Name: ____________________
PUID#: ____________________

Midterm 2 Version B– Math 527 (11/07/12)

Books, notes, calculators, and any electronic devices are NOT allowed during the exam. The exam is one hour and 15 minute long.

Please write your answers in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11
Answer	E	D	D	C	B	E	E	C	C	A	
Points											

103-105 90-95 80-85 70-75 60-65

21 11 8 7 4

50-55 45 30

5 1 1

1. (10pts) Let $f(t) = \begin{cases} t, & 0 \le t < 4, \\ t^2 + t, & 4 \le t < \infty. \end{cases}$ Then the Laplace transform $\mathcal{L}\{f\} =$

$A.\ \frac{2}{s^3}$; $B.\ \frac{1}{s^2} + e^{-4s}\left(\frac{2}{s^3} - \frac{1}{s^2}\right)$; $C.\ \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$; $D.\ \frac{1}{s^2} + e^{-4s}\left(\frac{2}{s^3} - \frac{8}{s^2} + \frac{16}{s}\right)$;

$E.\ \frac{1}{s^2} + e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$.

$$f(t) = t\left[1 - u(t-4)\right] + (t^2 + t)\,u(t-4)$$

$$= t + \left[(t-4)+4\right]^2 u(t-4)$$

$$= t + \left\{(t-4)^2 + 8(t-4) + 16\right\} u(t-4)$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} + e^{-4s}\left[\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right]$$

2. (10pts) Let $F(s) = \mathcal{L}\{t\cos t\}$, then $F(1) =$

A. -2;

B. -1;

C. $1/2$;

D. 0;

E. 1.

$$F(s) = \mathcal{L}\{t\cos t\} = -\left[\mathcal{L}\{\cos t\}\right]'$$

$$= -\left(\frac{s}{s^2+1}\right)' = \frac{s^2-1}{(s^2+1)^2}$$

$$F(1) = 0$$

3. (10pts) Let $F(s) = \frac{se^{-2s}}{s^2+4s+13}$. Then the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\} =$

A. $u(t-2)e^{-2t}\left[\cos(3(t-2)) - \sin(3(t-2))\right]$; B. $u(t-2)e^{-2(t+2)}\left[\cos(3(t+2)) - \frac{2}{3}\sin(3(t+2))\right]$;
C. $u(t-2)e^{-2t}\left[\cos(3t) - \frac{2}{3}\sin(3t)\right]$; D. $u(t-2)e^{-2(t-2)}\left[\cos(3(t-2)) - \frac{2}{3}\sin(3(t-2))\right]$;
E. $e^{-2(t+2)}\left[\cos(3(t+2)) - \frac{2}{3}\sin(3(t+2))\right]$.

$$F(s) = \frac{s+2}{(s+2)^2+3^2}e^{-2s} - \frac{2}{3}\frac{3}{(s+2)^2+3^2}e^{-2s}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} = e^{-2t}\cos 3t, \quad \mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} = e^{-2t}\sin 3t$$

$$\mathcal{L}^{-1}\{F(s)\} = u(t-2)e^{-2(t-2)}\left[\cos 3(t-2) - \frac{2}{3}\sin 3(t-2)\right]$$

4. (10pts) Let

$$f(t) = \int_0^t (t-\tau)^3\cos(3\tau)\,d\tau.$$

Then the Laplace transform $\mathcal{L}\{f\} =$

$A.\ \frac{6}{s^4} + \frac{3s}{s^2+9}$; $B.\ \frac{18s}{s^4(s^2+9)}$; $C.\ \frac{6s}{s^4(s^2+9)}$; $D.\ \frac{6}{s^4(s^2+9)}$; $E.\ \frac{s}{s^4(s^2+9)}$.

$$\mathcal{L}\{f\} = \mathcal{L}\{t^3\}\mathcal{L}\{\cos 3t\}$$

$$= \frac{3!}{s^4}\cdot\frac{s}{s^2+9} = \frac{6s}{s^4(s^2+9)}$$

5. (10pts) What is the Laplace transform of the solution of initial value problem

$$y''(t) + 2y'(t) + 2y(t) = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0?$$

$A.\ \dfrac{s+2}{s^2+2s+2}$; $B.\ \dfrac{e^{-\pi s}+2+s}{s^2+2s+2}$: $C.\ \dfrac{e^{-\pi s}+2}{s^2+2s+1}$; $D.\ \dfrac{e^{-\pi s}+2+s}{s^2+2s+1}$; $E.\ \dfrac{e^{-\pi s}+2}{s^2+2s+2}$.

