilde{k}_i$ will also converge to the limit $\tilde{k}_\infty$, which is the Bers-Royden extension of k_∞ on the set $S_\infty \supseteq D_\infty$ (see Figure 1). Thus, if a holomorphic motion K, defined on an open set $D \in \mathbf{C}^2$ can be decomposed into motions k_i defined on a decomposition $D = \cup D_i$ as above, the resulting Bers-Royden extensions will yield a continuous holomorphic motion on a larger set $S \supseteq D$. We make crucial use of these ideas in Section 4.

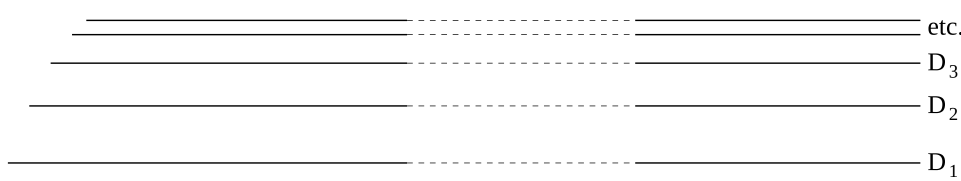

FIGURE 1. The motions k_i defined on the one (complex) dimensional sets D_i (represented by the solid lines) can be extended along each of the dotted lines. The resulting motions $\hat{k}_i$ will converge to a limit motion $\hat{k}_\infty$, which is the extension of the limiting motion k_∞ defined on D_∞.

3. The d_f Metric and Graph Transform

In this section we develop the tools necessary to construct conjugacies in a neighborhood of J for a hyperbolic Hénon map f (Theorem 1.1). In particular, we define a metric which

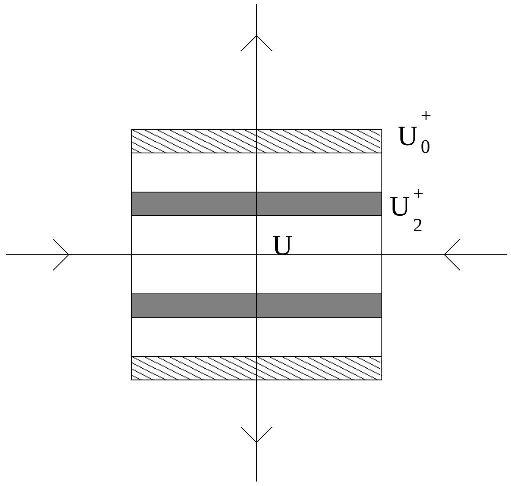

FIGURE 2. The neighborhood U and the sets U_0^+ and U_2^+.

is invariant under the map f, and define the graph transform. We then use the techniques of Robinson to demonstrate stability of f in a neighborhood of J.

We suppose that $\{f_\lambda\}$ is a family of hyperbolic Hénon maps, depending holomorphically on the parameter λ (where say, $|\lambda| < r$), and we define $J_\lambda = J(f_\lambda)$ to be the Julia set of f_λ.

We denote f_0 by f, and we denote its Julia set J_0 by J. We consider an adapted metric for f in a neighborhood of J. Also, we define stable and unstable manifolds of size r associated to f as in [HP, Theorem 3.2].

Given a neighborhood, U, of J, and an integer $m \geq 0$, we define sets of points that stay in U for a given number of iterates as

$$U_m^+ = \{p : f^j(p) \in U \text{ for } 0 \leq j \leq m \text{ and } f^{m+1}(p) \notin U\}. \tag{3.1}$$

Also, define $U_{-1}^+ = \emptyset$ and $U_\infty^+ = \{p : f^j(p) \in U, j \geq 0\}$. Analogously, we define U_j^-, replacing iterates of f by iterates of f^{-1}. Then U is the disjoint union of U_m^+ over $m \geq 0$, $m = \infty$ and the disjoint union of U_m^- over $m \geq 0$, $m = \infty$. Also, $U_\infty^+ \subset J^+$ and $U_\infty^- \subset J^-$.

As usual, we define the distance d between two sets V and W as $d(V, W) = \inf |v - w|$, where $v \in V$ and $w \in W$. For a well-chosen neighborhood U of J, the distance between the sets U_0^+ and U_2^+ has a fixed lower bound, as is seen in the following lemma (see Figure 2):

Lemma 3.1. *Given a neighborhood V of J, there exists a neighborhood U of J and $\delta_0 > 0$ such that $\overline{U} \subseteq V$ and $d(U_0^+, U_2^+) > \delta_0$ and $d(U_0^-, U_2^-) > \delta_0$.*

Proof. Let $W_r^u(J) = \cup_{p\in J} W_r^u(p)$, and likewise for $W_r^s(J)$. For sufficiently small r, the statement is true for the set $U' = W_r^u(J) \cup W_r^s(J)$ by the definition of $W^s(J)$ and the hyperbolicity of f on J. The statement is true for all sufficiently small neighborhoods of U' by uniform continuity of f near J. $\square$

Now, we define a version of Robinson's d_f metric for use in a neighborhood of a hyperbolic set (Robinson's definition of this metric for use on compact manifolds does not satisfy the triangle inequality in our setting). The idea here is to use iteration of the mapping f to separate points as much as possible. However, because we restrict ourselves to a compact neighborhood U of J, some technicalities arise.

Given U, δ_0 as in Lemma 3.1, we choose $0 < \delta < \delta_0$. For a nonnegative integer n and $p, q \in U_n^{\pm} \cup U_{n-1}^{\pm}$, define

$$d_n^{\pm}(p,q) = \min\{\delta, \sup\{|f^{\pm j}(p) - f^{\pm j}(q)| : 0 \leq j \leq n\}\}.$$

For $p, q \in U_\infty^{\pm}$, define

$$d_\infty^{\pm}(p,q) = \min\{\delta, \sup\{|f^{\pm j}(p) - f^{\pm j}(q) : 0 \leq j\}\}.$$

For $p, q \in U$, define

$$d_f^{\pm}(p,q) = \begin{cases} d^{\pm}_{\max\{n,m\}}(p,q) & \text{if } p \in U_n^{\pm}, q \in U_m^{\pm}, n, m \in \mathbf{N}, |n-m| \leq 1 \\ d_\infty^{\pm}(p,q) & \text{if } p, q \in U_\infty^{\pm} \\ \delta & \text{otherwise.} \end{cases}$$

For $p, q \in U$, define $d_f(p,q) = \max\{d_f^+(p,q), d_f^-(p,q)\}$.

The following proposition shows that d_f is indeed a metric, and that this metric can be used to give a lower bound on the maximum distance between iterates within the set U. For the remainder of the paper, we denote $L(f)$ to be the Lipschitz constant of the mapping f.

Proposition 3.2. *d_f is a metric on U and $d_f(p,q) = d_f(f(p), f(q))$ whenever p, q, $f(p)$, $f(q)$ are all in U. If $L_0 > \max\{L(f|U), L(f^{-1}|U)\}$, then for all $p, q \in U$, there is an integer n so that $f^j(p), f^j(q) \in U$ for j between 0 and n, inclusive, and so that $|f^n(p) - f^n(q)| \geq d_f(p,q)/L_0$.*

Proof. We show first that d_f^+ is a metric. For each n a nonnegative integer or ∞, d_n^+ is the supremum of metrics, hence is itself a metric. Hence $d_f^+(p,q) \geq 0$ with $d_f^+(p,q) = 0$ if and only if $p = q$. Also, $d_f^+(p,q) = d_f^+(q,p)$. For the triangle inequality, let $p \in U_m$, $q \in U_n$, and $r \in U_k$. If $|k-n| > 1$ then $d_f^+(q,r) = \delta$, so $d_f^+(p,q) \leq d_f^+(p,r) + d_f^+(r,q)$. Likewise if $|k-m| > 1$. Thus we may assume $|k-n| \leq 1$ and $|k-m| \leq 1$.

If $n, m < \infty$ and $|n-m| \leq 1$, then without loss we may assume $n \geq m$, and hence $d_f^+(p,q) = d_n^+(p,q)$. If $k = n$, then $d_f^+(p,r) = d_n^+(p,r)$ and $d_f^+(r,q) = d_n^+(r,q)$. Since d_n^+ is a metric, we have $d_f^+(p,q) \leq d_f^+(p,r) + d_f^+(r,q)$. If $k = m$, then $d_f^+(p,r) = d_n^+(p,r)$ and $d_f^+(r,q) = d_m^+(r,q) \leq d_n^+(r,q)$, so again the triangle inequality holds since d_n^+ is a metric.

If $n, m < \infty$ and $|n-m| > 1$, then $d_f^+(p,q) = \delta$. However, in this case, there are two possibilities. First, it may be that one of $|n-k|$ or $|m-k|$ is also larger than 1, in which case the corresponding distance is δ and hence the triangle inequality holds. Otherwise, both $|n-k|$ or $|m-k|$ equal 1, and hence $|n-m| = 2$. Thus we may assume $n = m+2$ and $k = m+1$. Then $f^m(q) \in U_0^+$, $f^m(r) \in U_1^+$, and $f^m(p) \in U_2^+$. Hence from lemma 3.1,

$$\begin{aligned} d_f^+(p,r) + d_f^+(r,q) &\geq d(f^m(p), f^m(r)) + d(f^m(r), f^m(q)) \\ &\geq d(f^m(p), f^m(q)) \\ &> \delta. \end{aligned}$$

Thus $d_f^+(p,q) \leq d_f^+(p,r) + d_f^+(r,q)$.

If exactly one of n or m is ∞, then $d^+_f(p,q) = \delta$, and r must have d^+_f-distance δ to at least one of p or q since either k is ∞ or it is not. If both n and m are ∞, then either k is not ∞ and r has distance δ to both p and q, or $k = \infty$ and

$$d^+_f(p,q) = d^+_\infty(p,q) \leq d^+_\infty(p,r) + d^+_\infty(r,q) = d^+_f(p,r) + d^+_f(r,q).$$

Thus d^+_f is a metric on U. Likewise, d^-_f is a metric on U. Hence d_f is a metric on U.

The invariance of d_f under f is clear from the definition of d_f. The existence of n with the stated properties follows from the fact that $d_f(p,q)$ is essentially obtained by taking the sup of $|f^k(p) - f^k(q)|$ over all j so that $f^k(p)$ and $f^k(q)$ are contained in $U \cup f(U) \cup f^{-1}(U)$. The factor of L_0 is needed to give the bound when $f^k(p)$ and $f^k(q)$ are not contained in U. □

Here and throughout, we will denote the standard Euclidean metric as $|\cdot|$, while reserving $||\cdot||$ for a metric defined on $T\mathbf{C}^2$, in hopes of avoiding confusion.

Now that we have constructed the metric for use in our results, we consider another question. As stated in Section 2, one can find a continuous splitting $A^s \oplus A^u$ over the hyperbolic set J which is invariant in the sense given there. We now wish to extend this splitting to a splitting over the neighborhood U of J. The splitting should be smooth, and "nearly invariant," with respect to the derivative maps Df_λ and Df_λ^{-1}.

To be more precise, let π denote the projection map from $A^s \oplus A^u$ to the corresponding base point in $\mathbf{C}^2$, and let π^u (resp. π^s) denote projection from $A^s \oplus A^u$ to A^s (resp. A^u). Then we have the following proposition:

Proposition 3.3. *Given $\epsilon^{su} > 0$, there exists a C^∞ splitting, $E^s \oplus E^u$, over a neighborhood, U, of J, a constant $0 < \tau < 1$, a continuous metric, $\|\cdot\|$, on $T\mathbf{C}^2|U$, and $\rho > 0$ so that for all $n \geq 1$ and all $|\lambda| < \rho$,*

$$\max\{\|(\pi^s Df^n_\lambda)|E^s\|, \|(\pi^u Df^{-n}_\lambda)|E^u\|\} < \tau^n$$

and

$$\max\{\|(\pi^u Df_\lambda)|E^s\|, \|(\pi^s Df_\lambda)^{-1})|E^u\|\} < \epsilon^{su},$$
$$\max\{\|(\pi^s Df_\lambda)|E^u\|, \|(\pi^u Df^{-1}_\lambda)|E^s\|\} < \epsilon^{su}.$$

Moreover, this metric is invariant under multiplication by i in the natural complex structure of E^u_p and E^s_p.

