Complete holomorphic vector fields and time-1 maps

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1 Introduction

In this paper we examine some questions related to holomorphic vector fields on $\mathbb{C}^2$ and their associated time-1 maps. We say that a holomorphic vector field on $\mathbb{C}^2$ is complete if it is integrable for all complex values of time. By [F1], this is equivalent to being integrable for all real values of time.

Note that any time-1 map of a complete holomorphic vector field is an automorphism of $\mathbb{C}^2$, i.e. a holomorphic diffeomorphism of $\mathbb{C}^2$. We denote the space of such maps by $\mathrm{Aut}(\mathbb{C}^2)$ and endow it with the topology of uniform convergence on compacts applied both to the map and its inverse.

In this paper we show that there is an open dense subset $\mathcal{S}$ of $\mathrm{Aut}(\mathbb{C}^2)$ such that each element of $\mathcal{S}$ is not the time-1 map for any complete holomorphic vector field on $\mathbb{C}^2$. We also give a precise classification of those polynomial automorphisms of $\mathbb{C}^2$ which are the time-1 map of a complete holomorphic vector field.

Along other lines we give examples of holomorphic vector fields in $\mathbb{C}^2$ which cannot be approximated by complete holomorphic vector fields and noncomplete vector fields which can be approximated by complete ones.

2 Background

In this section we provide some background on the ideas of complete holomorphic vector fields.

A time independent holomorphic vector field X on $\mathbb{C}^2$ is simply a holomorphic map $X = (X_1, X_2) : \mathbb{C}^2 \to \mathbb{C}^2$, where we identify X with the vector field $X_1\partial/\partial z + X_2\partial/\partial w$. The flow of X is a map ϕ which is a solution of the ordinary differential equation

$$\frac{d}{ds}\phi_s(p)|_{s=t} = X(\phi_t(p)), \quad \phi_0(p) = p.$$

It is standard [H] that for each $p \in \mathbb{C}^2$, the solution ϕ_t exists for t in some neighborhood of the origin in $\mathbb{C}$, is unique, and is holomorphic in (t, p). Moreover, such ϕ satisfies the group property $\phi_s\phi_t = \phi_{s+t}$ for s and t near 0.

If $\phi_1(p)$ and $\phi_{-1}(p)$ are defined for all $p \in \mathbb{C}^2$, then we see at once that X is complete in real time, hence in complex time by [F1]. In such a case, ϕ_1 is an automorphism of $\mathbb{C}^2$, and we say that ϕ_1 is the time-1 map of X.

Fixing p, we can think of $\phi_t(p)$ as an analytic map from a neighborhood of the origin in $\mathbb{C}$ to $\mathbb{C}^2$, and we can extend the domain of definition of $\phi_t(p)$ in the time plane using analytic continuation along paths starting at the origin. The maximal domain of definition is then a Riemann surface, R_p, spread over $\mathbb{C}$ and is multiply sheeted in general. For more details and further results, see [F1].

3 Automorphisms as time-1 maps

In this section we collect some results about the set of automorphisms which can arise as the time-1 map of a holomorphic vector field.

We say that a point p is periodic of minimal period d for a map F if $F^d(p) = p$ but $F^k(p) \neq p$ for $d \in \{1, \ldots, d-1\}$.

The following proposition is a simple consequence of well-known ideas and applies in the differentiable case as well. We include the proof for completeness.

PROPOSITION 3.1 *Let F be an automorphism of $\mathbb{C}^2$ and suppose that F has a periodic point p of minimal period $d \geq 2$ which is isolated in the set of periodic points of period d. Then F is not the time-1 map of a time independent holomorphic vector on $\mathbb{C}^2$.*

Proof: Suppose that F is the time-1 map of a holomorphic vector field X, and let ϕ_t be the associated flow. Since p has period d and $\phi_d = F^d$, we see that $\phi_d(p) = p$, and hence $\phi_t(p)$ describes a closed curve for $t \in [0, d]$. Since $\phi_1(p) \neq p$ by assumption, this curve is nondegenerate.

