

Study Guide for Exam 3

1. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$ and the values on the critical value(s) $f(c)$'s. You should know what the definition of a critical value is.

Example Problems:

1.1. Find the absolute maximum/minimum and local maximum/minimum of the function defined by

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on the closed interval $[-1, 4]$.

1.2. Find the absolute maximum and absolute minimum values of the function f on the given interval.

- (a) $f(x) = x^{2/3}$ on $[-1, 8]$
- (b) $f(x) = (15 - x^2)e^{-x}$ on $[-4, 6]$
- (c) $f(x) = (x^2 - 1)^3$ on $[-1, 3]$
- (d) $f(t) = 2 \cos t + \sin 2t$ on $[0, 2\pi]$
- (e) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$
- (f) $f(x) = x - 2 \ln x$ on $[1, 3]$

2. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

Example Problems:



2.1. The first derivative of a function f , is given by f'

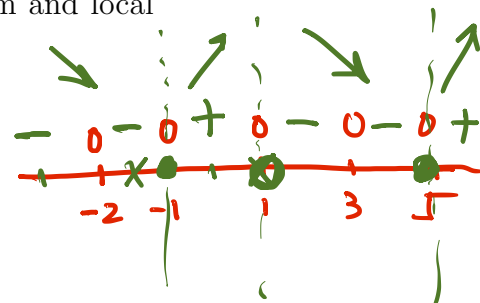
$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5)$$

Find the values of x for which the function f takes

- (a) local maximum, and
- (b) local minimum.

2.2.

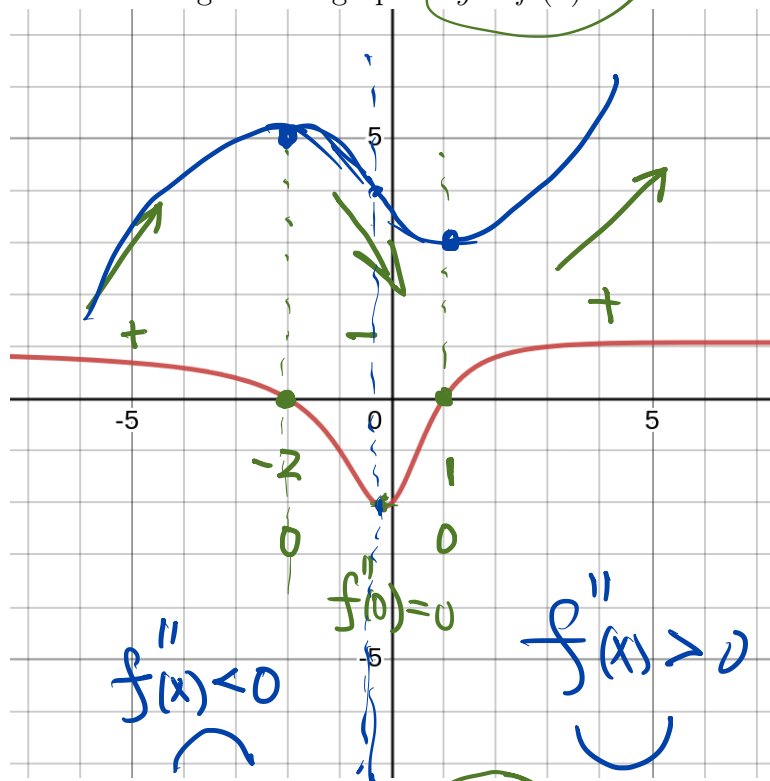
- (a) Find the critical numbers of the function $f(x) = x^8(x-4)^7$.
- (b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?
- (c) What does the First Derivative Test tell you that the Second Derivative test does not?



3. You are supposed to be able to draw the (rough) graph of a function, given the graph of its derivative.

Example Problems:

→ 3.1. Below is given the graph of $y = f'(x)$.



Draw the (rough) graph of $y = f(x)$.

4. You are supposed to tell whether the graph of a function is concave up/down, and find its inflection points, by looking at the second derivative of the function.

Example Problems:

4.1. Determine how the concavity changes for the function

$$f(x) = \frac{1}{2}x - \sin(x)$$

on the interval $(0, 3\pi)$.

→ 4.2. The second derivative of a function f is given by

$$f''(x) = (x+5)^3(x+2)^2(x-2)^5(x-3)^3(x-6)^2 = 0$$

find the x -coordinates of all the inflection points.

4.3. How many inflection points does the graph of the function $y = f(x) = x^5 - 5x^4 + 25x$ have?

4.4. We have a function whose first derivative is given by the formula $f'(x) = (x-1)^2(x+3)^3$. Find the local extrema and the inflection points of the function.

5. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.

Example Problems:

5.1.

(a) $\lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}}$

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}$

(c) $\lim_{x \rightarrow 0} \frac{e^{7x} - \cos 2x}{\tan(3x)}$

(d) $\lim_{x \rightarrow 0} \frac{\sin x}{1 - x^2}$

(e) $\lim_{x \rightarrow 0} \frac{3x - \sin(3x)}{3x - \tan(3x)}$

(f) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^3}$

→ (g) $\lim_{x \rightarrow 0} \frac{\ln\left(\frac{\sin x}{x}\right)}{\frac{x^2}{x^2}} \rightarrow \frac{0}{0} = \lim_{x \rightarrow 0} \frac{\left[\ln\left(\frac{\sin x}{x}\right)\right]'}{[x^2]'} = \lim_{x \rightarrow 0} \frac{\frac{1}{\frac{\sin x}{x}} \cdot \left(\frac{\sin x}{x}\right)'}{2x}$

$$= \lim_{x \rightarrow 0} \frac{\frac{x}{\sin x} \cdot \frac{x \cos x - \sin x}{x^2}}{2x} = \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{2x^2 \sin x} = \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{2[2x \sin x + x^2 \cos x]}$$

→
$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{-x \sin x}{2x \sin x + x^2 \cos x} \rightarrow 0$$

$$= -\frac{1}{2} \lim_{x \rightarrow 0} \frac{1}{2 \sin x + \frac{x}{\sin x} \cos x} = -\frac{1}{2} \cdot \frac{1}{2 + 1 \cdot 1} = -\frac{1}{4}$$

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6. You are supposed to know how to compute the limits of the form $\pm\infty \times 0, \infty - \infty$.

Example Problems:

6.1. Compute the following limits:

(a) $\lim_{x \rightarrow 0^+} x \ln(2x)$

→ (b) $\lim_{x \rightarrow \infty} 2x \tan\left(\frac{1}{3x}\right) = 2 \lim_{x \rightarrow \infty} \frac{\tan\left(\frac{1}{3x}\right)}{\frac{1}{x}} = 2 \lim_{x \rightarrow \infty} \frac{\frac{1}{\cos^2\left(\frac{1}{3x}\right)} \cdot \left(\frac{1}{3x}\right)'}{\left(\frac{1}{x}\right)'}$

(c) $\lim_{x \rightarrow \left(\frac{\pi}{2}\right)^-} (2x - \pi) \cdot \tan(x)$

(d) $\lim_{x \rightarrow \infty} (\sqrt{x^2 - 5x + 7} - x)$

$= \frac{2}{3} \lim_{x \rightarrow \infty} \frac{1}{\cos^2\left(\frac{1}{3x}\right)} = \frac{2}{3}$

(e) $\lim_{x \rightarrow 1} \left(\frac{x}{x-1} - \frac{1}{\ln(x)} \right)$

→ (f) $\lim_{x \rightarrow 4} \left(\frac{1}{\sqrt{x}-2} - \frac{4}{x-4} \right) = \lim_{x \rightarrow 4} \frac{\sqrt{x}+2-4}{x-4} = \lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x \rightarrow 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

$\left[\frac{1}{f(x)} \right]' = -\frac{1}{f^2(x)} f'(x)$ (g) $\lim_{x \rightarrow \infty} x^2 \tan\left(\frac{1}{5x^2+2}\right)$ (g*) $\lim_{x \rightarrow 0} x^2 \tan\left(\frac{1}{5x^2+2}\right)$

7. You are supposed to be able to compute the limits $\lim_{x \rightarrow a} [f(x)]^{g(x)}$

of the form $0^0, \infty^0, 1^\infty$. $\lim_{x \rightarrow \infty} f(x)^{g(x)} = e^{\lim_{x \rightarrow \infty} g(x) \ln f(x)}$ $0 \cdot \infty \rightarrow \frac{0}{0} \text{ or } \frac{\infty}{\infty}$

Example Problems:

7.1. Compute the following limits:

(a) $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x} \right)^{7x}$

(b) $\lim_{x \rightarrow \infty} (4x+1) \ln \frac{x+3}{x-2} = e^{\lim_{x \rightarrow \infty} \frac{(4x+1) \ln \frac{x+3}{x-2}}{1}} = e^{\lim_{x \rightarrow \infty} \frac{4x+1}{1} \cdot \frac{\frac{1}{x+3} - \frac{1}{x-2}}{1}} = e^{\lim_{x \rightarrow \infty} \left[\frac{4x+1}{1} \cdot \frac{x-2-(x+3)}{(x+3)(x-2)} \right]} = e^{\lim_{x \rightarrow \infty} \frac{5(4x+1)}{4(x+3)(x-2)}} = e^{\frac{5}{4} \cdot 16} = e^{20}$

