TAs 5, 6, 8, 11, 12

Study Guide for Final Exam

(Version 11/25/2022 11:00 AM)

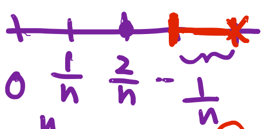
New materials after Exam 3

Remark: There will be 5 questions regarding the new materials after Exam 3 in the Final Exam. Look also at the "NOTE" for the subject matter 14 about computing the total distance.

$\sum_{i=1}^{n} i^{-2} = ?$

1. You are supposed to know how to compute the integral using the Riemann Sum. Conversely, you should know how to compute the limit in the form of Riemann Sum using the integration and Fundamental Theorem of Calculus. You are also supposed to know how to compute the integral knowing its geometrical meaning.

$\frac{i}{n}$ $\frac{i+1}{n}$



Example Problems

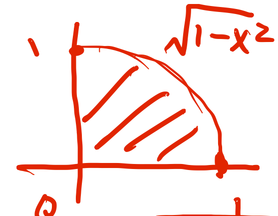
$\Delta x = \frac{1}{n}$, $0 = \frac{1}{n} < \frac{2}{n} < \cdots < \frac{i}{n} < \cdots < \frac{n-1}{n} < 1$

1.1. (i) Write down the formula for approximating the integration $\int_0^1 \sqrt{1-x^2}\, dx$ as the Riemann Sum dividing the interval $[0,1]$ into n equal subintervals and using the right end points.

$= \frac{1}{4}\pi \cdot 1^2 = \frac{1}{4}\pi$

$\lim_{\Delta x \to 0} \sum_{i=1}^{n} \sqrt{1-(x_i^*)^2}\, \Delta x =$

(ii) Compute the integral using the geometric interpretation.



1.2. Evaluate the following definite integral $\int_1^3 \sqrt{4^2-(x-3)^2}\, dx$.

$y = \sqrt{4^2-(x-3)^2}$

$y^2 + (x-3)^2 = 4^2$

$u = x-3 : -2 \to 0$

$du = dx$

$\int_{-2}^{0} \sqrt{4^2-u^2}\, du$

$= 4\int_{-2}^{0} \sqrt{1-\left(\frac{u}{4}\right)^2}\, du$

$v = \frac{u}{4} : -\frac{1}{2} \to 0$

$dv = \frac{1}{4}du \Rightarrow du = 4dv$

$= 4^2 \int_{-\frac{1}{2}}^{0} \sqrt{1-v^2}\, dv$

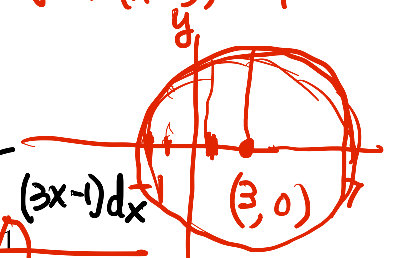
1.3. Evaluate the following definite integral $\int_0^5 |3x-1|\, dx$.

$3x-1=0 \Rightarrow x = \frac{1}{3}$

$0 \quad \frac{1}{3} \quad 1$

$= \int_0^{\frac{1}{3}} (1-3x)\, dx + \int_{\frac{1}{3}}^{5} (3x-1)\, dx$

$3\cdot 1 - 1 > 0$



(3, 0)

1.4. Compute the following limits:

(i) $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k}{n^2+k^2} \right)$

HINT: Use the equation

$$\begin{aligned} \sum_{k=1}^{n} \frac{k}{n^2+k^2} &= \sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2} \cdot \frac{n}{n^2} = \sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2} \frac{1}{n} \\ &= \sum_{k=1}^{n} f(x_k^*)\Delta x \end{aligned}$$

where

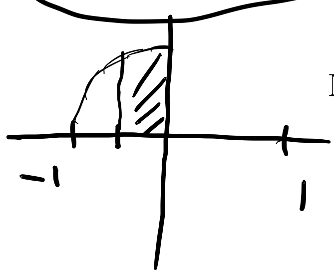
$$\left\{ \begin{aligned} f(x) &= \frac{x}{1+x^2} \\ x_k^* &= k/n \\ \Delta x &= \frac{1}{n}. \end{aligned} \right.$$

