

Study Guide for Exam 2

1. You are supposed to be able to compute the basic limits related to the trigonometric functions (such as $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$ and $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta}$) and via these limits, you are supposed to know how to compute the (higher) derivatives of the trigonometric functions.

Example problems:

1.1. Compute the following limits:

- (i) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{5x}$
- (ii) $\lim_{x \rightarrow \infty} \frac{\sin(3x)}{5x}$
- (iii) $\lim_{x \rightarrow 0} \frac{\sin(1/x)}{1/x}$
- (iv) $\lim_{x \rightarrow 0} \frac{\sin(4x)^2}{2x^2 \cos x}$
- (v) $\lim_{x \rightarrow -\infty} \cos(3x)e^x$
- (vi) $\lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{\theta}$

Handwritten notes and calculations:

- $\lim_{x \rightarrow 0} \frac{\sin f(x)}{f(x)} = 1$
- If $\lim_{x \rightarrow 0} f(x) = 0$
- $\frac{0}{0} = 1$
- $\frac{\infty}{\infty} = 0$
- For (iv): $\frac{\sin(4x)^2}{(4x)^2} \cdot \frac{(4x)^2}{2x^2 \cos x} \rightarrow \frac{16}{2} = 8$

1.2. Compute the derivative of the following function, using the defining formula, the basic limits related to the trigonometric functions, and the product and quotient rules.

- (i) $y = \sin x$
- (ii) $y = \cos x$
- (iii) $y = \tan x$
- (iv) $y = \csc x$
- (v) $y = \sec x$
- (vi) $y = \cot x$

1.3. (i) Consider the function $y = \cos x$.

What is its 1996-th derivative $\left(\frac{d}{dx}\right)^{1996} y$ evaluated at $x = \pi/4$?

(ii) Consider the function $y = \sin(2x)$.

What is its 2001-th derivative $\left(\frac{d}{dx}\right)^{2001} y$ evaluated at $x = \pi/3$?

2. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are supposed to understand the relation of the derivative of a one-to-one function and that of its inverse. You are supposed to be able to compute the derivatives of the inverse trigonometric functions.

Example problems:

2.1. Compute the derivative of the following function:

(i) $y = \sin(\sin(\sin x))$

(ii) $y = \cos(2\pi \cdot 3^x)$

(iii) $y = \left(\frac{t-2}{2t+1} \right)^9$

(iv) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

→ (v) $y = e^{2x^3}$

(vi) $y = \sin^{-1}(\sqrt{x})$

(vii) $y = \tan^{-1}(\sqrt{x})$

(viii) $y = \ln |\sec(3\theta) + \tan(3\theta)|$

(ix) $y = \ln(e^{\sin x} + e^{-\sin x})$

(x) $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$

(xi) $y = \ln(x \ln x)$

$$y' = e^{2x^3} \cdot [2x^3]' = e^{2x^3} \cdot 2x^2 \ln 2 \cdot [x^3]' = 3x^2 \ln 2 \cdot 2x^2 \cdot e^{2x^3} \quad (\ln|x|)' = \frac{1}{x}$$

$$y' = \frac{1}{2} \left[\frac{1}{1 + \frac{1}{2}(x+x^{\frac{1}{2}})^{\frac{1}{2}}} \cdot \left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right) \right]'$$

$$y' = \frac{1}{\sec 3\theta + \tan 3\theta} \cdot [\sec 3\theta + \tan 3\theta]' = \frac{e^x}{e^x + 1} - \frac{e^x}{e^x - 1}$$

$$y' = \frac{1}{x \ln x} \cdot (x \ln x)' = \frac{\ln x + 1}{x \ln x}$$

→ 2.2. Suppose that $F(x) = \frac{1}{g(f^{-1}(x))}$ and that the functions f (which

is one-to-one, and hence has its inverse) and g , satisfy the following conditions. Find $F'(5)$.

$$x=5 \quad f^{-1}(5) = 3$$

$$g(3)$$

$$\begin{cases} f(3) = 5 \\ f'(3) = 7 \\ g(3) = 2 \\ g'(3) = 11 \\ g(5) = 4 \\ g'(5) = 9 \end{cases}$$

$$F'(x) = -[g(f^{-1}(x))]^{-2} \cdot g'(f^{-1}(x)) \cdot [f^{-1}(x)]'$$

$$= -[g(3)]^{-2} \cdot g'(3) \cdot \frac{1}{f'(3)}$$

$$= -2^{-2} \cdot 11 \cdot \frac{1}{7} \quad f^{-1}(5) = 3$$

→ 2.3. Suppose that $F(x) = f(x)f(g^{-1}(x))$ and that the functions f and g satisfy the following conditions:

$$\begin{cases} f(3) = 5, & f(4) = 3, & f(5) = -1 \\ f'(3) = 4, & f'(4) = 3, & f'(5) = -2 \\ g(3) = 3, & g(4) = 2, & g(5) = 3 \\ g'(3) = 2, & g'(4) = 3, & g'(5) = -2 \end{cases}$$

Find $F'(3)$.

$$f(1) = 1 + 6 + \sin \pi = 7$$

→ 2.6. Consider the function $y = f(x) = x^3 + 6x + \sin(\pi x)$. It is one-to-one, and hence has its inverse function f^{-1} . Observe that the point $(1, 7)$ is on the graph of $y = f(x)$.