Proof. The first bound follows from hyperbolicity. The last two are proved in [R2]. We recall the sketch of both here. Since $f = f_0$ is hyperbolic on J, there is a continuous invariant (complex) splitting on J and a continuous adapted metric on $\mathbf{C}^2$, and this metric can be chosen to respect the complex structure on A^s and A^u, i.e., both the estimates given above hold on J with 0 in place of ϵ^{su}. This splitting extends to a continuous splitting in a neighborhood, U, of J. By restricting to a sufficiently small neighborhood of J, the two estimates above hold with $\epsilon^{su}/3$ in place of ϵ^{su}. By approximating this splitting sufficiently closely with a differentiable splitting and possibly increasing τ, the estimates hold with $\epsilon^{su}/2$ in place of ϵ^{su}. Finally, since f_λ depends holomorphically on λ, Theorem 7.1 of [HP] implies that there exists $\rho > 0$ so that if $|\lambda| < \rho$, then f_λ is hyperbolic on J_λ and that the first estimate applies to f_λ (again possibly increasing τ). The remaining estimates follow from continuity. $\square$

Without loss of generality, we may assume that if $v \in E^s_p \oplus E^u_p$, then $\|v\| = \max\{\|\pi^s_p v\|, \|\pi^u_p v\|\}$. Given $r > 0$, let $E^u(r)$ (resp. $E^s(r)$) denote the disk bundle of radius r in E^u (resp. E^s)), that is, the vectors $v \in E^u$ with $|v| \le r$. We now give some definitions.

Definition 3.4. *Fix $p \in U$, and define a linear map $T_{p,\lambda} : E^s_p \oplus E^u_p \to E^s_{f(p)} \oplus E^u_{f(p)}$ by*

$$T_{p,\lambda}(v) = \pi^s_{f(p)}(D_p f_\lambda)\pi^s_p v + \pi^u_{f(p)}(D_p f_\lambda)\pi^u_p v.$$

Define $T^{u/s}_{p,\lambda} = \pi^{u/s} T_{p,\lambda}$.

Note that the mapping $T_{p,\lambda}$ is diagonal on the tangent space $T_p(U)$ with respect to the splitting $T_p(U) = E^s_p \oplus E^u_p$. Moreover, proposition 3.6 below shows that this mapping approximates $D_p f - \lambda$.

Since the tangent space of $\mathbf{C}^2$ at a point p is naturally equivalent to $\mathbf{C}^2$, we can regard the exponential map at p to be translation by p. That is, $\exp_p : T_p\mathbf{C}^2 \to \mathbf{C}^2$ is given by $\exp_p(v) = p + v$. With this, we make the following definition, which corresponds to applying $f = f_0$ to the base point and f_λ to a stable or unstable disk.

Definition 3.5. *Define*

$$f_{p,\lambda} = \exp^{-1}_{f(p)} \circ f_\lambda \circ \exp_p$$

and

$$f^{u/s}_{p,\lambda} = \pi^{u/s}_{f(p)} \circ f_{p,\lambda}.$$

Proposition 3.6. *Given f_λ and U as above, there exists $\rho > 0$ and $C > 0$ such that if $|\lambda| < \rho$, then the Lipschitz constant of $f_{p,\lambda} - T_{p,\lambda}$ on $E^s_p(r) \times E^u_p(r)$, satisfies*

$$L(f_{p,\lambda} - T_{p,\lambda}) \le \epsilon^{su} + Cr$$

and so that if $v \in E^s(r) \times E^u(r)$, then

$$\|f_{p,\lambda}(v) - T_{p,\lambda}(v)\| \le \|f_\lambda(p) - f_0(p)\| + (\epsilon^{su} + Cr)\|v\|.$$

Proof. Choose ρ as in Proposition 3.3. The definition of $T_{p,\lambda}$ and the Taylor expansion of $f_{p,\lambda}$ imply that if $v \in E^s(r) \times E^u(r)$, then

$$(f_{p,\lambda} - T_{p,\lambda})(v) = f_\lambda(p) - f_0(p) + (D_p f_\lambda - T_{p,\lambda})(v) + H(v), \tag{3.2}$$

where $\|H(v)\| \le C\|v\|^2$ for some $C > 0$ depending only on r and ρ. The last estimate of the proposition follows immediately from this using Proposition 3.2 together with the definition of $T_{p,\lambda}$. The Taylor expansion of H implies that if $\|v_1\|, \|v_2\| < r$, then $\|H(v_1) - H(v_2)\| \le Cr\|v_1 - v_2\|$ for some possibly larger C, still depending only on r and ρ. Let $G_{p,\lambda} = f_{p,\lambda} - T_{p,\lambda}$. Then (3.2) and the estimate on H imply that

$$\|G_p(v_1) - G_p(v_2)\| \le \|(D_p f_\lambda - T_{p,\lambda})(v_1 - v_2)\| + Cr\|v_1 - v_2\|.$$

Hence by Proposition 3.3, $L(f_{p,\lambda} - T_{p,\lambda}) \le \epsilon^{su} + Cr$. $\square$

We now consider the graph transform, and construct the semi-invariant disk families for use in constructing the conjugacy. For future reference, let $L(f_\lambda)$ denote the Lipschitz constant of the mapping f_λ under the standard Euclidean norm, and let $L_f(f_\lambda)$ denote the Lipschitz constant with respect to the d_f metric. Write $D_j f$ as the derivative of f with respect to the j^{th} variable.

Definition 3.7. *Given $r, \rho > 0$, $0 < r_0 < r$, and $0 < \tau < 1$, let $\mathcal{D}^u = \mathcal{D}^u(r, r_0, \rho, \tau)$ be the family of continuous maps $\Delta : E^u(r) \times \mathbb{D}(\rho) \to E^s(r)$ such that the following hold:*

1. *For each fixed $p \in U$, Δ maps $E^u_p(r) \times \mathbb{D}(\rho)$ to $E^s_p(r)$, and is holomorphic on this set.*
2. *Δ is smooth on $E^u_p(r)|V \times \mathbb{D}(\rho)$ for each V disjoint from J^-, and*
3. *$\Delta_{p,\lambda} = \Delta|E^u_p(r) \times \{\lambda\}$ satisfies $L(\Delta_{p,\lambda}) \le \tau$ and $\|\Delta_{p,\lambda}(0)\| \le r_0$ for each $p \in U$, $\lambda \in \mathbb{D}(\rho)$.*

The topology on $\mathcal{D}^u$ is that induced by the supremum norm.

Let $\hat{\Delta}_{p,\lambda}$ denote the graph of $\Delta : E^u_p(r) \times \{\lambda\} \to E^s_p(r)$, viewed as a subset of $\mathbf{C}^2$. That is, the graph in $T_p\mathbf{C}^2$ is considered as a subset of $\mathbf{C}^2$ via the exponential map, $\exp_p$. We define a norm on such maps via the sup norm over $E^u(r) \times \mathbb{D}(\rho)$.

Later, in proposition 3.13, we will construct a semi-invariant Δ. Once this is done, the graph $\hat{\Delta}_{p,\lambda}$ should be viewed as a local unstable manifold for f_λ that moves holomorphically as λ varies (and hence as f_λ varies). In general, this graph will not contain the point p except when $\lambda = 0$. If $p = p_0$ is in J and p_λ is the corresponding point in J_λ, then the graph $\hat{\Delta}_{p,\lambda}$ will pass through the point p_λ. We will construct the conjugacy from f to f_λ near J by mapping p to the intersection of the corresponding stable and unstable disks $\hat{\Delta}_{p,\lambda}$ and $\hat{\Delta}^s_{p,\lambda}$. In order for this procedure to work correctly, we need to be able to make these disks semi-invariant under f_λ.

To construct an appropriate family of stable and unstable disks near J, we use the same outline as that used by Robinson. That is, we first construct the unstable disks in a neighborhood of a fundamental domain of the form $\overline{U \setminus f(U)}$, where U is a small neighborhood of J, then use a graph transform and contraction mapping argument to extend to U.

We define the graph transform on these holomorphic motions as follows: we apply f_λ to the disk and f to the base point. That is, $((f^\#_\lambda)(\Delta))_{p,\lambda}$ is obtained by taking $\hat{\Delta}_{f^{-1}(p),\lambda}$, applying f_λ to this graph, then expressing the image as a graph over $E^u_p(r)$ (See Figure 3).

Definition 3.8. *Given $\Delta \in \mathcal{D}^u$ and $p \in f(U)$, define $((f^\#_\lambda)(\Delta))_{p,\lambda}$ to be the function $\Delta^\# : E^u_p(r) \to E^s_p(r)$ such that the image under $\exp_p$ of the graph of $\Delta^\#$ is equal to $f_\lambda(\hat{\Delta}_{f^{-1}(p),\lambda})$.*

Definition 3.9. *Let $U_f = \overline{U_0^- \cup U_1^-}$, fix a map $\Sigma \in \mathcal{D}^u$ that is invariant under $f^\#$ when restricted to U_f. I.e., if $p, f^{-1}(p) \in U_f$, then $\Sigma^\#_{p,\lambda} = \Sigma_{p,\lambda}$. Let $\mathcal{D}^u_U$ denote the maps in $\mathcal{D}^u$ that equal Σ when restricted to U_f.*

A priori, it is not clear that $\mathcal{D}^u_U$ is nonempty; we shall prove this below (proposition 3.12). Assuming this for the moment, we first define the graph transform $f^\# : \mathcal{D}^u_U \to \mathcal{D}^u_U$ by using

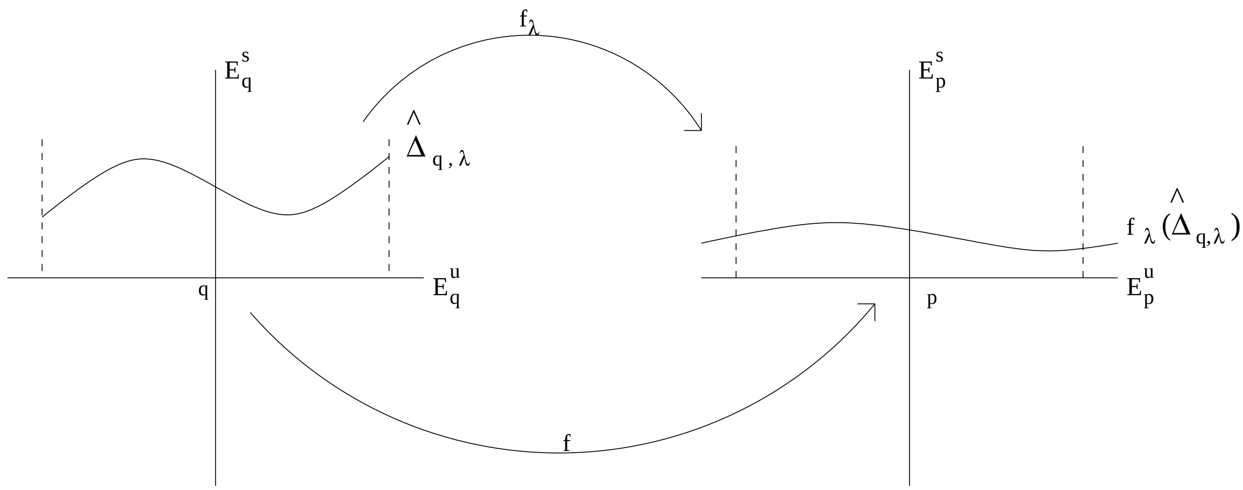

FIGURE 3. The graph transform illustrates the idea, alluded to in the introduction, of "applying f_λ to the graph, but f to the base point". Here, $q = f^{-1}(p)$, and the dotted lines denote the set $E^u_q(r)$.

the graph transform as above to define $\Delta^{\#}_{p,\lambda}$ for $p \in f(U)$, then defining $\Delta^{\#}_{p,\lambda} = \Delta_{p,\lambda}$ for $p \in U_f$. We now show that this graph transform is a contraction on $\mathcal{D}^u_U$.