(b) $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x-2} \right)^{4x+1}$

$\lim_{x \rightarrow 0} \frac{\ln(1-5x)}{x} = \lim_{x \rightarrow 0} \frac{1-5x \cdot (-5)}{1} = -5 = e^{-5}$

(c) $\lim_{x \rightarrow 0^+} (1-5x)^{1/x}$

(d) $\lim_{x \rightarrow \frac{\pi}{2}^-} (5 \tan x)^{\cos x}$

8. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply the following corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise: If $f'(x) = 0$ for all values of $x \in (a, b)$, then a continuous function f on the closed interval $[a, b]$ is actually a constant.

Example Problems:

8.1. Consider the function $f(x) = x^4 - 2x^2 + 7x - 2$ over the interval $[-2, 2]$. Does it satisfy the conditions for the Mean Value Theorem to hold? If it does, find the value(s) $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)}.$$

- 8.2. Consider the function $y = f(x) = x^{2/3}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem to hold?

Do we have any value $c \in (-1, 1)$ such that $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$?

- 8.3. Determine the exact value of $\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)$.

- 8.4. Determine the exact value of $\sin^{-1}\left(\frac{3}{7}\right) - \cos^{-1}\left(-\frac{3}{7}\right)$.

- 8.5. Determine the exact value of $\tan^{-1}(7) - \tan^{-1}\left(-\frac{1}{7}\right)$.

- 8.6. Consider the equation $f(x) = x^3 + x - 1 = 0$.

Determine how many solutions are there for the above equation on the interval $[0, 1]$, using the Intermediate Value Theorem and the Mean Value Theorem.

9. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down), and also by determining the horizontal/vertical asymptotes and x -intercept (y -intercept).

Example Problems:

9.1. Draw the graph of the following function:

(a) $y = f(x) = \frac{1}{x^2 - 16}$

(b) $y = f(x) = \frac{x}{x^2 - 16}$

(c) $y = f(x) = \frac{x^2}{x^2 - 16}$

(d) $y = f(x) = \frac{x^3}{x^2 - 16}$

(e) $y = f(x) = \frac{x}{x^2 + 16}$

(f) $y = f(x) = \frac{x^3}{x^2 + 1}$

(g) $y = f(x) = e^{-x} \sin x$ on $[0, 2\pi]$

(h) $y = f(x) = \ln(x^2 - 10x + 24)$

$$\begin{aligned} f(x) &= \tan^{-1} x - \tan^{-1}\left(-\frac{1}{x}\right) \\ f'(x) &= \frac{1}{1+x^2} - \frac{\left(-\frac{1}{x}\right)'}{1+\left(-\frac{1}{x}\right)^2} \\ &= \frac{1}{1+x^2} - \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} \\ &= \frac{1}{1+x^2} - \frac{1}{x^2+1} \\ &= 0 \\ \Rightarrow f(x) &= \text{const.} \\ &= f(7) = f(0) \\ &= 0 - \tan^{-1}\left(-\frac{1}{0}\right) \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} x^3 - x \cdot 16 + 16x \\ = x + \frac{16x}{x^2 - 16} \end{aligned}$$

$\lim_{x \rightarrow \pm \infty} \frac{1}{x^2 - 16}$

$\lim_{x \rightarrow -4} f(x) =$

$\lim_{x \rightarrow 4} f(x) =$

10. You are supposed to understand the idea of the linear approximation of a function $f(x)$ at $x = a$

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

and apply it to approximate the value of a function at a given point.

Example Problems:

10.1. Find the formula for the linear approximation to $f(x) = \sqrt{x}$ at $x = a = 1$ and use it to approximate $\sqrt{1.1}$.

→ 10.2. Find the formula for the linear approximation to $f(x) = e^x$ at $x = a = 0$ and use it to approximate $e^{-0.05}$.

11. Total of 3 Optimization Problems will be given in Exam 3.

Of particular importance are:

- Cylinder/Box Problem
- Maximizing or minimizing the area of a figure (rectangle, isosceles, right triangle etc.) under the given restrictions
- Coffee Cup problem
- Circular Cone Problem
- Walk Way Problem

Example Problems:

- 11.1. A rectangular box has to be made with the width being twice as long as the length (with a bottom but WITHOUT a top). If the surface area of the box is 400 cm^2 , what is the height of the box with the largest volume?