Find the equation of the tangent line to the graph of $y = f^{-1}(x)$ at the point $(7, 1)$.

$$[f^{-1}(7)]' = \frac{1}{f'(1)} = \frac{1}{9 - \pi}$$

$$f' = 3x^2 + 6 + \pi \cos(\pi x)$$

$$f'(1) = 3 + 6 + \pi \cos \pi = 9 - \pi$$

$$y-1 = \frac{1}{9-\pi} (x-7)$$

3

3. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems:

3.1. Find the derivative of the following function.

(i) $y = x^{\cos x}$

→ (ii) $y = (\ln x)^{\tan 3x}$ $f = \ln x, g = \tan 3x$

(iii) $y = (\sqrt{x})^{\sin x}$

→ (iv) $y = \left(1 + \frac{2}{x}\right)^x$ $f = 1 + \frac{2}{x}, g = x$

$$y = f^g = e^{g \ln f}$$

$$y' = f^g \cdot [g \ln f]'$$

$$= f^g [g' \ln f + g \cdot \frac{f'}{f}]$$

4. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems:

4.1. Suppose that f is a differentiable function defined on $(-\infty, \infty)$, satisfying the following equation

$$f(x) + x^2 f(x)^3 = 10$$

and that $f(1) = 2$. Find $f'(1)$.

$$0 = f' + 2x f^3 + x^2 \cdot 3 f^2 \cdot f'$$

$$f' = - \frac{2x f^3}{1 + 3x^2 f^2} \Big|_{x=1} = \frac{-2 \cdot 2^3}{1 + 3 \cdot 2^2}$$

4.2. Find the slope of the tangent to the curve given by the equation

$$x^2 + 2xy - y^2 + x = 2$$

$$= \frac{-16}{13}$$

at point $(x, y) = (1, 2)$

4.3. Find the equation of the tangent line to the curve defined by

$$4 + 3\sqrt{xy+1} = x^2 + y^2$$

$$\left(\frac{x}{y}\right)' = \frac{y - x \cdot y'}{y^2}$$

$$y' = \frac{y^{-1} e^{\frac{x}{y}} - 7}{x y^{-2} e^{\frac{x}{y}} - 1}$$

at the point $(3, 1)$.

4.4. Find the equation of the tangent line to the curve defined by

$$2(x+y)^{1/3} = y$$

at the point $(4, 4)$

→ 4.5. Find $\frac{dy}{dx}$ given $e^{x/y} = 7x - y$

$$7 - y' = e^{\frac{x}{y}} \cdot \frac{y - x y'}{y^2} = \frac{1}{y} e^{\frac{x}{y}} - \frac{x}{y^2} e^{\frac{x}{y}} y'$$

5. You are supposed to be able to compute the derivatives of the exponential functions and logarithmic functions, as well as to understand the shape of the graphs of these functions.

Example Problems:

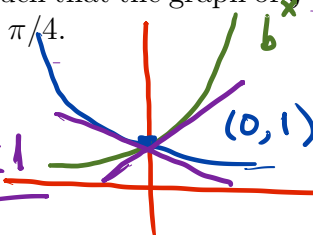
→ 5.1. Find all such values of $b > 0$ that the graph of $y = b^x$ and the graph of $y = b^{-x}$ intersect perpendicularly

→ 5.2. Find the value $b (> 1)$ such that the graph of $y = \log_b x$ intersects with the x -axis at an angle of $\pi/4$.

$$(-\ln b)(\ln b) = -1$$

$$\Rightarrow (\ln b)^2 = 1 \Rightarrow \ln b = \pm 1$$

$$b = e^{-1} \text{ or } e$$

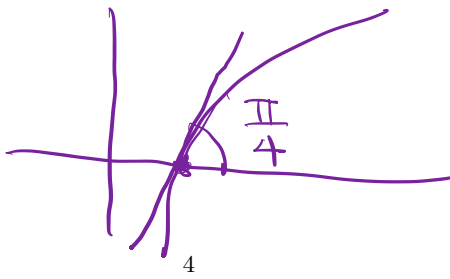


$$m_1 = (b^x)' \Big|_{x=0} = b \ln b \Big|_{x=0}$$

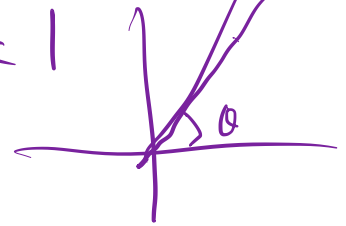
$$= \ln b$$

$$m_2 = (b^{-x})' \Big|_{x=0} = b^{-x} \ln b (-1) \Big|_{x=0}$$

$$= -\ln b$$



$$m = \frac{1}{\ln b} \Big|_{x=1} = \frac{1}{\ln b} = 1$$



6. You are supposed to be able to compute the derivative of a function involving the logarithmic functions, first simplifying the formula using the laws of the logarithms.