Now by the group property, for any $t \in [0, d]$ we have $\phi_d\phi_t(p) = \phi_t\phi_d(p) = \phi_t(p)$. Hence each $\phi_t(p)$ is a periodic point of period d, which contradicts the fact that p is isolated in the set of periodic points of period d. Thus F cannot be the time-1 map of a holomorphic vector field. ■

We say that a periodic point p of minimal period d for a map F is hyperbolic if none of the eigenvalues of $(DF^d)(p)$ has modulus 1. By the inverse function theorem, such a point is always isolated in the set of periodic points of period d.

For notation, let $\mathbf{B}(0; R)$ denote the ball of radius R centered at 0 in $\mathbb{C}^2$.

THEOREM 3.2 *There is an open dense subset $\mathcal{S} \subseteq \mathrm{Aut}(\mathbb{C}^2)$ such that no element of $\mathcal{S}$ is the time-1 map of a holomorphic vector field on $\mathbb{C}^2$.*

Proof: We let $\mathcal{S}$ be the set of automorphisms of $\mathbb{C}^2$ which have a hyperbolic periodic point of minimal period 2. By the implicit function theorem, hyperbolic periodic points are persistent under small perturbations of the map, so $\mathcal{S}$ is open, and from the previous proposition we see that no element of $\mathcal{S}$ is the time-1 map of a holomorphic vector field on $\mathbb{C}^2$.

It remains to show that $\mathcal{S}$ is dense. Let $F \in \mathrm{Aut}(\mathbb{C}^2)$ with $F \not\equiv I$, and let $R, \epsilon, \delta > 0$. Since the set of fixed points of F is an analytic set of dimension at most 1, we can choose a point $p \in \mathbb{C}^2 - (\mathrm{B}(0; 3R) \cup F^{-1}(\mathrm{B}(0; 3R)) \cup F(\mathrm{B}(0; 3R)))$ such that $p \neq F(p)$. Using techniques like those in [RR, corollary 1.3], we can find $\Psi \in \mathrm{Aut}(\mathbb{C}^2)$ such that $\Psi(p) = p$, $\Psi(F(p)) = F^{-1}(p)$, and such that Ψ is within δ of the identity on $\mathrm{B}(0; 2R)$. Then p is a periodic point of minimal period 2 for the map ΨF, and using [B, lemma 2.6], we can find $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ arbitrarily near the identity such that p is a hyperbolic periodic point of minimal period 2 for $\Phi\Psi F$.

For δ small and Φ near the identity, we see that $\Phi\Psi F$ and $(\Phi\Psi F)^{-1}$ will be near F and F^{-1}, respectively, on $\mathrm{B}(0; R)$. Since $\Phi\Psi F \in \mathcal{S}$, we see that $\mathcal{S}$ is dense. ∎

In the case of polynomial automorphisms, that is, automorphisms such that each coordinate function is a polynomial, we can give a precise description of those maps which can be the time-1 map of a holomorphic vector field.

From [FM], we know that a polynomial automorphism of $\mathbb{C}^2$ is conjugate either to an affine linear map, an elementary map preserving sets of the form $w = \text{const}$, or to a generalized Hénon map. In the following theorem we show that a polynomial automorphism of $\mathbb{C}^2$ which is the time-1 map of a holomorphic vector field must be conjugate to one of a few types of elementary maps.

THEOREM 3.3 *Let F be a polynomial automorphism of $\mathbb{C}^2$, and suppose that F is the time-1 map of a vector field X on $\mathbb{C}^2$. Then F is conjugate via a polynomial automorphism to one of the following maps.*

(a) $(z, w) \mapsto (\alpha z, \beta w)$

(b) $(z, w) \mapsto (\alpha z, w + 1)$

(c) $(z, w) \mapsto (\beta^d (z + w^d), \beta w),\ d \geq 1$

(d) $(z, w) \mapsto (z + w^\mu q(w), w)$,

where in each case $\alpha, \beta \neq 0$, and in case (d), μ is a nonnegative integer and $q(w) = w^k + q_{k-1} w^{k-1} + \cdots + q_1 w + 1$ with $k \geq 1$ and $q_{k-1} = 0$ if $k \geq 2$. Moreover, each of these maps can be realized as the time-1 map of a flow ϕ such that ϕ_t is a polynomial automorphism of $\mathbb{C}^2$ for all t.