$\frac{k}{n^2} = \frac{k}{n} \cdot \frac{1}{n}$

$1 + \left(\frac{k}{n}\right)^2$

$\int_0^1 \frac{x}{1+x^2}\, dx$

$\frac{1}{n}$, $\frac{n}{n} = 1$



Now

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 f(x)\, dx.$$

$u = 1+x^2 : 1 \to 2$

$du = 2x\,dx$

$\int_1^2 \frac{1}{2}\frac{1}{u}\, du$

$= \frac{1}{2}\ln|u|\Big|_1^2$

$= \frac{1}{2}\ln 2$

$$\sum_{k=1}^{n} \boxed{f(x_k^*)\,\frac{1}{n}}$$

$$\sum_{k=1}^{n} \frac{1}{1+2\frac{k}{n}} \cdot \frac{1}{n}$$

$x_k^* = \frac{k}{n}$

$x_1^* = \frac{1}{n} \to 0 \quad x_n^* = \frac{n}{n} = 1$

(ii) $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \dfrac{1}{n+2k} \right)$

HINT: Use the equation

$$\begin{aligned} \textstyle\sum_{k=1}^{n} \frac{1}{n+2k} &= \textstyle\sum_{k=1}^{n} \frac{1}{1+2(k/n)} \cdot \frac{1}{n} \\ &= \textstyle\sum_{k=1}^{n} f(x_k^*)\Delta x \end{aligned}$$

$$\int_0^1 \frac{1}{1+2x}\,dx$$

where

$$\begin{cases} f(x) &= \dfrac{1}{1+2x} \\ x_k^* &= k/n \\ \Delta x &= \dfrac{1}{n}. \end{cases}$$

$$= \frac{1}{2} \ln \left| 1+2x \right| \Big|_0^1$$

Now

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_k^*)\Delta x = \int_0^1 f(x)\ dx.$$

(iii) $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \dfrac{n}{n^2+k^2} \right)$

HINT: Use the equation

$$\begin{aligned} \textstyle\sum_{k=1}^{n} \frac{n}{n^2+k^2} &= \textstyle\sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \cdot \frac{n}{n^2} = \sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \cdot \frac{1}{n} \\ &= \textstyle\sum_{k=1}^{n} f(x_k^*)\Delta x \end{aligned}$$

where

$$\begin{cases} f(x) &= \dfrac{1}{1+x^2} \\ x_k^* &= k/n \\ \Delta x &= \dfrac{1}{n} \end{cases}$$

Now

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 f(x)\ dx.$$

(iv) $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\dfrac{1}{\sqrt{4n^2-k^2}}\right)$

HINT: Use the equation

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{\sqrt{4n^2-k^2}} &= \sum_{k=1}^{n}\frac{1}{\sqrt{1-(k/2n)^2}}\cdot\frac{1}{2n}=\sum_{k=1}^{n}\frac{1}{\sqrt{1-\left(k\cdot\frac{1/2}{n}\right)^2}}\cdot\frac{1/2}{n}\\ &= \sum_{k=1}^{n} f(x_k^*)\Delta x\end{aligned}$$

where

$$\begin{cases} f(x) &= \dfrac{1}{\sqrt{1-x^2}}\\ x_k^* &= k\cdot\dfrac{1/2}{n}\\ \Delta x &= \dfrac{1/2}{n}.\end{cases}$$

Now

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x=\int_0^{1/2} f(x)\ dx=\int_0^{1/2}\frac{1}{\sqrt{1-x^2}}\ dx.$$