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + 2Y(s)$$

$$= (s^2+2s+2)Y(s) - (s+2) = \mathcal{L}\{\delta(t-\pi)\} = e^{-\pi s}$$

$$Y(s) = \frac{e^{-\pi s} + s + 2}{s^2+2s+2}$$

6. (10pts) Let $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t. \end{cases}$ $= 3u(t-2)$ What is the Laplace transform of the solution of initial value problem

$$y''(t) + y(t) = f(t), \quad y(0) = 2, \quad y'(0) = 1.$$

$A.\ \dfrac{3e^{-2s}+2s^2}{s(s^2+1)}$; $B.\ \dfrac{3e^{-2s}+s}{s(s^2+1)}$; $C.\ \dfrac{3+s+2s^2}{s(s^2+1)}$; $D.\ \dfrac{3e^{-2s}+s^2+2s}{s(s^2+1)}$; $E.\ \dfrac{3e^{-2s}+s+2s^2}{s(s^2+1)}$.

$$s^2Y(s) - sy(0) - y'(0) + Y(s)$$

$$= (s^2+1)Y(s) - (2s+1) = \mathcal{L}\{3u(t-2)\} = 3\frac{e^{-2s}}{s}$$

$$Y(s) = \frac{1}{s^2+1}\left[\frac{3}{s}e^{-2s} + 2s + 1\right] = \frac{3e^{-2s} + 2s^2 + s}{s(s^2+1)}$$

7. (10pts) Let $f(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & 1 < x < 2. \end{cases}$ Then the Fourier-Sine series satisfies which of the following?

$A.\ b_3 = 0,\ b_4 = -\frac{1}{\pi}$; $B.\ b_3 = \frac{2}{3\pi},\ b_4 = \frac{1}{\pi}$; $C.\ b_3 = \frac{2}{3\pi},\ b_4 = -\frac{1}{\pi}$ $D.\ b_3 = \frac{2}{\pi},\ b_4 = \frac{1}{\pi}$;

$E.$ None of the above.

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx = \int_1^2 \sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2 = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_3 = \frac{2}{3\pi}\left[\cos\frac{3\pi}{2} - \cos 3\pi\right] = \frac{2}{3\pi}$$

$$b_4 = \frac{1}{2\pi}\left[\cos 2\pi - \cos 4\pi\right] = 0$$

8. (10pts) Let $f(x) = \begin{cases} 0, & -1 < x < 0, \\ 2, & 0 < x < 1 \end{cases}$ and $f(x+2) = f(x)$ for all x. Then the Fourier series satisfies which of the following?

$A.\ a_0 = 1,\ a_2 \cancel{=} \frac{4}{\pi}$; $B.\ a_0 \cancel{=} 0,\ b_2 \cancel{=} \frac{4}{3\pi}$; $C.\ a_1 = 0,\ b_1 = \frac{4}{\pi}$; $D.\ a_0 = 1,\ b_1 \cancel{=} \frac{2}{\pi}$;

$E.\ a_1 \cancel{=} 1,\ b_2 = \frac{4}{3\pi}$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_0^1 2\,dx = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx = 2\int_0^1 \cos n\pi x\,dx = \frac{2\sin n\pi x}{n\pi}\Big|_0^1 = 0$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = 2\int_0^1 \sin n\pi x\,dx = -\frac{2\cos n\pi x}{n\pi}\Big|_0^1$$

$$= \frac{2}{n\pi}(1 - \cos n\pi), \quad b_1 = \frac{4}{\pi}, \quad b_2 = \frac{1}{\pi}(1-1) = 0$$