Proposition 3.10. *For ϵ^{su} and r sufficiently small, $r_0 < r$ sufficiently small, and some $\rho > 0$, the graph transform $f^{\#}_\lambda$ is well-defined and a contraction on $\mathcal{D}^u_U$. Hence there is a unique fixed point.*

Proof. We use a slight modification of the ideas of Hirsch and Pugh. Choose $\epsilon \in (0, (1-\tau)/3(1+\tau))$. Then $\tau + 3\epsilon < 1$, $\tau^{-1} - \epsilon > 1$, and $(\tau+\epsilon)/(1-\tau\epsilon) < 1$. Using Proposition 3.6, choose U, ϵ^{su}, the splitting, r, and ρ small enough that $\epsilon^{su} + Cr < \epsilon$. Choose $r_0 < r$ small enough that $r_0 + r\tau < r$ and $r(\tau^{-1} - \epsilon) - 2\epsilon r_0 > r$. If needed, decrease ρ so that $\|f_\lambda(p) - f_0(p)\| < \epsilon r_0$ for all $p \in U$.

Form $\mathcal{D}^u_U$ with these constants, and let $\Delta \in \mathcal{D}^u_U$ and $p \in U$. Then $\Delta_{p,\lambda} : E^u_p(r) \times \mathbb{D}(\rho) \to E^s_p(r)$.

Define $\Gamma_{p,\lambda}(v^u) = \Delta_{p,\lambda}(v^u) + v^u$. Since $L(\Delta_{p,\lambda}) \le 1$ and the norm on E_p is the max of the norms on E^s_p and E^u_p, we see that $L(\Gamma_{p,\lambda}) = 1$. Consider the components of the graph transform of Δ:

$$\Psi_{p,\lambda}(v^u) = f^u_{p,\lambda}\Gamma_{p,\lambda}(v^u),$$

and

$$\Phi_{p,\lambda}(v^u) = f^s_{p,\lambda}\Gamma_{p,\lambda}(v^u).$$

Then $\Psi_{p,\lambda} : E^u_p(r) \to E^u_{f(p)}$ and $\Phi_{p,\lambda} : E^u_p(r) \to E^s_{f(p)}$. We need to show that $\Delta^{\#}_{p,\lambda} = \Phi_{f^{-1}(p),\lambda}\Psi^{-1}_{f^{-1}(p),\lambda}$ is well-defined.

Since $T_{p,\lambda}$ is diagonal with respect to the splitting, $T^u_{p,\lambda}\Gamma_{p,\lambda} - T^u_{p,\lambda}$ is identically 0. Hence using Proposition 3.6 plus properties of Lipschitz constants as in Proposition 1.3 of [HP], we have on $E^u(r)$ that

$$\begin{aligned} L(\Psi_{p,\lambda} - T^u_{p,\lambda}) &\leq L(f^u_{p,\lambda}\Gamma_{p,\lambda} - T^u_{p,\lambda}\Gamma_{p,\lambda}) + L(T^u_{p,\lambda}\Gamma_{p,\lambda} - T^u_{p,\lambda}) \\ &\leq L(f_{p,\lambda} - T_{p,\lambda})L(\Gamma_{p,\lambda}) \\ &\leq \epsilon^{su} + Cr < \epsilon. \end{aligned} \tag{3.3}$$

Hence by the Lipschitz inverse function theorem of [HP],

$$L((\Psi_{p,\lambda})^{-1}) \leq [L((T^u_{p,\lambda})^{-1})^{-1} - L(\Psi_{p,\lambda} - T^u_{p,\lambda})]^{-1} \leq (\tau^{-1} - \epsilon)^{-1}.$$

By the size estimate of [HP], we have

$$\Psi_{p,\lambda}(E^u_p(r)) \supset E^u_{f(p)}(r(\tau^{-1} - \epsilon)) + \Psi_{p,\lambda}(0). \tag{3.4}$$

However, $\Psi_{p,\lambda}(0) = f^u_{p,\lambda}\Delta_{p,\lambda}(0) \in f^u_{p,\lambda}(E^s_p(r_0))$. Since $T^u_{p,\lambda}(v) = 0$ for $v \in E^s_p(r_0)$, we have by Proposition 3.3 that

$$\begin{aligned} \|f^u_{p,\lambda}(v)\| &\leq \|f^u_{p,\lambda}(v) - T^u_{p,\lambda}(v)\| \\ &\leq \|f_\lambda(p) - f_0(p)\| + (\epsilon^{su} + Cr)\|v\| \\ &\leq 2\epsilon r_0. \end{aligned} \tag{3.5}$$

Since $r(\tau^{-1} - \epsilon) - 2\epsilon r_0 > r$, we have

$$\Psi_{p,\lambda}(E^u_p(r)) \supset E^u_{f(p)}(r).$$

Thus, $\Psi^{-1}_{p,\lambda} : E^u_{f(p)}(r) \to E^u_p(r)$ is well-defined with Lipschitz constant at most $(\tau^{-1} - \epsilon)^{-1} < 1$.

Thus, writing $\Delta^\#_{p,\lambda} = (f^\#_\lambda(\Delta))_{p,\lambda}$, we define

$$\Delta^\#_{p,\lambda} = \begin{cases} \Delta_{p,\lambda} & \text{if } p \in U_f \\ \Phi_{f^{-1}(p),\lambda}\Psi^{-1}_{f^{-1}(p),\lambda} & \text{if } p \in U \setminus U^-_0. \end{cases}$$

The invariance of Δ on U_f and the estimates given above imply that $\Delta^\#_{p,\lambda}$ is well-defined as a map from $E^u_p(r)$ to $E^s_p(r)$. Also, since $\Phi_{p,\lambda} = f^s_{p,\lambda}\Gamma_{p,\lambda}$ and $\Gamma_{p,\lambda}$ has Lipschitz constant 1, we have

$$L(\Phi_{p,\lambda}) \leq L(T^s_{p,\lambda}) + L(f^s_{p,\lambda} - T^s_{p,\lambda}) \leq \tau + \epsilon. \tag{3.6}$$

Hence, $L(\Delta^\#_{p,\lambda}) \leq (\tau + \epsilon)/(\tau^{-1} - \epsilon) < \tau$ if $p \in U \setminus U^-_0$. Moreover, $\Delta^\#$ is continuous and is holomorphic on $E^u_p(r) \times \mathbb{D}(\rho)$ for each fixed p.

To show that $\Delta^\#$ is in $\mathcal{D}^u_U$, we need only to show that $\|\Delta^\#_{p,\lambda}(0)\| \leq r_0$ for each $p \in U \setminus U^-_0$ and $\lambda \in \mathbb{D}(\rho)$. The relation in (3.4) implies that if $t > 0$ and $\|\Psi_{p,\lambda}(0)\| < t(\tau^{-1} - \epsilon)$, then $\Psi_{p,\lambda}(E^u_p(t))$ contains $E^u_{f(p)}(t(\tau^{-1} - \epsilon) - \|\Psi_{p,\lambda}(0)\|)$, and hence $\|\Psi^{-1}_{p,\lambda}(0)\| < t$. The minimal such t is $t = \|\Psi_{p,\lambda}(0)\|/(\tau^{-1} - \epsilon)$. From (3.6), we have $\|\Psi_{p,\lambda}(0)\| \leq 2\epsilon r_0$. Hence

$$\|\Psi^{-1}_{p,\lambda}(0)\| \leq \frac{2\epsilon r_0}{\tau^{-1} - \epsilon}. \tag{3.7}$$

Also, since $\Phi_{p,\lambda}(0) = f^s_{p,\lambda}(\Delta_{p,\lambda}(0))$, we have

$$\|\Phi_{p,\lambda}(0)\| \leq \|T^s_{p,\lambda}(\Delta_{p,\lambda}(0))\| + \|(f^s_{p,\lambda} - T^s_{p,\lambda})(\Delta_{p,\lambda}(0))\| < \tau r_0 + \epsilon r_0.$$

Hence by (3.6) and (3.7),

$$\begin{aligned}\|\Phi_{p,\lambda}(\Psi_{p,\lambda}^{-1}(0))\| &\le \|\Phi_{p,\lambda}(0)\| + L(\Phi_{p,\lambda})\|\Psi_{p,\lambda}^{-1}(0)\| \\ &\le (\tau+\epsilon)r_0 + (\tau+\epsilon)2\epsilon r_0(\tau^{-1}-\epsilon)^{-1} \\ &\le (\tau+3\epsilon)r_0 < r_0.\end{aligned}$$

Thus, $\Delta^{\#}$ is in $\mathcal{D}_U^u$.

To show that the graph transform is a contraction on $\mathcal{D}_U^u$, let $\Delta_1, \Delta_2 \in \mathcal{D}_U^u$. Then, dropping p, λ for clarity, and using Ψ_j, Φ_j to represent the coordinate functions for Δ_j, we have on $E_p^u(r)$ that

$$\begin{aligned}\|\Delta_1^{\#} - \Delta_2^{\#}\| &\le \|\Phi_1\Psi_1^{-1} - \Phi_1\Psi_2^{-1}\| + \|\Phi_1\Psi_2^{-1} - \Phi_2\Psi_2^{-1}\| \\ &\le L(\Phi_1)\|\Psi_1^{-1} - \Psi_2^{-1}\| + \|\Phi_1 - \Phi_2\|.\end{aligned}$$

To continue, note that by Proposition 3.6 and the definition of T, we have $L(f^s) \le \tau + \epsilon$. Since $\Phi_j = f^s(v+\Delta_j(v))$, we have $L(\Phi_1) \le \tau+\epsilon$ and $\|\Phi_1-\Phi_2\| \le (\tau+\epsilon)\|\Delta_1-\Delta_2\|$. We use this together with Proposition 1.4(b) of [HP], which implies that $|g^{-1}-h^{-1}| \le L(g^{-1})|g-h|$ when g and h are invertible maps between vector spaces. Hence

$$\begin{aligned}\|\Delta_1^{\#} - \Delta_2^{\#}\| &\le (\tau+\epsilon)L(\Psi_1^{-1})\|\Psi_1 - \Psi_2\| + (\tau+\epsilon)\|\Delta_1 - \Delta_2\| \\ &\le (\tau+\epsilon)(\tau^{-1}-\epsilon)^{-1}\|\Psi_1 - \Psi_2\| + (\tau+\epsilon)\|\Delta_1 - \Delta_2\|\end{aligned}$$

Also, by Proposition 3.6 and the fact that $T^u|E^s \equiv 0$, we have

$$\begin{aligned}\|\Psi_1 - \Psi_2\| &\le \|(f^u - T^u)(v + \Delta_1(v)) - (f^u - T^u)(v + \Delta_2(v))\| \\ &\le \epsilon\|\Delta_1 - \Delta_2\|\end{aligned}$$

Hence

$$\begin{aligned}\|\Delta_1^{\#} - \Delta_2^{\#}\| &\le \frac{(\tau+\epsilon)(\epsilon + (\tau^{-1}-\epsilon))}{\tau^{-1}-\epsilon}\|\Delta_1 - \Delta_2\| \\ &= \frac{\tau+\epsilon}{1-\tau\epsilon}\|\Delta_1 - \Delta_2\|.\end{aligned}$$

Since $(\tau+\epsilon)/(1-\tau\epsilon) < 1$, we see that the graph transform is a contraction. Since $\mathcal{D}_U^u$ is closed, there is a unique fixed point. □

In order to show that the map defined by taking the intersection of stable and unstable disks is a homeomorphism, we need to know also that the fixed point of the graph transform is Lipschitz as a map on all of E^u and not just on each fiber. That is, following Robinson, we regard E^u and E^s each as subsets of $U \times \mathbf{C}^2$ with the natural embedding and define a metric on E^u by $d_f((p,v),(q,w)) = \max\{d_f(p,q), \|v-w\|\}$. Note that f is an isometry under d_f.