Alternatively, one may use the equation

$$\sum_{k=1}^{n}\frac{1}{\sqrt{4n^2-k^2}} = \sum_{k=1}^{n}\frac{1}{\sqrt{4-(k/n)^2}}\cdot\frac{1}{n}=\sum_{k=1}^{n} f(x_k^*)\Delta x$$

where

$$\begin{cases} f(x) &= \dfrac{1}{\sqrt{4-x^2}}\\ x_k^* &= k/n\\ \Delta x &= \dfrac{1}{n}.\end{cases}$$

Now

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x=\int_0^{1} f(x)\ dx=\int_0^{1}\frac{1}{\sqrt{4-x^2}}\ dx.$$

We observe

$$\int_0^1\frac{1}{\sqrt{4-x^2}}\ dx=\int_0^1\frac{1/2}{\sqrt{1-(x/2)^2}}\ dx=\int_0^{1/2}\frac{1}{\sqrt{1-u^2}}\ du,$$

where the last equality is obtained by the substitution $u=x/2$.

We observe also that the last definite integral is identical to the one obtained in the first part.

2. You are supposed to understand the meaning of the Fundamental Theorem of Calculus, and use it to compute the derivative of a function given in the form of an integration. (This includes the requirement that you are supposed to know how to take an anti-derivative of a function.) You are also supposed to understand the special features of the integration involving even/odd functions.

$$\boxed{\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)}$$

Example Problems

2.1. Set $J(x) = \displaystyle\int_{x}^{2x^2} \ln t\ dt$. Compute $J'(1)$.

$$=\left[\ln(2x^2)\cdot 4x - \ln x\cdot 1\right]_{x=1}$$

2.2. Compute the following.

(i) $\displaystyle \frac{d}{dx}\left(\int_0^{x^4} \sec(t)dt\right)$

(ii) $\displaystyle \frac{d}{dx}\left(\int_{2x}^{3x} \frac{t^2-1}{t^2+1}dt\right)$

(iii) $\displaystyle \frac{d}{dx}\left(\int_{e^{-2x}}^{e^{2x}} \ln t\ dt\right)$

$$(2^x)' = 2^x\cdot \ln 2$$

2.3. Compute the following integrals.

(i) $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2(x)dx$ (ii) $\displaystyle\int_0^4 2^x dx = \frac{1}{\ln 2}\int_0^4 (2^x)'\,dx = \frac{1}{\ln 2}\, 2^x\Big|_0^4$

(iii) $\displaystyle\int_{-2}^{2} (1-|x|^3)dx$ (iv) $\displaystyle\int_{-1}^{1} \frac{\tan(x)}{1+x^2+x^4}dx = 0$

$$2\int_0^2 \left(1-|x|^3\right)dx = 2\int_0^2 \left(1-x^3\right)dx$$

3. You are supposed to know how to use the Substitution Rule to compute the indefinite and/or definite integrals.

Example Problems

3.1. Compute the following integrals:

(i) $\displaystyle\int \tan x\ dx$ (ii) $\displaystyle\int \frac{\ln x}{x}\ dx$

(iii) $\displaystyle\int_0^4 \sqrt{1+2x}\ dx$ (iv) $\displaystyle\int_0^2 x^5\sqrt{1+x^2}\ dx$

$x^2 = u-1$

$u = 1+x^2 : 1\to 5$

$du = 2x\,dx$

$$\int_1^5 x^4 \sqrt{u}\ \frac{1}{2}du$$

$$=\frac{1}{2}\int_1^5 (u-1)^2\sqrt{u}\,du$$

$$=\frac{1}{2}\int_1^5 (u^2-2u+1)\sqrt{u}\,du$$

$$=\frac{1}{2}\int_1^5 \left(u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$$

(v) $\displaystyle\int_0^{\pi/4} \sec^4 x\tan x\ dx$ (vi) $\displaystyle\int_0^{\pi/4} \tan^3 x\sec^2 x\ dx$

$u = \tan x : 0\to 1$

$du = \sec^2 x\, dx$

$$\int_0^1 \sec^2 x\ u\,du$$

$$1+\tan^2 x = \sec^2 x$$

$$=1+\frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

$$\int_0^1 (1+u)u\,du$$

(vii) $\displaystyle\int_{\pi/2}^{\pi} (1-\sin^2 x)\cos x\ dx$ (viii) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\ dx$