- 11.2. What is the largest area of an isosceles triangle inscribed in a circle of radius 1?

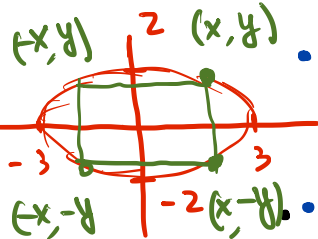
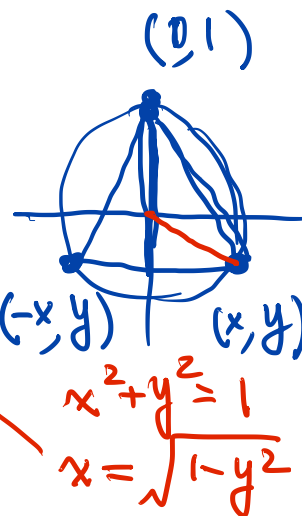
- 11.3. What is the largest area of the rectangle inscribed in an ellipse

- 11.4. Find the equation of a line passing through the point $(3, 2)$, which cuts off the least amount of area from the first quadrant.

11.5. What is the maximum area of a rectangle whose base is on the x-axis, having the remaining two vertices on the graph of $y = 9 - x^2$, and lying above the x-axis?

11.6. What is the largest area of a right triangle whose hypotenuse has length 5?

11.7. A cone-shaped drinking cup is made from a circular piece of paper of fixed radius R by cutting out a sector and joining the edges CA and CB. Find the maximum capacity of such a cup.

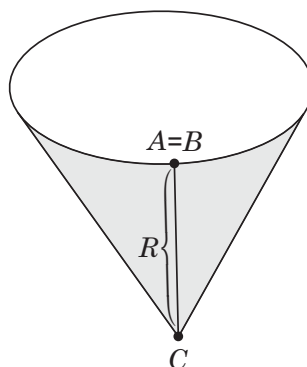
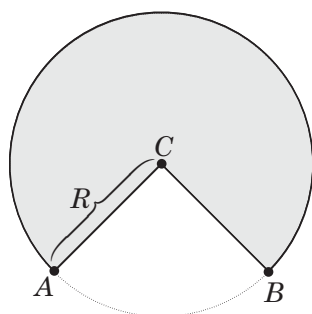


$$A = \frac{1}{2}(1+y) \cdot 2x = x(1+y)$$

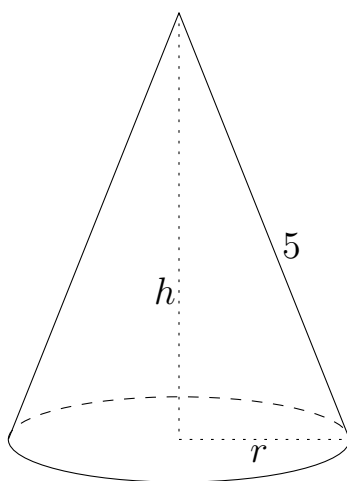
$$A = 4xy$$

$$= 4x \cdot 2\sqrt{1-x^2}$$

$$\left(\frac{x^2}{9} + \frac{y^2}{4} = 1 \right) = \frac{8}{3} x \sqrt{9-x^2}$$



11.8. The slant height of a right circular cone is the distance from the edge of the base of the cone to the vertex of the cone.



What is the maximum volume of a right circular cone with slant height 5 cm.

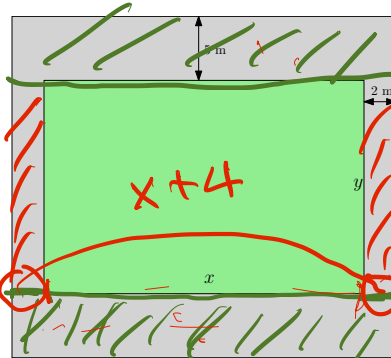
11.9. Suppose a rectangular area is to be surrounded by a concrete walkway that is 2 meter wide on the East and West and 5 meters wide on the North and South.

$$A = 5(x+4) \cdot 2$$

$$+ 2y \cdot 2$$

$$= 10x + 4y + 40$$

$$xy = 100 \Rightarrow y = \frac{100}{x}$$



$$A = 10x + \frac{400}{x} + 40$$

$$\min A$$

If the area inside the walkway is to be 100 square meters, what should the interior width of the enclosed area be (labeled x in the diagram above) in order to minimize the amount of concrete used. (Note minimizing the amount of concrete used is equivalent to minimizing the area covered by concrete.)