Proposition 3.11. *Suppose that $\Sigma \in \mathcal{D}^u$ is invariant under $f^{\#}$ on U_f and that $L(\Sigma_\lambda) \le L_0$ on $E^u(R)$ for some $L_0 > 1$, $R > 0$ independent of λ. Then there exists $L > 0$ and sufficiently small ϵ^{su}, r, r_0 and ρ such that the fixed point, Δ, obtained by the previous proposition, satisfies $\|\Delta_\lambda(p,v) - \Delta_\lambda(q,w)\| \le L d_f((p,v),(q,w))$ whenever $v \in E_p^u(r)$, $w \in E_q^u(r)$, $|\lambda| < \rho$, and $|p-q| < 2r$.*

Note: As in [R2], it is possible to show that the fixed point actually has global Lipschitz constant L_0.

Proof. Choose ϵ as in the proof of the previous proposition and so that $(\tau+5\epsilon)(1+6\epsilon L_0) < 1$. Choose U, ϵ^{su}, r, r_0 and ρ as in the previous proof. In the rest of the proof, we decrease r and ρ a number of times, but in each case we can then decrease r_0 and ρ again in order to preserve the relations from the previous proof.

Cover $\overline{U}$ by finitely many open balls, $\{U_j\}$, so that in each U_j, there is a smooth trivialization of $E^{u/s}|U_j$ to $U_j \times \mathbf{C}$ given by $(p, v) \mapsto (p, \sigma_j^{u/s}(p, v))$ so that each $\sigma_j^{u/s}$ is a holomorphic linear isometry on each fiber. Define a norm on $U_j \times \mathbf{C}$ as the max of the usual norm on each factor. Let σ stand for either σ^u or σ^s. Since $\sigma_j(p, 0) = 0$, for $\epsilon > 0$, there is r small enough that if $v \in E_p^u(r)$, then $|D_1\sigma_j(p, v)| < \epsilon$, hence $|\sigma_j(p, v) - \sigma_j(q, v)| \le \epsilon|p - q|$. Using the fact that $|p - q| \le d_f(p, q)$, we have for $v \in E_p^u(r)$, $w \in E_q^u(r)$, $p, q \in U_j$, that

$$\begin{aligned}|\sigma_j(p, v) - \sigma_j(q, w)| &\le |\sigma_j(p, v) - \sigma_j(q, v)| + |\sigma_j(q, v) - \sigma_j(q, w)| \\ &\le \epsilon d_f(p, q) + \|v - w\|.\end{aligned}$$

Hence σ_j has Lipschitz constant at most $1 + \epsilon$ on $E^u(r)|U_j$. Likewise we may assume $(p, v) \mapsto (p, (\sigma_{j,p})^{-1}(v))$ has Lipschitz constant at most $1 + \epsilon$. Hence for the remainder of the proof, it suffices to use local coordinates given by the $\sigma_j^{u/s}$. Thus, we identify $E^u|U_j$ with $U_j \times \mathbf{C}$. Then all previous estimates on Lipschitz constants on fibers remain the same, but now we can subtract vectors lying above different base points in the same U_j. In these coordinates, Σ has Lipschitz constant at most $L_0(1+2\epsilon)$. Hence we replace L_0 by $L_0(1+2\epsilon)$ and show that the resulting fixed point has Lipschitz constant $L = L_0$. Also, we use the fact that if f and g are functions of two variables and are Lipschitz using the maximum of the distance on each of the two factors, then $L(f(Id, g)) \le L(f)(1 + L(g))$.

Let $\delta > 0$ so that if $|p - q| < \delta$, then there exists j so that $p, q \in U_j$. Decrease r if necessary so that if $p, q \in U$ and $|p - q| < 2r$, then $|f^k(p) - f^k(q)| < \delta$ for $k = -1, 0$ while if $|p - q| < 2r$ and $f^{-1}(p) \notin U$, then $p, q \in U_f$. For the remainder of the proof, assume $|p - q| < 2r$, $p, q \in U_j$, $v \in E_p^u(r)$, $w \in E_q^u(r)$, $|\lambda| < \rho$.

As in the proof of Proposition 3.6, let $G_\lambda^u(p, v) = f_{p,\lambda}^u(v) - T_{p,\lambda}^u(v)$, although now in the local coordinates. Note that G^u is smooth in p, λ, v. Also, since $f_{p,0}^u(0) = T_{p,0}^u(0) = 0$, we have $G_0^u(p, 0) \equiv 0$, independent of p, hence $D_1G_0^u(p, 0) = 0$. Hence for r and ρ sufficiently small, we have $|D_1G_\lambda^u(p, v)| \le \epsilon$. Using Proposition 3.6 we have

$$\begin{aligned}|G_\lambda^u(p, v) - G_\lambda^u(q, w)| &\le |G_\lambda^u(p, v) - G_\lambda^u(q, v)| + |G_\lambda^u(q, v) - G_\lambda^u(q, w)| \\ &\le \epsilon|p - q| + \epsilon|v - w| \\ &\le 2\epsilon d_f((p, v), (q, w)).\end{aligned}$$

Then with the same estimates as (3.3) only applied with varying base point and using the Lipschitz bound on Σ, we have $L(\Psi_\lambda - T_\lambda^u) < 2\epsilon(1 + L_0)$.

An estimate similar to that for G^u implies that if ρ is sufficiently small and $x \in E_p^s$, $y \in E_q^s$, then

$$|T_{\lambda,p}^s(x) - T_{\lambda,q}^s(y)| \le \tau|x - y| + \epsilon d_f(p, q), \tag{3.8}$$

and if also r is sufficiently small then

$$|T^u_{\lambda,p}(v) - T^u_{\lambda,q}(w)| \geq \tau^{-1}|v-w| - \epsilon d_f(p,q).$$

Hence

$$\begin{aligned}|\Psi_{p,\lambda}(v) - \Psi_{q,\lambda}(w)| &\geq |T^u_{p,\lambda}(v) - T^u_{q,\lambda}(w)| - |G^u_{p,\lambda}\Gamma_{p,\lambda}(v) - G^u_{q,\lambda}\Gamma_{p,\lambda}(w)| \\ &\geq \tau^{-1}|v-w| - \epsilon d_f(p,q) - 2\epsilon(1+L_0)d_f((p,v),(q,w)) \\ &\geq (\tau^{-1} - 2\epsilon(1+L_0))|v-w| - 3\epsilon(1+L_0)d_f(p,q).\end{aligned}$$

Let $x = \Psi_{p,\lambda}(v)$ and $y = \Psi_{q,\lambda}(w)$. Then this last inequality and $L_0 > 1$ implies that

$$|x-y| \geq (\tau^{-1} - 4\epsilon L_0)|\Psi^{-1}_{p,\lambda}(x) - \Psi^{-1}_{q,\lambda}(y)| - 6\epsilon L_0 d_f(p,q),$$

or, using the fact that $\tau^{-1} - 4\epsilon L_0 > 1$,

$$|\Psi^{-1}_{p,\lambda}(x) - \Psi^{-1}_{q,\lambda}(y)| \leq |x-y| + 6\epsilon L_0 d_f(p,q).$$

Next, since $v \in E^u_{p,\lambda}$, we have $T^s_{p,\lambda}(v) = 0$. Let

$$d_1 = d_f((p,v),(q,w)) \qquad d_2 = d_f((p, v+\Sigma_{p,\lambda}(v)),(q, w+\Sigma_{q,\lambda}(w))).$$

Then using (3.8), we have

$$\begin{aligned}|\Phi_{p,\lambda}(v) - \Phi_{q,\lambda}(w)| &= |f^s_{p,\lambda}(v+\Sigma_{p,\lambda}(v)) - f^s_{q,\lambda}(w+\Sigma_{q,\lambda}(w))| \\ &\leq |T^s_{p,\lambda}(\Sigma_{p,\lambda}(v)) - T^s_{q,\lambda}(\Sigma_{q,\lambda}(w))| \\ &\quad + |G^s_{p,\lambda}(v+\Sigma_{p,\lambda}(v)) - G^s_{q,\lambda}(w+\Sigma_{q,\lambda}(w))| \\ &\leq \tau L_0 d_1 + \epsilon d_f(p,q) + 2\epsilon d_2 \\ &\leq (\tau L_0 + \epsilon + 2\epsilon(1+L_0))d_1 \\ &\leq (\tau + 5\epsilon)L_0 d_1\end{aligned}$$

Hence, for the partially invariant Σ,

$$\begin{aligned}|\Sigma^{\#}_{p,\lambda} - \Sigma^{\#}_{q,\lambda}| &= |\Phi_{f^{-1}(p),\lambda}\Psi^{-1}_{f^{-1}(p),\lambda}(v) - \Phi_{f^{-1}(q),\lambda}\Psi^{-1}_{f^{-1}(q),\lambda}(w)| \\ &\leq L(\Phi_\lambda)d_f((f^{-1}(p), \Psi^{-1}_{f^{-1}(p),\lambda}(v)),(f^{-1}(q), \Psi^{-1}_{f^{-1}(q),\lambda}(w))) \\ &\leq (\tau+5\epsilon)L_0 \max\{d_f(p,q), |v-w| + 6\epsilon L_0 d_f(p,q)\} \\ &\leq (\tau+5\epsilon)(1+6\epsilon L_0)L_0 d_f((p,v),(q,w)),\end{aligned}$$

whenever $f^{-1}(p), f^{-1}(q) \in U$. By choice of ϵ, this Lipschitz constant is less than L_0, hence each iteration of the graph transform of Σ has Lipschitz constant at most L_0 on the set where $f^{-1}(p), f^{-1}(q) \in U$. If p and q do not satisfy this, then $p, q \in U_f$ by choice of r. In this case, $\Sigma^{\#} = \Sigma$ and has Lipschitz constant at most L_0 by assumption.

Thus, each graph transform of Σ satisfies the Lipschitz condition of the proposition and is invariant on U_f. Hence the limit function, which exists by the previous proposition, also satisfies the condition of the proposition. □

Note that the proof that the limit function is Lipschitz relies heavily on the fact that f is an isometry with respect to the d_f-metric. This shows up in the last set of inequalities when we change to the distance between p and q from the distance between their inverse images. If we used the usual distance, this would introduce an extra factor of τ^{-1}, which would yield a Lipschitz constant larger than L_0. Note also that since the fixed point, Δ, is bounded by r, it has global Lipschitz constant at most $\max\{2, L_0\}$.

Using this, we obtain the following proposition.