Proof: Suppose F is not conjugate to an affine or elementary map. Then by [FM], F is cyclically reduced, and hence has dynamical degree $d \geq 2$ in the sense of [BLS].

For $n \geq 1$, let Fix_n denote the set of fixed points of F^n, and let Per_n be the set of points in Fix_n which are not in Fix_j for $j < n$. By [FM], we have $\#\mathrm{Fix}_n \leq d^n$, and by [BLS], we have

$$\lim_{n\to\infty} \frac{1}{d^n} \#\mathrm{Per}_n = 1.$$

In particular, we can choose $p \in \mathrm{Per}_d$ for some $d \geq 2$, in which case p is isolated in the set of fixed points of period d since there are only finitely many such points. Hence by proposition 3.1 we see that F is not the time-1 map of a holomorphic vector field.

In the remaining cases, F is either an affine linear map or conjugate to an elementary map. If F is affine, we may use a linear change of coordinates to make DF upper triangular, so we see that F is conjugate to an elementary map in this case also. Hence by [FM], F is conjugate either to one of the maps in (a) - (c), or to

(d′) $(z, w) \mapsto (\beta^\mu(z + w^\mu q(w^r)), \beta w)$,

where μ is a nonnegative integer, β is a primitive rth root of unity, and $q(w) = w^k + q_{k-1}w^{k-1} + \cdots q_1 w + 1$. Here $k \geq 1$, and $q_{k-1} = 0$ if $\beta = r = 1$ and $k \geq 2$. By [BM], the maps in (a) - (d) can all be realized as the time-1 map of a flow on $\mathbb{C}^2$ as in the statement of the theorem, so all that remains is to show that (d′) cannot be realized as a time-1 map if $r > 1$.

Let $F(z, w) = (\beta^\mu(z + w^\mu q(w^r)), \beta w)$, be as in (d′) with $r > 1$ and hence $\beta \neq 1$. Then the fixed point set of $F^r(z, w) = (z + rw^\mu q(w^r), w)$ is the set of points of the form (z, w_j), where the w_j's are the roots of $rw^\mu q(w^r) = 0$. Since q has a nonzero root, we may assume $w_1 \neq 0$.

Suppose that F is the time-1 map of a vector field with flow ϕ_t. Using an argument like that in proposition 3.1, we see that each point of the form $\phi_t(z, w_1)$ for $t \geq 0$ is a fixed point for F^r, hence is contained in one of the sets $\mathbb{C} \times \{w_j\}$ for each t. Since the w_j's form a discrete set, we see that the set $\mathbb{C} \times \{w_1\}$ is invariant under ϕ_t, hence under F. But this is impossible since $\beta w_1 \neq w_1$. Thus F is not the time-1 map of a holomorphic vector field. ■

4 Approximability of vector fields by complete vector fields

We say that a holomorphic vector field X can be approximated by complete holomorphic vector fields if there is a sequence X_j of holomorphic vector fields converging to X uniformly on compact sets in $\mathbb{C}^2$. In [F1], the question was raised as to whether every holomorphic vector field on $\mathbb{C}^2$ can be approximated by complete ones.

In this section we give two examples showing that such approximation is not always possible and provide some general obstructions to such approximation. We also give an example of a noncomplete holomorphic vector field which is approximable by complete ones.

We say that a fixed point p of a vector field X is attracting if there is a neighborhood U of p such that $\lim_{t\to+\infty} \phi_t(q) = p$ uniformly for all $q \in U$. The maximal domain with this property is called the basin of attraction of p. A fixed point p is a saddle point if $(D\phi_1)(p)$ has one eigenvalue larger than 1 in modulus and one smaller than 1. In this case, there are 1-dimensional immersed complex submanifolds $W^s(p)$ and $W^u(p)$ defined by

$$\begin{aligned} W^s(p) &= \{q \in \mathbb{C}^2 : \lim_{t\to+\infty} \phi_t(q) = p\} \\ W^u(p) &= \{q \in \mathbb{C}^2 : \lim_{t\to-\infty} \phi_t(q) = p\}, \end{aligned}$$

where in each case we restrict to time values $t \in \mathbb{R}$.

