

6.8 Laplace Transform: General Formulas

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st}f(t) dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ $\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$ $\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{(n-1)}f(0) - \dots$ $\dots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s}\mathcal{L}\{f\}$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ $= \int_0^t f(t - \tau)g(\tau) d\tau$ $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$	Convolution	6.5
$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(\tilde{s}) d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}\{f\} = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st}f(t) dt$	f Periodic with Period p	6.4 Project 16

6.9 Table of Laplace Transforms

For more extensive tables, see Ref. [A9] in Appendix 1.

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	} 6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \dots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\frac{1}{s-a}$	e^{at}	} 6.1
8	$\frac{1}{(s-a)^2}$	te^{at}	
9	$\frac{1}{(s-a)^n} \quad (n = 1, 2, \dots)$	$\frac{1}{(n-1)!} t^{n-1} e^{at}$	
10	$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)} t^{k-1} e^{at}$	
11	$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{a-b} (e^{at} - e^{bt})$	
12	$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{a-b} (ae^{at} - be^{bt})$	
13	$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin \omega t$	} 6.1
14	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$	
15	$\frac{1}{s^2 - a^2}$	$\frac{1}{a} \sinh at$	
16	$\frac{s}{s^2 - a^2}$	$\cosh at$	
17	$\frac{1}{(s-a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sinh \omega t$	
18	$\frac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$	
19	$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2} (1 - \cos \omega t)$	} 6.2
20	$\frac{1}{s^2(s^2 + \omega^2)}$	$\frac{1}{\omega^3} (\omega t - \sin \omega t)$	

(continued)

Table of Laplace Transforms (continued)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
21	$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	} 6.6
22	$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$	
23	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	
24	$\frac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2}(\cos at - \cos bt)$	
25	$\frac{1}{s^4 + 4k^4}$	$\frac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$	
26	$\frac{s}{s^4 + 4k^4}$	$\frac{1}{2k^2} \sin kt \sinh kt$	
27	$\frac{1}{s^4 - k^4}$	$\frac{1}{2k^3}(\sinh kt - \sin kt)$	
28	$\frac{s}{s^4 - k^4}$	$\frac{1}{2k^2}(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	
30	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2}t\right)$	I 5.5
31	$\frac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$	J 5.4
32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}} e^{at}(1 + 2at)$	
33	$\frac{1}{(s^2 - a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	I 5.5
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\frac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$	J 5.4
37	$\frac{1}{\sqrt{s}} e^{-k/s}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
38	$\frac{1}{s^{3/2}} e^{k/s}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$	

(continued)

Table of Laplace Transforms (continued)

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
40	$\frac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	γ 5.5
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$	
42	$\ln \frac{s^2 + \omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t} \sin \omega t$	
45	$\frac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$	App. A3.1

CHAPTER 6 REVIEW QUESTIONS AND PROBLEMS

- State the Laplace transforms of a few simple functions from memory.
- What are the steps of solving an ODE by the Laplace transform?
- In what cases of solving ODEs is the present method preferable to that in Chap. 2?
- What property of the Laplace transform is crucial in solving ODEs?
- Is $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$? $\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$? Explain.
- When and how do you use the unit step function and Dirac's delta?
- If you know $f(t) = \mathcal{L}^{-1}\{F(s)\}$, how would you find $\mathcal{L}^{-1}\{F(s)/s^2\}$?
- Explain the use of the two shifting theorems from memory.
- Can a discontinuous function have a Laplace transform? Give reason.
- If two different continuous functions have transforms, the latter are different. Why is this practically important?

11-19 LAPLACE TRANSFORMS

Find the transform, indicating the method used and showing the details.

- $5 \cosh 2t - 3 \sinh t$
- $e^{-t}(\cos 4t - 2 \sin 4t)$
- $\sin^2(\frac{1}{2}\pi t)$
- $16t^2 u(t - \frac{1}{4})$

15. $e^{t/2}u(t - 3)$

17. $t \cos t + \sin t$

19. $12t * e^{-3t}$

16. $u(t - 2\pi) \sin t$

18. $(\sin \omega t) * (\cos \omega t)$

20-28 INVERSE LAPLACE TRANSFORM

Find the inverse transform, indicating the method used and showing the details:

20. $\frac{7.5}{s^2 - 2s - 8}$

22. $\frac{\frac{1}{16}}{s^2 + s + \frac{1}{2}}$

24. $\frac{s^2 - 6.25}{(s^2 + 6.25)^2}$

26. $\frac{2s - 10}{s^3} e^{-5s}$

28. $\frac{3s}{s^2 - 2s + 2}$

21. $\frac{s + 1}{s^2} e^{-s}$

23. $\frac{\omega \cos \theta + s \sin \theta}{s^2 + \omega^2}$

25. $\frac{6(s + 1)}{s^4}$

27. $\frac{3s + 4}{s^2 + 4s + 5}$

29-37 ODEs AND SYSTEMS

Solve by the Laplace transform, showing the details and graphing the solution:

29. $y'' + 4y' + 5y = 50t, \quad y(0) = 5, \quad y'(0) = -5$

30. $y'' + 16y = 48(t - \pi), \quad y(0) = -1, \quad y'(0) = 0$