Proposition 3.12. *Let f_λ be a one-parameter family of hyperbolic polynomial automorphisms of $\mathbf{C}^2$ depending holomorphically on $\lambda \in \mathbb{D}$ and satisfying the conditions of proposition 3.3 for some $\tau < 1$. Then there are a neighborhood, U, of J_0, a smooth splitting $E^s \times E^u$ over U, r, r_0, ρ, and elements $\Delta^u \in \mathcal{D}^u_U(r, r_0, \rho, \tau)$, $\Delta^s \in \mathcal{D}^s_U(r, r_0, \rho, \tau)$ which are invariant under $f^\#_\lambda$ and $(f^{-1}_\lambda)^\#$, respectively. Moreover, we have that $\Delta^{u/s}_0(p, 0) = 0$ for each $p \in U$, and there is $L > 0$ so that for each fixed $|\lambda| < \rho$, $L(\Delta^{u/s}_\lambda) < L$ on $U \times E^{u/s}(r)$.*

Proof. Given the preceding propositions, it suffices to show that there exists an element of the set $\mathcal{D}^u_U(r, r_0, \rho, \tau)$ that is Lipschitz and invariant under $f^\#_\lambda$ on U_f, and likewise for $\mathcal{D}^s_U(r, r_0, \rho, \tau)$. This uses the same ideas as those in [R2] (see section 4), which we recall here. The set $D = \overline{U^-_0 \setminus f_0(U^-_0)}$ is a fundamental domain for points in $U \setminus J^-$. I.e., each point in $U \setminus J^-$ has at least one preimage in D and exactly one preimage unless some preimage is contained in $B = D \cap f_0(D)$. Moreover, B and $f^{-1}_0(B)$ are disjoint. Let N_0 and N_1 be neighborhoods of $f^{-1}_0(B)$ and B, respectively, so that their closures are disjoint. Define $\Sigma^u_{p,\lambda}(v) = 0$ for p in N_0 and apply the graph transform $f^\#_\lambda$ to define $\Sigma^u_{p,\lambda}$ for $p \in f_0(N_0) \cap N_1$. The same estimates as in proposition 3.11 imply that $L(\Sigma^u_{p,\lambda}) \leq \tau$. Let χ be a cutoff function with support in $f_0(N_0) \cap N_1$ and equal to 1 in a neighborhood of B. Define Σ^u on D by $\chi(p)\Sigma^u_{p,\lambda}(v)$. Extend to a neighborhood of $\overline{U_f}$ by applying $f^\#_\lambda$ again. Finally, use another cutoff function equal to 1 on U_f and with support contained in the neighborhood of $\overline{U_f}$ where Σ^u is currently defined. Then $\Sigma^u \in \mathcal{D}^u_U$ and is smooth with compact support, hence globally Lipschitz, and still $L(\Sigma^u_{p,\lambda}) \leq \tau$. Hence the previous two propositions imply that there exists $\Delta^u \in \mathcal{D}^u_U$ which is invariant under $f^\#_\lambda$ and is globally Lipschitz. Replacing f with f^{-1} gives the corresponding result for Δ^s. □

In the end, the conjugacy between f_0 and f_λ will be obtained by mapping the point p to the point of intersection between the graphs of $\Delta^u_{p,\lambda}$ and $\Delta^s_{p,\lambda}$. In the following proposition, we determine some of the properties of this map. Later we will show that with the d_f metric, this map is a homeomorphism.

Proposition 3.13. *Let $A \subset \mathbf{C}^2$ and let $r > 0$. Suppose $\Delta^{u/s} : A \times \mathbb{D}(r) \times \mathbb{D} \to \mathbb{D}(r)$ is continuous and is holomorphic for $(u, \lambda) \in \mathbb{D}(r) \times \mathbb{D}$ for each fixed $p \in A$. Suppose also that there are a metric, d, on A and constants $K > 0$ and $0 < \tau < 1$, such that if $p \in A$, $u, v \in \mathbb{D}(r)$ and $\lambda \in \mathbb{D}$, then*

$$|\Delta^{u/s}(p, u, \lambda) - \Delta^{u/s}(q, v, \lambda)| \leq K(d(p, q) + |u - v|),$$

$$|\Delta^{u/s}(p, u, \lambda) - \Delta^{u/s}(p, v, \lambda)| \leq \tau|u - v|.$$

Finally, suppose that $\Delta^{u/s}(p, 0, 0) = 0$ for each $p \in A$. Write $\Delta^{u/s}_{p,\lambda}(v) = \Delta^{u/s}(p, v, \lambda)$. Then there is $\rho > 0$ such that for all $|\lambda| < \rho$ there is a unique point $v_p(\lambda)$ satisfying

$$v_p(\lambda) = \Delta^u_{p,\lambda}(\Delta^s_{p,\lambda}(v_p(\lambda))).$$

Moreover, $v_p(\lambda)$ is holomorphic in λ for $p \in A$ fixed, and is continuous in the usual metric on A for λ fixed, and there exists $C > 0$ such that for $|\lambda| < \rho$,

(1) $(v_p(\lambda), \Delta^s_{p,\lambda}(v_p(\lambda)))$ *is the unique point in the intersection of the graphs* $\{(v,w) : w = \Delta^s_{p,\lambda}(v)\}$ *and* $\{(v,w) : v = \Delta^u_{p,\lambda}(w)\}$
(2) $|v_p(\lambda)| \le C|\lambda|$ *for all* $p \in A$
(3) $|v_p(\lambda) - v_q(\lambda)| \le Cd(p,q)|\lambda|$ *for all* $p, q \in A$.

By symmetry, the same estimates apply if we interchange Δ^u *and* Δ^s.

Note: The same result holds with the same proof if $\lambda \in \mathbb{D}$ is replaced by $\lambda \in \mathbb{D}^k$ and derivatives with respect to λ are replaced by gradients with respect to λ.

Proof. The assumptions on $\Delta^{u/s}$ imply that $v_p(0) = 0$ and $|D_2\Delta^{u/s}| \le \tau < 1$. Hence, we may shrink r and the domain of λ, then re-scale in λ to assume that $\Delta^{u/s}$ is continuous on $A \times \overline{\mathbb{D}(r)} \times \overline{\mathbb{D}}$ and maps into $\mathbb{D}(\kappa r)$ for some $\kappa < 1$. Let $\rho < 1$.

The point (v,w) is in the intersection of the graphs as in (a) if and only if $v = \Delta^u_{p,\lambda}\Delta^s_{p,\lambda}(v)$ and $w = \Delta^s_{p,\lambda}(v)$. Since $|\Delta^{u/s}_{p,\lambda}(u) - \Delta^{u/s}_{p,\lambda}(v)| \le \tau|u - v|$, the contraction mapping theorem implies that $v_p(\lambda)$ exists and is unique for each fixed p and λ. The implicit function theorem implies that $v_p(\lambda)$ is holomorphic in λ for p fixed. The contracting map theorem [HP, 1.1] implies that $v_p(\lambda)$ is continuous in p for λ fixed.

In the following, we use C to indicate a positive constant that may increase from line to line. That is, on a given line C is chosen large enough to satisfy any previous conditions plus any conditions required for the current line. Unless stated otherwise, all estimates are for $p, q \in A$, $v, w \in \mathbb{D}(r)$, $|\lambda| < \rho$. Also, several times we use the inequality $|ab - a'b'| \le |a||b - b'| + |b'||a - a'|$.

Let $\phi_p(v,\lambda) = \Delta^u_{p,\lambda}\Delta^s_{p,\lambda}(v)$. Then $\phi_p(v,\lambda)$ is continuous in p, v, λ and is holomorphic in v, λ for fixed p. Moreover, $|D_2\Delta^{u/s}| \le \tau$ implies that $|D_1\phi_p(v,\lambda)| \le \tau^2$. Also, since

$$D_2\phi_p(v,\lambda) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\phi_p(v,\zeta)}{(\zeta-\lambda)^2}d\zeta$$

and $|\phi_p| < r$, we have $|D_2\phi_p(v,\lambda)| \le r/(1-\rho)^2$. As noted above, $v_p(\lambda) - \phi_p(v_p(\lambda),\lambda) = 0$, so taking the derivative with respect to λ and solving gives

$$v_p'(\lambda) = \frac{(D_2\phi_p)(v_p(\lambda),\lambda)}{1 - (D_1\phi_p)(v_p(\lambda),\lambda)}. \tag{3.9}$$

Thus, $v_p(\lambda)$ satisfies this differential equation in λ, with initial condition $v_p(0) = 0$. Using the bounds on $|D_1\phi_p|$ and $|D_2\phi_p|$ and integrating from 0 to λ, we get

$$|v_p(\lambda)| \le \frac{r}{(1-\rho)^2(1-\tau^2)}|\lambda| \le C|\lambda|.$$

Thus property (b) holds.

For property (c), first note that two applications of the Lipschitz assumption on $\Delta^{u/s}$ imply that

$$|\phi_p(v,\lambda) - \phi_q(w,\lambda)| \le C(d(p,q) + |v - w|).$$

Also,

$$\begin{aligned}|D_2\phi_p(v,\lambda) - D_2\phi_q(w,\lambda)| &\leq \frac{1}{2\pi}\int_{\partial\mathbb{D}}\left|\frac{\phi_p(v,\zeta)-\phi_q(w,\zeta)}{(\zeta-\lambda)^2}\right| d|\zeta| \\ &\leq \frac{C}{(1-\rho)^2}(d(p,q)+|v-w|).\end{aligned}$$

Likewise, letting $\hat{\phi}_p = \phi_p(\zeta,\lambda)$, we have for $|v|, |w| < \kappa r$ that

$$\begin{aligned}|D_1\phi_p(v,\lambda) - D_1\phi_q(w,\lambda)| &\leq \frac{1}{2\pi}\int_{\partial\mathbb{D}(r)}\left|\frac{\hat{\phi}_p}{(\zeta-v)^2} - \frac{\hat{\phi}_q}{(\zeta-w)^2}\right| d|\zeta| \\ &\leq \frac{1}{2\pi r^4(1-\kappa)^4}\int_{\partial\mathbb{D}(r)} |(\zeta-w)^2\hat{\phi}_p - (\zeta-v)^2\hat{\phi}_q|\; d|\zeta| \\ &\leq \frac{C}{r^4(1-\kappa)^4}\int_{\partial\mathbb{D}(r)}(|\zeta|^2|\hat{\phi}_p - \hat{\phi}_q| + 2|\zeta||\hat{\phi}_p w - \hat{\phi}_q v| \\ &\qquad + |\hat{\phi}_p w^2 - \hat{\phi}_q v^2|)\; d|\zeta| \\ &\leq \frac{2\pi r C}{r^4(1-\kappa)^4}(r^2 C d(p,q) + 2r(r|v-w| + rCd(p,q)) \\ &\qquad + (r|v^2-w^2| + r^2 C d(p,q))) \\ &\leq C(d(p,q)+|v-w|).\end{aligned}$$

Write $D_j\phi_p = D_j\phi_p(v_p(\lambda),\lambda)$. Then

$$\begin{aligned}\left|\frac{D_2\phi_p}{1-D_1\phi_p} - \frac{D_2\phi_q}{1-D_1\phi_q}\right| &\leq |D_2\phi_p|\left|\frac{1}{1-D_1\phi_p} - \frac{1}{1-D_1\phi_q}\right| \\ &\qquad + \left|\frac{1}{1-D_1\phi_q}\right| |D_2\phi_p - D_2\phi_q| \\ &\leq \frac{r}{(1-\rho)^2}\frac{1}{(1-\tau^2)^2}|D_1\phi_p - D_1\phi_q| \\ &\qquad + \frac{1}{1-\tau^2}\frac{C}{(1-\rho)^2}(d(p,q)+|v_p(\lambda)-v_q(\lambda)|) \\ &\leq C(d(p,q)+|v_p(\lambda)-v_q(\lambda)|).\end{aligned}$$

Using this with (3.9), we have

$$\begin{aligned}|v_p(\lambda)-v_q(\lambda)| &\leq \int_0^\lambda |v_p'(\zeta)-v_q'(\zeta)|\; d|\zeta| \\ &\leq Cd(p,q)|\lambda| + \int_0^\lambda C|v_p(\zeta)-v_q(\zeta)|\; d|\zeta|.\end{aligned}$$

For fixed λ and $t\in[0,1]$, let $h_{p,q}(t) = |v_p(t\lambda)-v_q(t\lambda)|$. Then

$$h_{p,q}(t) \leq Cd(p,q)|\lambda| t + \int_0^t C h_{p,q}(s)ds.$$

Gronwall's inequality implies that

$$h_{p,q}(t) \le Cd(p,q)|\lambda|t + Cd(p,q)|\lambda| \int_0^t C\exp(Cs)ds.$$

Hence

$$|v_p(\lambda) - v_q(\lambda)| = h_{p,q}(1) \le Cd(p,q)|\lambda|,$$

so property (c) holds. $\square$

We may now give a proof of Theorem 1.1:

Proof. Given f_λ and the semi-invariant disk families $\Delta^{u/s}$ as in 3.12, cover $\overline{U}$ with finitely balls as in 3.11 to get local trivializations of E^u and E^s. In these local coordinates, the disk families satisfy the conditions for proposition 3.13 with $d = d_f$. Let $\phi_\lambda(p)$ be the intersection of the graphs of the disks $\Delta^u_{p,\lambda}$ and $\Delta^s_{p,\lambda}$ in the original splitting $T\mathbf{C}^2|U = E^s \times E^u$, given by proposition 3.13. Since the metric from proposition 3.3 is equivalent to the standard metric, and since the local trivialization is bilipschitz, proposition 3.13 implies that there is a constant $C > 0$ so that in the usual metric on $T\mathbf{C}^2$, $|\phi_\lambda(p)| \le C|\lambda|$ and $|\phi_\lambda(p) - \phi_\lambda(q)| \le Cd_f(p,q)|\lambda|$.