9. (10pts) Eigenvalues and eigenfuctions for the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$$

are for $k = 0, 1, 2, \cdots$,

A. $\lambda_k = \left(\frac{2k+1}{2}\right)^2$ and $y_k(x) = \sin \frac{2k+1}{2} x$;

B. $\lambda_k = k^2$ and $y_k(x) = \cos kx$;

C. $\lambda_k = \left(\frac{2k+1}{2}\right)^2$ and $y_k(x) = \cos \frac{2k+1}{2} x$;

D. $\lambda_k = k^2$ and $y_k(x) = \sin kx$;

E. None of the above.

$$0 = s^2 + \lambda \Rightarrow s = \pm\sqrt{-\lambda}$$

$\underline{\lambda = -\nu^2}$ $\quad s = \pm \nu, \quad y = c_1 e^{-\nu t} + c_2 e^{\nu t} \Rightarrow y(t) \equiv 0$

$\underline{\lambda = 0}$ $\quad y = c_1 + c_2 t \Rightarrow y(t) \equiv 0$

$\underline{\lambda = \nu^2}$ $\quad s = \pm \nu i, \quad y = c_1 \cos \nu t + c_2 \sin \nu t$

$$y' = -\nu c_1 \sin \nu t + \nu c_2 \cos \nu t$$

$$0 = y'(0) = \nu c_2 \Rightarrow c_2 = 0$$

$$0 = y(\pi) = c_1 \cos \nu \pi \Rightarrow \cos \nu \pi = 0 \Rightarrow \nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

$$= \frac{2k+1}{2}, \quad k = 0, 1, 2, \ldots$$

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2 \quad \text{for } k = 0, 1, 2, \ldots$$

$$y_k(t) = \cos \frac{2k+1}{2} x$$

10. (10pts) Let $f(x) = \begin{cases} 2x/\pi, & 0 < x < \pi/2, \\ 2(\pi - x)/\pi, & \pi/2 < x < \pi \end{cases}$ and let $f(x)$ be **even** and **periodic** function with period 2π. Fourier series of $f(x)$ is

$$f(x) = \frac{1}{2} - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos(6x) + \frac{1}{10^2}\cos 10x + \cdots\right).$$

Then the sum of the series $\frac{1}{2^4} + \frac{1}{6^4} + \frac{1}{10^4} + \cdots$ is

A. $\frac{\pi^4}{6 \times 16^2}$; B. $\frac{13\pi^4}{24 \times 16^2}$; C. $\frac{2\pi^4}{3 \times 16^2}$; D. $\frac{5\pi^4}{12 \times 16^2}$; E. None of the above.

(Hint: Using the Parseval's identity)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\frac{2}{\pi}\right)^2 x^2\,dx + \int_{\frac{\pi}{2}}^{\pi}\left(\frac{2}{\pi}\right)^2(\pi - x)^2\,dx\right]$$

$$= \left(\frac{2}{\pi}\right)^3\left[\frac{1}{3}x^3\Big|_0^{\frac{\pi}{2}} - \frac{1}{3}(\pi - x)^3\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{3}\left(\frac{2}{\pi}\right)^3\left[\left(\frac{\pi}{2}\right)^3 + \left(\frac{\pi}{2}\right)^3\right] = \frac{2}{3}$$

$$2a_0^2 + \sum_{n=1}^{\infty} a_n^2 = 2\cdot\left(\frac{1}{2}\right)^2 + \left(\frac{16}{\pi^2}\right)^2\left[\frac{1}{2^4} + \frac{1}{6^4} + \frac{1}{10^4} + \cdots\right]$$

$$\Longrightarrow \frac{1}{2^4} + \frac{1}{6^4} + \frac{1}{10^4} + \cdots$$

$$= \left[\frac{2}{3} - \frac{1}{2}\right]\frac{\pi^4}{16^2} = \frac{\pi^4}{6 \times 16^2}$$

11. (5pts) Prove that $\mathcal{L}^{-1}\{\ln \frac{s}{s-1}\} = (e^t - 1)/t$.

$F(s) = \ln \frac{s}{s-1} = \ln s - \ln(s-1)$. Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$F'(s) = \frac{1}{s} - \frac{1}{s-1}$

$\mathcal{L}^{-1}\{F'(s)\} = 1 - e^t$

$\Rightarrow 1 - e^t = -t\, f(t)$

$\mathcal{L}^{-1}\{F'(s)\} = -t\, f(t)$

$f(t) = \frac{e^t - 1}{t}$