If X is complete, there is a bijective holomorphic map $H : \mathbb{C} \to W^s(p)$ with $H(0) = p$, and likewise a map from $\mathbb{C}$ to $W^u(p)$. Moreover, if q is contained in the stable manifold of p, then for $s \in \mathbb{C}$, we have from the group property that

$$\begin{aligned} \lim_{t\to+\infty} \phi_t(\phi_s(q)) &= \phi_s(\lim_{t\to+\infty} \phi_t(q)) \\ &= p, \end{aligned}$$

where the last equality follows from $\phi_s(p) = p$. Hence $\phi_s(q) \in W^s(p)$ for all $s \in \mathbb{C}$, and analogous statements are true for the unstable manifold or for the basin of attraction of a fixed point.

PROPOSITION 4.1 *The vector field $X = (z(z-1), -w)$ cannot be approximated by complete holomorphic vector fields on $\mathbb{C}^2$.*

Proof: Note that the associated flow for initial value (z, w) has the form

$$\phi_t(z, w) = \left(\frac{z}{z + e^t(1-z)}, e^{-t}w\right).$$

Hence X has an attracting fixed point at $(0,0)$ and a saddle fixed point at $(1,0)$. In particular, for any z in the interval $(0,1)$, we see that $\phi_t(z,0)$ is defined for all $t \in \mathbb{R}$ and $\lim_{t\to+\infty} \phi_t(z,0) = (0,0)$ and $\lim_{t\to-\infty} \phi_t(z,0) = (1,0)$. Thus the unstable manifold for $(1,0)$ intersects the basin of attraction for $(0,0)$.

By the stable manifold theorem, we see that for a holomorphic vector field Y sufficiently near X, the flow of Y will have an attracting fixed point p_1 near $(0,0)$ and a saddle fixed point p_2 near $(1,0)$, and the unstable manifold for p_2 will intersect the basin of attraction for p_1.

Suppose now that Y is a complete holomorphic vector field near X with these properties, and let ψ_t be the corresponding flow. Pick a point p contained in the basin of attraction of p_1 and in the unstable manifold of p_2. Since Y is complete, there is an injective holomorphic map H from $\mathbb{C}$ onto the unstable manifold of p_2 sending 0 to p_2. Since the set $\{\phi_s(p) : s \in \mathbb{C}\}$ is contained in the unstable manifold of p_2 but does not include p_2, we see that $H^{-1}\phi_t(p)$ is entire, nonzero, and nonconstant, hence maps onto $\mathbb{C} - \{0\}$. Thus every point of $W^u(p_2) - \{p_2\}$ is contained in $\{\phi_s(p) : s \in \mathbb{C}\}$, hence in the basin of attraction of p_1.

Now, letting γ be the image of the unit circle under the map H, we see that if $q \in \gamma$, then $\|\phi_t(q) - p_1\| < \epsilon$ for t sufficiently large and positive. By compactness, we can choose t large enough so that this is true uniformly on γ. But then for ϵ sufficiently small, the map $\phi_t H(z) - p_1$ will violate the maximum principle. Hence no such Y exists. ■

We give a second example of a holomorphic vector field which is not approximable by complete ones using methods which generalize to many holomorphic vector fields.

PROPOSITION 4.2 *The vector field $X = (z^2, 0)$ is not approximable by complete holomorphic vector fields on $\mathbb{C}^2$.*

Proof: Note that the flow of X is given by

$$\phi_t(z, w) = \left(\frac{z}{1 - tz}, w\right).$$

In particular, for $p = (1, 0)$, the maximal domain of definition of $\phi_t(p)$ is the domain $R_p = \mathbb{C} - \{1\}$. Let γ_1 be the boundary of the disk centered at 1 with radius 1, and let γ_2 be the interval $[0, 3/4]$ on the real axis. Given $\epsilon > 0$, it is standard that if Y is a holomorphic vector field sufficiently close to X, then the flow $\psi_t(p)$ for Y will be defined for $t \in \gamma_1 \cup \gamma_2$ and will be within ϵ of $\phi_t(p)$ on that set.