By abuse of notation, let $\Phi_\lambda(p) = p + \phi_\lambda(p)$, where '+' stands for the standard exponential map. Since the disk families are invariant under the graph transform, Φ_λ gives a semiconjugacy. That is, the point of intersection of $\hat{\Delta}^u_{p,\lambda}$ and $\hat{\Delta}^s_{p,\lambda}$ is mapped by f_λ to the point of intersection of $\hat{\Delta}^u_{f(p),\lambda}$ and $\hat{\Delta}^s_{f(p),\lambda}$. Hence, $f_\lambda\Phi_\lambda(p) = \Phi_\lambda f(p)$, so $f^j_\lambda\Phi_\lambda(p) = \Phi_\lambda f^j(p)$ whenever both sides are defined.

Then following the ideas of [R2], we have

$$|p - q| \le |\Phi_\lambda(p) - \Phi_\lambda(q)| + |\phi_\lambda(p) - \phi_\lambda(q)|,$$

hence

$$|\Phi_\lambda(p) - \Phi_\lambda(q)| \ge |p - q| - Cd_f(p,q)|\lambda|.$$

Let L_0 be the max of $L(f|U)$ and $L(f^{-1}|U)$. Reduce ρ so that $C\rho/L_0 < 1$.

Let $p, q \in U$. By proposition 3.2 there is n so that $f^j(p)$ and $f^j(q)$ lie in U for j between 0 and n and

$$|f^n(p) - f^n(q)| \ge d_f(p,q)/L_0 = d_f(f^n(p), f^n(q))/L_0.$$

Then,

$$\begin{aligned}|\Phi_\lambda(f^n(p)) - \Phi_\lambda(f^n(q))| &\ge |f^n(p) - f^n(q)| - C\rho d_f(f^n(p), f^n(q)) \\ &\ge |f^n(p) - f^n(q)|(1 - C\rho/L_0).\end{aligned}$$

Thus $\Phi_\lambda(f^n(p)) = \Phi_\lambda(f^n(q))$ exactly when $f^n(p) = f^n(q)$. By the semiconjugacy property and the fact that f is injective, we see that Φ_λ is injective. Also, shrinking U if needed, Φ_λ is continuous on the compact set $\overline{U}$, hence is a homeomorphism. Finally, the smoothness part of the definition of $\mathcal{D}^{u/s}$ implies that Φ_λ is smooth outside $J^+ \cup J^-$. $\square$

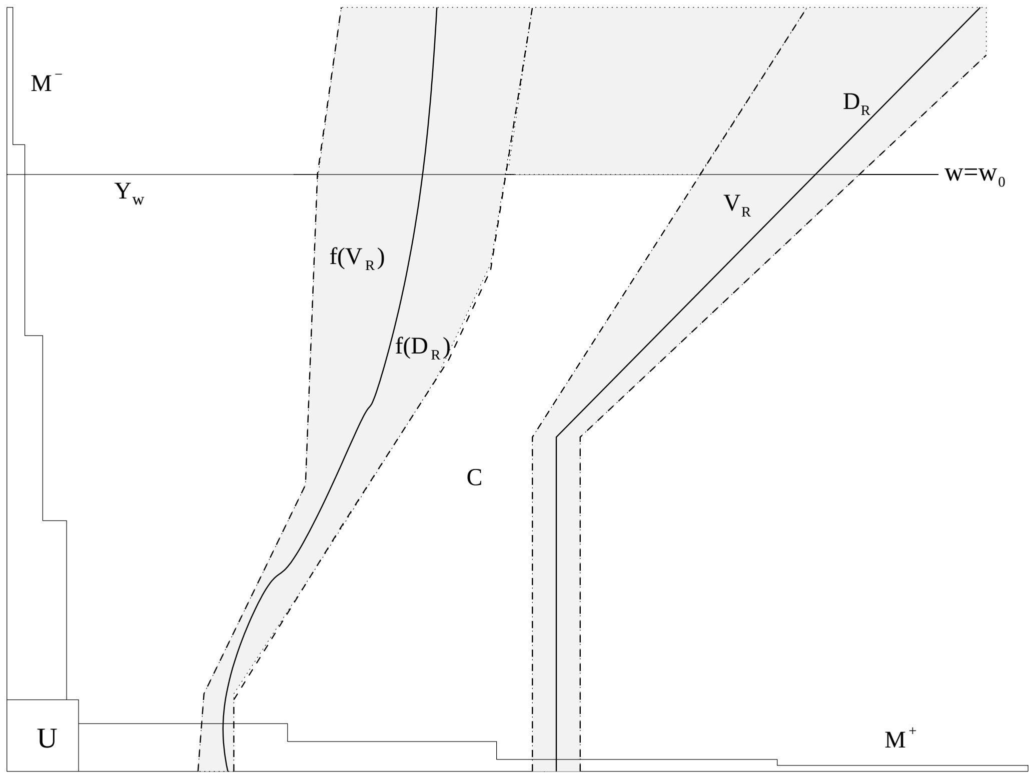

Figure 4. The conjugacy Φ_λ was constructed on U in section 3. The construction on the sets $M^\pm$ is given by proposition 4.1. It is then defined on a fundamental domain, first on the shaded region and then across the set C. Here, D_R and $f(D_R)$ are denoted by solid lines, while V_R and $f(V_R)$ are bounded by dotted lines.

4. Global conjugacy

We now use the conjugacy in a neighborhood of J to construct a conjugacy on all of $\mathbf{C}^2$ and prove Theorems 1.2 and 1.3. The conjugacy will be constructed in numerous parts over a fundamental domain V, with the exception of a compact set C. Figure 4 describes the sets in question:

Our first goal is to construct an extension of such a conjugacy in a neighborhood of all of J^+. Essentially, such a conjugacy is already predetermined. To see this, we note that we have already constructed a conjugacy in a neighborhood U of J (section 3). Let

$$M^+ = \bigcup_{n=0}^{\infty} f_0^{-n}(U).$$

The idea is simply to iterate f_0 enough times so that we land in U, apply our conjugacy, and then iterate by f_λ^{-1} to return. More precisely, we have the following proposition:

Proposition 4.1. *Suppose that $\{f_\lambda\}$ is a family of hyperbolic polynomial automorphisms depending holomorphically on the parameter λ, and let J_λ be the associated Julia sets. Let U be a neighborhood of $J = J_0$, and suppose that, for $|\lambda| < \rho$ (where ρ is sufficiently small),* *there are homeomorphisms Φ_λ defined on U satisfying the relation $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$. Then, for the neighborhood M^+ of J^+ defined above, there are extensions of these homeomorphisms to Φ_λ defined on M^+, so that $\Phi_\lambda f_0 = f_\lambda \Phi_\lambda$.*

Proof. We define Φ_λ as follows: given any $p \in M^+$, define $n_p \in \mathbf{N} \cup \{0\}$ to be the smallest such number so that $f_0^{n_p} \in U_0$. Then, for any $p \in M^+$, define $\Phi_\lambda(p) = f_\lambda^{-n_p} \Phi_\lambda f_0^{n_p}(p)$. This map is well-defined for all $p \in M^+$. Furthermore, it is continuous and injective. The arguments are similar to those in [R2]; we give them for the reader's convenience.

To see that Φ_λ is continuous, note that for $\epsilon > 0$ sufficiently small, we have that $f_0^{n_p}(q) \in U_0$ if $|q - p| < \epsilon$, and hence we must have for any such q that $n_q \leq n_p$. Since Φ_λ is a conjugacy on U, we have $f_\lambda^{-n_q} \Phi_\lambda f_0^{n_q}(q) = f_\lambda^{-n_p} \Phi_\lambda f_0^{n_p}(q)$. Hence, continuity follows, since all of f_0, Φ_λ, and f_λ^{-1} are continuous on their respective domains.

To see that Φ is injective, let $p, q \in M^+$, and define $\hat{p} = \Phi_\lambda f_0^{n_p}(p)$ and $\hat{q} = \Phi_\lambda f_0^{n_q}(q)$. If $n_p = n_q$, there is nothing to prove, so without loss of generality, suppose that $n_p > n_q$ and write $n = n_p - n_q$. If $\Phi_\lambda(p) = \Phi_\lambda(q)$, then we must have $f_\lambda^{-n_p}(\hat{p}) = f_\lambda^{-n_q}(\hat{q})$, and so $f_\lambda^n(\hat{q}) = \hat{p}$. Hitting both sides of this equation with Φ_λ^{-1}, and using the fact that $\Phi_\lambda^{-1} f_\lambda = f_0 \Phi_\lambda^{-1}$ on U_λ, we have $f_0^n \Phi_\lambda^{-1}(\hat{q}) = \Phi_\lambda^{-1}(\hat{p})$. But this yields $f_0^{n_p}(p) = f_0^{n_p}(q)$, and so $p = q$. □

We now prove Theorem 1.2:

Proof. Proposition 4.1 defines Φ_λ on M^+, and a nearly-identical procedure will construct Φ_λ on M^-. The smoothness of Φ_λ on an open set not intersecting $J_0^+ \cup J_0^-$ follows immediately from the construction of Φ_λ on the set J (recall the definition of $\mathcal{D}^u$ from definition 3.7), plus the holomorphicity of f_λ. □

Now, with this extension Φ_λ to $M^+ \cup M^-$ in hand, we begin the process of extending this conjugacy to all of $\mathbf{C}^2$. The goal here is as follows: we will define a holomorphic motion which has controlled behavior as $|p| \to \infty$. We do this by constructing a holomorphic motion on a connected, unbounded set which sends certain (complex) lines to themselves, in a sense to be explained later. We then use the Bers-Royden extension for one-variable motions, together with the dynamics of the given polynomial automorphisms, to extend this motion to a conjugacy which is defined on some fundamental domain, with the exception of a compact set C. We extend to this set C via cutoffs, and use the dynamics to extend the resulting conjugacy to all of $\mathbf{C}^2$.

Let $R > 0$, and define the sets $D_R = \{(x, y) \in \mathbf{C}^2 : |z| = |w| \text{ and } |z| \geq R\} \cup \{(x, y) \in \mathbf{C}^2 : |z| = R \text{ and } |w| \leq R\}$ (see Figure 4). We prove the following:

Lemma 4.2. *For the holomorphic family of hyperbolic polynomial automorphisms $\{f_\lambda\}$, let $\{\Phi_\lambda\}$ be the conjugacies on M^+ as given by Proposition 4.1. Then, for a fixed $R > 0$ sufficiently large, there are neighborhoods $V_1 \subseteq V_2$ of $J^+ \cap D_R$ and a holomorphic motion $\{K_\lambda\}$ defined on $D_R \cup V_2$ so that $K_\lambda = Id$ on $D_R \backslash V_2$ and $K_\lambda = \Phi_\lambda$ on V_1 (after possibly shrinking $|\lambda|$).*

Proof. We choose the neighborhoods V_1 and V_2 so that $V_1 \subseteq \overline{V_1} \subset V_2 \subseteq \overline{V_2} \subseteq M^+$. Choose a C^∞ function $\mu : C^2 \to [0, 1]$ so that $\mu = 1$ on all of V_1, but the support of μ lies in V_2. Now, if the set V_2 is chosen small enough, there is a single $n \in \mathbf{N}$ so that we can write the homeomorphism given by theorem 1.2 as $f_\lambda^{-n} \Phi_\lambda f_0^n$. Let $\epsilon > 0$. From the proof of theorem

1.1, for any point $p \in U$, we can write $\Phi_\lambda(p) = p + \phi_\lambda(p)$, where $|\phi_\lambda(p)| < \epsilon$ and $L(\phi_\lambda) < \epsilon$ (if $|\lambda|$ is sufficiently small). Similarly, the mappings $f_\lambda^{-n} f_0^n$ can be made arbitrarily close to the identity, in the sense that $f_\lambda^{-n} f_0^n(p) = p + g_\lambda(p)$, where g_λ and $L(g_\lambda)$ get arbitrarily small as $|\lambda| \to 0$. But our homeomorphism is simply $f_\lambda^{-n}(f_0^n + \phi_\lambda f_0^n)$. Thus, we can write this map as $Id + \tilde{\phi}_\lambda$, where $|\tilde{\phi}_\lambda|$ and $L(\tilde{\phi}_\lambda)$ can be made arbitrarily small by shrinking $|\lambda|$.