Example Problems:

6.1. Compute the derivatives of the following functions.

(i) $y = \ln(x\sqrt{x^2 - 10})$

(ii) $y = \ln(e^x + xe^x)$

(iii) $y = \frac{(x^3 - 1)^4 e^x}{(x^2 + 4)^3}$

→ 6.2. Consider the function $h(x) = \ln(\sqrt[3]{g(x)})$. Assume $g(3) = 7$ and $g'(3) = -5$. Compute $h'(3)$.

$$h' = g^{-\frac{1}{3}}(x) \cdot \frac{1}{3} \cdot g^{-\frac{2}{3}} \cdot g' = \frac{1}{3} [g(x)]^{-\frac{2}{3}} \cdot g'(x)$$

7. You are supposed to be able to compute the limit of some indeterminate form, by relating its computation to the definition of the derivative.

$$h'(3) = \frac{1}{3} [g(3)]^{-\frac{2}{3}} g'(3) = \frac{1}{3} \cdot \frac{1}{7} \cdot (-5)$$

Example Problems:

7.1. Compute the following limits.

$$\frac{f(x) - f(0)}{x} \rightarrow f'(0) \quad \lim_{h \rightarrow 0} \frac{\left[\sin\left(\frac{\pi}{2} + h\right) \right]^{\frac{\pi}{2} + h} - 1}{(3+h)^{5-h} - 3^5}$$

(ii) $\lim_{h \rightarrow 0} \frac{(3+h)^{5-h} - 3^5}{(3+2h)^{5+3h} - 3^5}$

(iii) $\lim_{h \rightarrow 0} \frac{(3+2h)^{5+3h} - 3^5}{(3+h)^{5-h} - 3^5} \cdot 2$

$$\frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\frac{f(x+2h) - f(x)}{(x+2h) - x}$$

$$f(x) = x^{x+2}$$

$$\frac{f(3+2h) - f(3)}{(3+2h) - 3}$$

$$= \frac{(3+2h)^{3+2h+2} - 3^{3+2}}{2h} \Rightarrow f'(3)$$

FOUR "Related Rates" problems will be given in Exam 2. Of particular importance are:

- Distance problem
- Boat being pulled to the dock problem
- Conical tank problem
- Ladder problem
- Light house problem
- Volume or surface area of a sphere problem
- Particle moving along the graph of a function problem

$$f(x) = \left[\sin\left(\frac{\pi}{2} + x\right) \right]^{\frac{\pi}{2} + x}$$

$$f(0) = \left(\sin\frac{\pi}{2} \right)^{\frac{\pi}{2}} = 1$$

$$f(x) = (3+x)^{5-x}, \quad f(0) = 3^5$$

$$f(x) = (3+2x)^{5+3x}, \quad f(0) = 3^5$$

10.1. (speed) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 2 m higher than the bow of the boat. If the rope is pulled in at a rate of 3 m/sec, how fast is the boat approaching the dock when it is 10 m away from the dock ?

10.2. (angle) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/sec, what is the rate of change of the angle between the rope and the horizontal line when the bow is $\sqrt{3}$ m away from the dock ?

10.3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2 \text{ m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.

→ 10.4. (speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

10.5. (angle) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the angle θ between the top of the ladder and the wall increasing when the bottom of the ladder is 6 ft from the wall ?

→ 10.6. (speed) A light house is located on a small island 4 km away from the nearest point P on a straight shoreline and its light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

→ 10.7. (revolution) A lighthouse is located on an island 4 km away from the nearest point P on a straight shoreline. The beam of light is moving along the shoreline at the speed of 25π km/min when it is 3 km away from the point P . How many revolutions per minute is the beam making at the lighthouse ?

10.8. Suppose the surface area of a sphere is increasing at a constant rate of $10 \text{ in}^2/\text{sec}$. What will the radius of the sphere be at the instant when the radius of the sphere is increasing at 5 in/sec?

10.9. A particle is moving along the curve $xy = 12$. As it reaches the point $(6, 2)$, the x -coordinate is decreasing at a rate of 5 cm/sec. What is the rate of change of the y -coordinate of the particle at that instant ?

10.10. A particle moves along the curve $y = x^2$. As it passes through the point $(2, 4)$, its x -coordinate increases at a rate of $\sqrt{5}$ cm/sec. How fast is the distance from the particle to the origin changing ?

Note: The unit for measuring the coordinate length is given by "cm".