For ϵ sufficiently small, we will have $\|\psi_{3/4}(p)\| > \sup\{\psi_t(p) : |1 - t| = 1\}$, and hence $\psi_t(p)$ cannot be holomorphic throughout the disk of radius 1 centered at 1 by the maximum principle. Thus Y cannot be complete, so X is not approximable by complete holomorphic vector fields. ■

Suppose X is a holomorphic vector field with flow ϕ_t. If the Riemann surface R_p is multiply connected for some $p \in \mathbb{C}^2$, then we can choose arcs $\gamma_1, \gamma_2 : [0, 1] \to R_p$ such that $\gamma_1(0) = \gamma_2(0) = 0$ and such that $\gamma_1(1)$ and $\gamma_2(1)$ project to the same point in $\mathbb{C}$ but $\phi_{\gamma_1(1)}(p) \neq \phi_{\gamma_2(1)}(p)$.

In this case, for $\epsilon > 0$ and Y sufficiently close to X, the flow $\psi_t(p)$ of Y can be extended along the image of γ_j in $\mathbb{C}$ for $j = 1, 2$ and will be within ϵ of $\phi_t(p)$ there. For ϵ small we see that $\psi_t(p)$ is not single valued in $\mathbb{C}$, and hence Y cannot be complete. Thus X cannot be approximated by complete holomorphic vector fields.

Finally, if R_p is a multiply connected domain in the plane, then we can surround a compact boundary component of R_p with a Jordan curve γ_1 in R_p, then join γ_1 to this compact boundary component with an arc γ_2 contained in R_p union the boundary component. Then $\phi_t(p)$ must be unbounded along γ_2 since otherwise we could extend it past the boundary of R_p. Hence we can use the same argument as in proposition 4.2 to show that X cannot be approximated by complete holomorphic vector fields on $\mathbb{C}^2$. Thus we obtain the following.

THEOREM 4.3 *Suppose that X is a holomorphic vector field on $\mathbb{C}^2$ with associated flow ϕ_t and that X is approximable by complete holomorphic vector fields. Then for each $p \in \mathbb{C}^2$, the maximal domain of definition of $\phi_t(p)$ is a simply connected domain in the plane.*

Using the same ideas, this theorem is proved in [F2] for holomorphic vector fields on Stein manifolds. That paper also exhibits various classes of vector fields which cannot be approximated by complete holomorphic vector field.

Given any holomorphic vector field X on $\mathbb{C}^2$ and $R, \epsilon > 0$, we can find a vector field Y which is within ϵ of X on the set $\mathbb{B}(0; R)$ and which has an unstable manifold for one fixed point intersecting the basin of attraction for another fixed point as in proposition 4.1. In this case, Y cannot be approximated by complete holomorphic vector fields, so we see that the set of nonapproximable holomorphic vector fields is dense in the space of all holomorphic vector fields. Since the set of approximable ones is by definition the closure of the complete vector fields, we obtain the following theorem.

THEOREM 4.4 *The set of holomorphic vector fields on $\mathbb{C}^2$ which are not approximable by complete holomorphic vector fields is an open dense subset of the space of all holomorphic vector fields.*

We conclude with an example of a holomorphic vector field which is not complete but which is approximable by complete ones. In [F1], a vector field X is constructed by using a nonzero constant vector field on $\mathbb{C}^2$ and pulling this field back to $\mathbb{C}^2$ using a biholomorphic map $F : \mathbb{C}^2 \to B$, where B is the basin of attraction of a fixed point of a polynomial automorphism. In this case, the vector field X is not complete since the constant vector field forces any point in the basin of attraction to reach the boundary of the basin in finite time.

However, the map F is approximable by automorphisms F_j of $\mathbb{C}^2$ and using these maps to pull back the constant vector field, we obtain complete holomorphic vector fields X_j which approximate X uniformly on compact sets.

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