Now, for any $p \in D_R \cup V_2$, we define $K_\lambda(p) = p + \mu(p)\tilde{\phi}_\lambda(p)$ (with the obvious interpretation if $p \notin V_2$). By a Lipschitz perturbation argument, for all fixed λ with $|\lambda|$ small enough, these mappings are both continuous and injective, and hence K_λ is a holomorphic motion. $\square$

Now, fix R as in lemma 4.2 and fix $0 < c_0 < 1$; our interest is when $c_0 \sim 1$. Define the set

$$V_R = \bigcup_{c_0 < |c| < \frac{1}{c_0}} \left(\{z = cw, |w| \geq R\} \cup \{z = cR, |w| \leq R\} \right)$$

Our goal here is to construct a holomorphic motion on (complex) 1-dimensional lines by considering the intersection of sets of the form $\{p \in \mathbf{C}^2 : \pi_2(p) = w\}$ with the set $f_\lambda(\{z = cw\})$, for c as above, while taking the motion to be the identity on V_R.

We need the following lemma:

Lemma 4.3. *Fix $0 < c_0 < 1$, and let $\{f_\lambda\}$ be a holomorphic family of Hénon maps. Then there exist constants $R > 0$ and $\rho > 0$ such that for the set V_R defined above and all $|\lambda| < \rho$, we have that $f_\lambda(V_R) \cap V_R = \emptyset$.*

Proof. We write $f(z,w) = f_1 \cdots f_n$, where $f_i(z,w) = (w, p_i(w) - a_i z)$, for p_i a monic polynomial of degree $m_i \geq 2$ and $|a_i| \neq 0$, $i = 1, \cdots, n$. Choose a single $\widetilde{R}$ so that if $|w| \geq c_0^2 \widetilde{R}$, and if either of $|w| > c_0|z|$, or $|z| < \frac{1}{c_0}\widetilde{R}$ are true, then $|p_i(w) - a_i z| \geq \frac{1}{2}|w|^2$ $(i = 1, \cdots, n)$. Define $R = \max(\frac{2}{c_0^3}, \widetilde{R})$. Our goal is to construct a filtration for a single Hénon map, and apply induction to complete the proof.

Thus, let $f(z,w) = (\tilde{z}, \tilde{w}) = (w, p(w) - az)$. We identify three cases (which contain all points of V_R).

Case 1:
Suppose that $|w| \geq c_0|z|$ and $|w| \geq c_0^2 R$. Then

$$|\tilde{w}| \geq \tfrac{1}{2}|w|^2 \geq \tfrac{1}{c_0}|w| = \tfrac{1}{c_0}|\tilde{z}|.$$

Thus, $(\tilde{z}, \tilde{w}) \notin V_R$. Note that $|\tilde{w}| \geq c_0|\tilde{z}|$ and $|\tilde{w}| \geq c_0^2 R$, so that the point $(\tilde{z}, \tilde{w})$ once again falls into Case 1.

Case 2:
Suppose that $|z| < \frac{1}{c_0}R$ and $|w| > c_0 R$, but $|w| < c_0|z|$. Then, we see that $|\tilde{w}| \geq \frac{1}{2}|w|^2 > \frac{1}{c_0}R$, while $|\tilde{z}| = |w| < c_0|z| < R$. Thus, $(\tilde{z}, \tilde{w}) \notin V_R$. Note that $|\tilde{w}| \geq c_0|z|$, and that $|\tilde{w}| \geq c_0^2 R$, so again the point $(\tilde{z}, \tilde{w})$ falls into Case 1.

Case 3:
Suppose that $|w| < c_0 R$. Then, it is clear that $|\tilde{z}| = |w| < c_0 R$, and again $(\tilde{z}, \tilde{w}) \notin V_R$.

Here, if $|\tilde{w}| \geq c_0 R$, then $|\tilde{w}| \geq c_0|\tilde{z}|$ (so $(\tilde{z}, \tilde{w})$ falls into Case 1), while if $|\tilde{w}| < c_0 R$, then $(\tilde{z}, \tilde{w})$ falls into Case 3.

This completes the proof for a single mapping. For a general f as above, one can simply apply each map f_i individually. Finally, since the sets V_R and $f(V_R)$ are open, any mapping f_λ sufficiently close to f must also satisfy the conclusions of the lemma. □

Now, we construct a holomorphic motion on the set $f_0(V_R)$. For any $c \in \mathbf{C}$, we define the sets $X_c = \{p \in \mathbf{C}^2 : p = (cw, w)\}$, and again, our interest will be in those c with $|c| \sim 1$. We also define $Y_w = \{p \in \mathbf{C}^2 : \pi_2(p) = w\}$. For the set $\{f_\lambda\}$, we assume that all mappings have degree d in the second component.

Lemma 4.4. *There exist constants $c_0 \in (0,1)$, $\rho > 0$ and $W > 0$ so that for all $c_0 < |c| < \frac{1}{c_0}$, $|\lambda| < \rho$, and $|w| > W$, the sets $f_\lambda(X_c \cap V_R)$ intersect Y_{w_0} transversally for all $|w_0| \geq |w_0|$. Moreover, $f_\lambda(X_c) \cap Y_{w_0}$ has d distinct intersection points.*

Proof. We consider first the mapping $f_0(z, w) = (p_1(z, w), w^d + p_2(z, w))$. We note that $deg(p_i) < d$ for $i = 1, 2$. Our interest here is in the second component, which we write $P_c(w) = w^d + p_2(cw, w)$ on the set $X_c \cap V_R$. We have the equation

$$P_c'(w) = dw^{d-1} + c\frac{\partial p_2}{\partial z}|_{z=cw} + \frac{\partial p_2}{\partial w}|_{z=cw}.$$

Consider first $c = 1$. By degree considerations, we can choose $R > 0$ so that for all points $(z, w) \in D_R$ with $z = w$, we have $P_1'(w) \neq 0$. Thus, we can choose $W > 0$ such that for all $|w_0| > W$, $P_1(w)$ has a nonempty, transversal intersection with the complex line $w = w_0$. Moreover, there are d distinct points of intersection (since the polynomial equation $P_1(w) - w_0 = 0$ has roots of multiplicity greater than 1 only if both $P_1(w) - w_0 = 0$ and $P_1'(w) = 0$). Now, we have

$$P_c(w) = w^d + \sum_{n=0}^{d-1} a_n(c) w^n.$$

Thus, given $0 < c_0 < 1$, this argument plus compactness implies that there is a $W > 0$ so that for all $c_0 < |c| < \frac{1}{c_0}$, the mappings $P_c(w)$ have a nonempty, transversal intersection with the complex lines of the form $w = w_0$, (since the dependence on c is continuous). Finally, compactness again implies that there exists $\rho > 0$ so that for all $|\lambda| < \rho$, the intersection remains transversal, with d distinct points.

□

We now define our holomorphic motion K_λ on the sets $S = \{p \in f_0(V_R) : |\pi_2(p))| \geq |w_0|\}$, where $w_0 \in \mathbf{C}$ is as given in lemma 4.4. In order to apply cutoffs to our motions to construct a conjugacy, we must have some control of the behavior of the motions as $|p| \to \infty$. Thus, we construct our motion so that it decomposes as a sequence of motions on complex one-dimensional subsets given by $S_w^\lambda = \{p \in f_\lambda(V_R) : \pi_2(p) = w\} = f_\lambda(V_R) \cap Y_w$, for $|w| > |w_0|$.

Proposition 4.5. *There is a holomorphic motion K_λ, defined on the set S, which can be decomposed into motions $k_\lambda : S_w^0 \to S_w^\lambda$.*

Proof. Given a point $p \in S_w^0$, note that $f_0^{-1}(p) \in V_R$, and in fact, $f_0^{-1}(p) \in X_c$ by lemma 4.4 and the choice of w_0. The definition of X_c implies that c is given by $\pi_1 f_0^{-1}(p)/\pi_2 f_0^{-1}(p)$. As shown in lemma 4.4, the mapping f_λ, defined on the complex line X_c will have d distinct points of intersection with the set S_w^λ. An application of the implicit function theorem yields a mapping $k(\lambda, p) = k_\lambda(p)$, defined on a neighborhood N of S_w, which is a holomorphic motion on the (complex one-dimensional) set S_w for $|w| \sim L$ (note that the maps are actually holomorphic in this case). This can be done for any w as above, and thus we have a holomorphic motion K_λ defined on the set $\{|w| \geq |w_0|\} \cap f_0(V_R)$, which restricts to a holomorphic motion k_λ defined on each of the complex lines S_w^0. $\square$

On the set V_R, we define the motion K_λ simply as the identity for all λ. We note here that by definition, $f_\lambda K_\lambda(X_c) = K_\lambda f_0(X_c)$ (when restricted to $|w| \geq |w_0|$). That is, K_λ is a conjugacy at the level of complex lines but not at the level of points.

In order to define our conjugacy Φ_λ, we must apply cutoffs to make the motion K_λ defined in proposition 4.5 compatible with the mapping defined in lemma 4.2.

Lemma 4.6. *Let K_λ be the holomorphic motion defined in proposition 4.5 on the set $f(V_R) \cap \{|w| \geq |w_0|\}$. Let $|w_0| \leq L_0' < L_0 < L_1 < L_1'$ be positive real numbers, and consider neighborhoods U_0, U_1 and U_2 of the set $\{L_0 < |w| < L_1\} \cap f(D_R)$, with $\overline{U_0} \subseteq U_1$, $\overline{U_1} \subseteq U_2$ and $U_2 = f(V_R) \cap \{L_0' < |w| < L_1'\}$. Then, there is a holomorphic motion $\widetilde{K}_\lambda$ defined on the set $f(V_R)$ so that $\widetilde{K}_\lambda = f_\lambda^{-1} f_0$ on $U_2 \backslash U_1 \cap \{|w| \leq L_0\}$, while $\widetilde{K}_\lambda = K_\lambda$ on $U_0 \cup \{|w| \geq L_1\}$.*

Proof. Let μ be a real-valued smooth function on U_2, $0 \leq \mu(p) \leq 1$, where $\mu = 1$ on $U_0 \cup \{|w| > L_1\}$, $\mu = 0$ on $(U_2 \backslash U_1) \cap \{|w| < L_0\}$. We will define $\widetilde{K}_\lambda(p) = \mu(p) K_\lambda + (1 - \mu(p)) f_\lambda^{-1} f_0(p)$ (with the obvious definition if $|\pi_2(p)| \leq |w_0|$). To see that this is an injective mapping, note that, on a compact set (and all the sets U_i have compact closure), we can write $f_\lambda^{-1} f_0(p) = p + g_\lambda(p)$ and $K_\lambda(p) = p + h_\lambda(p)$, where $|g_\lambda|$, $|h_\lambda|$, $L(g_\lambda)$ and $L(h_\lambda)$ can all be made small as $|\lambda|$ is made small. Thus, we have

$$\widetilde{K}_\lambda(p) = p + \mu(p) g_\lambda(p) + (1 - \mu(p)) h_\lambda(p).$$

Since the derivatives of μ are bounded on the set in question, we can choose λ sufficiently small so that the function $\mu(p) g_\lambda(p) + (1 - \mu(p)) h_\lambda(p)$ is small with small Lipschitz constant. Then, a Lipschitz perturbation argument allows us to conclude that the mapping $\widetilde{K}_\lambda$ is injective. $\square$

We shall drop the tilde notation and refer to the mapping in lemma 4.6 simply as K_λ.

Now, for complex lines Y_w with $|w|$ sufficiently large, we can extend the motion K_λ defined above, by appealing to ideas given in Buzzard and Verma. We give here the definition and theorem to which we shall refer, for the convenience of the reader. For details, please see [BV]:

Definition 4.7. *For each $i \in I$, let $\phi^i : \mathbb{D}^n \times \mathbf{C} \to \mathbf{C}^2$ be holomorphic with $\phi^i_\lambda = \phi^i(\lambda, \cdot)$ injective for each fixed λ, and suppose that ϕ^i converges to ϕ^∞ uniformly on compact sets. Let $E^i \subseteq \phi^i(0, \mathbf{C})$ for each i and let $\tau^i : \mathbb{D}^n \times E^i$ be a holomorphic motion on the leaves defined by ϕ^i. Then τ^i is said to converge uniformly to τ^∞ if the sets $A^i = (\phi^i_0)^{-1}(E^i)$ converge to A^∞ in the Hausdorff metric, and if the corresponding holomorphic motions in* *the plane converge uniformly on compact sets. That is, for each $\varepsilon > 0$, there exists $\delta > 0$ and $N > 0$ such that if $i > N$ and $|\lambda_i - \lambda_2| + d_s(z_1, z_2) < \delta$ for $z_1 \in A^i$ and $z_2 \in A^\infty$, then*

$$d_s((\phi^i_{\lambda_1})^{-1}\tau^i_{\lambda_1}\phi^i_0(z_1), (\phi^\infty_{\lambda_2})^{-1}\tau^\infty_{\lambda_2}\phi^\infty_0(z_2)) < \varepsilon.$$

Here, the metric d_s is the spherical metric defined on the Riemann sphere, and the Hausdorff metric on sets in the plane is defined in terms of d_s.

Then under then notion of convergence above, Buzzard and Verma have proven the following theorem [BV]:

Theorem 4.8. *Let ϕ^i and τ^i be as in the previous definition, and let $\hat{\tau}^i$ denote the Bers-Royden extension of τ^i. Then $\hat{\tau}^i$ converges uniformly to $\hat{\tau}^\infty$.*

In our case, the sets involved and the convergence as defined are very simple to see - they are simply copies of the plane. We give a proof of our extension below, but the idea of the proof is as follows: for $\{|w| > L_1\}$, we have mappings k_λ which are viewed as holomorphic motions defined in one complex variable; these mappings preserve sets of the form S_{w_k} defined above, and by including the set $V_R \cap \{w = w_k\}$, each of these sets may be viewed as concentric annuli in the plane. Viewed in this way, each motion k_λ may in turn be extended via the Bers-Royden process. Injectivity is guaranteed, and moreover, these extensions stack up to define a *continuous* holomorphic motion on the set $\{|w| \geq |w_0|\}$. However, our interest is in a neighborhood of the set bounded by D_R, $f_0(D_R)$ and $\{|w| \geq |w_0|\}$.

More precisely, we have

Lemma 4.9. *The motion K_λ, defined on the set $\{p \in V_R \cup f_0(V_R) : |\pi_2(p)| \geq |w_0|\}$, has a continuous extension to the set bounded by $\{w = w_0\}$, D_R and $f_0(D_R)$.*

Proof. Consider the mappings $\varphi^w : \mathbf{C} \to \mathbf{C}^2$, given by $\varphi^w(z) = (z, w)$. Define as before the sets S_w, and let $\widetilde{S}_w = \{p \in V_R : \pi_2(p) = w\}$. The holomorphic motion K_λ, which is defined as above on $\{|w| \geq L_0'\} \cap f(V_R)$, and defined to be the identity on $(\{|w| \leq L_0\} \cap f(V_R)) \cup V_R$, can be decomposed into holomorphic motions $\{\tau^w_\lambda\}$ defined on each of the sets $S_w \cup \widetilde{S}_w$ (recall that $K_\lambda(S_w \cup \widetilde{S}_w) \subseteq Y_w$, by definition). The motions given here obviously converge as in definition 4.7, and hence we may apply theorem 4.8 to the Bers-Royden extensions of these motions to yield a mapping $\widetilde{k}_\lambda$, which is a holomorphic motion on those portions of Y_w bounded by D_R and $f(D_R)$, and which is continuous on the set bounded by D_R, $f(D_R)$ and $\{(z, w) : w = w_0\}$. □

We are in a position to define the conjugacy on $U_R \backslash C$, where U_R is the set bounded by D_R and $f(D_R)$ (note that $\overline{U_R}$ is a fundamental domain for f) and C is compact. Once this is accomplished, we can extend to K via cutoff functions, and then to $\mathbf{C}^2$ by virtue of the dynamics.

We have defined a mapping K_λ on the set $f(V_R) \cup V_R$ which is the identity on V_R, which was then extended via the Bers-Royden theorem across lines of the form Y_w for w with modulus sufficiently large. We now redefine K_λ on the set V_R so that $f_\lambda K_\lambda(p) = K_\lambda f_0(p)$ for all $p \in D_R$. Of course, such a map is already predetermined - namely, given $p \in D_R$, we define $K_\lambda(p) = f_\lambda^{-1} K_\lambda f_0(p)$ (since K_λ is already defined at the point $f_0(p)$, via lemma 4.2, proposition 4.5 and lemma 4.6). We now augment this map so that outside V_R, $K_\lambda(p) = p$, so that the Bers-Royden extension defined in lemma 4.9 remains continuous (this argument is very similar to the one found in [B1], theorem 3.8).

The idea is as follows: since K_λ is a conjugacy at the level of complex lines X_c, it suffices to move points within X_c. Let $c_0 < c_1 < c_2 < 1$, where c_0 is the constant used in the definition of V_R. Let μ be a smooth function defined on $\mathbf{C}$, real-valued, with $\mu(c) = \mu(|c|)$ and $0 \leq \mu \leq 1$. We require that $\mu(c) = 1$ if $c_2 \leq |c| \leq \frac{1}{c_2}$, and $\mu(c) = 0$ if $|c| \leq c_1$ or $|c| \geq \frac{1}{c_1}$. Again, for any point $(z, w) \in V_R$ with $|z| > R$, we can write $p = (cw, w)$, where $c_0 < |c| < \frac{1}{c_0}$. Now, on these points $p \in V_R$, we define $K_\lambda(p) = K_\lambda(cw, w) = f^{-1}_{\mu(c)\lambda} K_{\mu(c)\lambda} f_0(cw, w)$. Note that this map preserves all lines of the form X_c, and that on any such line, μ is constant, so injectivity is guaranteed. Continuity is clear (in fact, the map is smooth away from J^+). Finally, note that $K_\lambda(p) = p$ if $p \notin V_R$, while for $p \in D_R$, the conjugacy relation holds, so that our map is compatible with the map defined by the Bers-Royden extension.

Finally, putting our motions together, we have constructed a homeomorphism on the set described in figure 4 which satisfies the necessary conjugacy relations on the sets D_R and $f(D_R)$ (in the end, our interest is only on the fundamental domain U_R defined above). In the same manner as the proof of Lemma 4.2, we may apply cutoffs to extend this homeomorphism across C (simply write $\Phi_\lambda = p + \phi_\lambda(p)$, near C, and note that ϕ_λ can be made uniformly small with uniformly small Lipschitz constant, provided that $|\lambda|$ is sufficiently small). Finally, via an argument like that of Proposition 4.1, we can then extend this map to all of $\mathbf{C}^2$ by appealing to the dynamics of the diffeomorphism. Thus, the conjugacy is constructed.

References

[BS] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$: currents, equilibrium measure and hyperbolicity, *Invent. Math.*, 103 (1991), no. 1, 69–99.

[BR] L. Bers and H.L. Royden, Holomorphic families of injections, *Acta Math.*, 157 (1986), 259-286

[B1] G. Buzzard, Extensions of Hénon maps to the closed 4-ball, *Ergod. Th. Dyn. Sys.*, 20 (2000), 1319-1334.

[B2] G. Buzzard, Infinitely many periodic attractors for holomorphic maps of 2 variables, *Ann. of Math.*, 145 (1997), 389-417.

[B3] G. Buzzard, Kupka-Smale theorem for automorphisms of $\mathbf{C}^n$, *Duke Math. J.*, 93 (1998), No. 3, 487-503.

[B4] G. Buzzard, Nondensity of stability for polynomial automorphisms of $\mathbf{C}^2$, *Indiana U. Math. J.*, 48 (1999), no. 3, 857-865.

[BHI] G. Buzzard, S. L. Hruska, Y. Ilyashenko, Kupka-Smale theorem for polynomial automorphisms of $\mathbb{C}^2$ and persistence of heteroclinic intersections, *Invent. Math.*, 161 (2003), 45-89.

[BV] G. Buzzard and K. Verma, Hyperbolic automorphisms and holomorphic motions in $\mathbf{C}^2$ *Mich. Math. J.*, 49 (2001), no. 3, 541-565.

[dMvS] W. de Melo and S. van Strien, Diffeomorphisms on surfaces with a finite number of moduli, *Ergod. Th. Dyn. Sys.*, 7 (1987), no. 3, 415-462.

[D] J. Diller, Dynamics of birational maps of $\mathbb{P}^2$, *Indiana U. Math. J.*, 45 (1996), 721-772.

[FM] S. Friedland and J. Milnor, Dynamical properties of polynomial automorphisms, *Ergod. Th. Dyn. Sys.*, 9 (1989), 67-99.

[FS] J. Fornaess and N. Sibony, Complex dynamics in higher dimension. I., *Astrisque*, No. 222 (1994), 5, 201-231.

[FS2] J. Fornaess and N. Sibony, Complex dynamics in higher dimension. II., Modern methods in complex analysis, (Princeton, NJ, 1992), 135-182.

[HO] J. Hubbard, R. Oberste-Vorth, Linked solenoid mappings and the non-transversality locus invariant, . *Indiana U. Math. J.*, 50 (2001), no. 1, 553-566.

[HP] M. Hirsch and C. Pugh, Stable manifolds and hyperbolic sets, Proc. Symp. Pure Math., AMS, 14 (1970), 133-164.

[HPV] J. Hubbard, P. Papadopol, V. Veselov, A compactification of Hénon mappings in $\mathbf{C}^2$ as dynamical systems, *Acta. Math.*, 184 (2000), 203-270.

[J] M. Jonsson, Holomorphic motions of hyperbolic sets, *Michigan Math. J.*, 45 (1998), no. 3, 409-415.

[M] R. Mañé, A proof of the C^1 stability conjecture, *IHES Publ. Math.*, no. 66 (1988), 161-210.

[MSS] R. Mañé, P. Sad, D. Sullivan, On the dynamics of rational maps, *Ann. Sci. Ec. Norm. Sup.*, 16 (1983), 193-217.

[MS] C. McMullen, D. Sullivan, Quasiconformal homeomorphisms and dynamics, III, *Adv. Math.*, 135 (1998), 351-395.

[NPT] S. Newhouse, J. Palis, F. Takens, Bifurcations and stability of families of diffeomorphisms, *Inst. Hautes Etudes Sci. Publ. Math.*, 57 (1983), 5-71.

[Ro] J. Robbin, A structural stability theorem, *Ann. Math.*, 99 (1971), 447-493

[R1] C. Robinson, *Dynamical systems : stability, symbolic dynamics, and chaos*, CRC Press, Boca Raton, 1995.

[R2] C. Robinson, Structural stability of C^1-diffeomorphisms, *J. Diff. Eq.*, 22 (1977), 28-73.

Department of Mathematics, Purdue University, West Lafayette, IN, 47907
E-mail address: `buzzard@math.purdue.edu`

Department of Mathematics, Purdue University, West Lafayette, IN, 47907
E-mail address: `majenkin@math.purdue.edu`