(ix) $\displaystyle\int_0^{\sqrt{3}/3} \frac{\tan^{-1} x}{1+x^2}\ dx$ (x) $\displaystyle\int_{-\ln 2}^{(\ln 3-\ln 4)/2} \frac{e^x}{\sqrt{1-e^{2x}}}\ dx$

$m(t)=m(0)e^{kt}$ $\quad \frac{1}{2}\cancel{m_0}=\cancel{m_0}e^{kT_{1/2}} \Rightarrow k=\frac{\ln(1/2)}{T_{1/2}}=\frac{\ln(1/2)}{h}=-\frac{\ln 2}{h}$

$=m_0 e^{-\frac{\ln 2}{h}\cdot t}=m_0 e^{\ln 2 \cdot \left(-\frac{t}{h}\right)}=m_0 2^{-\frac{t}{h}}$

4. You are supposed to understand that the differential equation $\boxed{\frac{dy}{dx}=ky}$ has a solution of the form $\boxed{y(t)=y(0)e^{kt}}$, and should be able to apply it to analyze the population growth and radioactive decay. In the case of the radioactive decay, you should also understand the formula $\boxed{m(t)=m(0)2^{-t/h}}$ in terms of the half-life h.

Example Problems

4.1. A culture of a single cell creature Amoeba is found to triple its population in three weeks. Find its relative growth rate k. The formula for the population $P(t)$ as a function of time t is given as follows using the relative growth rate: $P(t)=P(0)e^{kt}$.

$\boxed{3P(0)=P(3)}=P(0)e^{3k}$

$\Rightarrow 3=e^{3k}$

$\Rightarrow k=\frac{\ln 3}{3}$

4.2. The number of bacteria in a cell culture is initially observed to be 50. Three hours later the number is 100. Assuming that the bacteria grow exponentially, how many hours after the initial observation does the number of bacteria become equal to 700?

4.3. The half-life of cesium-137 is 30 years. Suppose we have a 120-mg sample at the beginning. How long will it take until the remain of the sample becomes 2-mg ?

$h=30,\ m_0=120$

$2=m(T)=m_0 2^{-\frac{T}{30}}$

$\boxed{2=120\, 2^{-\frac{T}{30}}}$

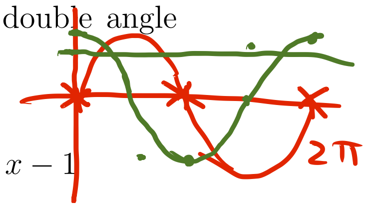
4.4. Initially there was 15 gm of a radioactive substance. After 3 years, only 7 gm remained. What is the half-life of this substance ?

$\frac{1}{60}=2^{-\frac{T}{30}} \Rightarrow T=30\cdot\log_2 60 = 30\cdot\frac{\ln 60}{\ln 2}$

Materials up to Exam 3 (including Exam 3)

Remark: There will be 20 questions regarding the materials up to Exam 3 (including Exam 3) in the Final Exam. Look also at the "NOTE" for the subject matter 14 about computing the total distance.

5. You are supposed to be able to solve the equations involving the trigonometric functions and find solutions on the given interval, using the basic formulas of the trigonometric functions (e.g., double angle formulas for sine and cosine



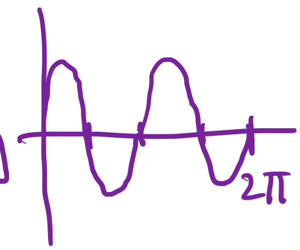
$$\begin{cases} \sin 2x &= 2\sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1 \end{cases}$$

and $\sin^2 x + \cos^2 x = 1$, etc.).

$\Rightarrow \begin{cases} \sin x = 0 \\ \text{or } \cos x = \frac{\sqrt{3}}{2} \end{cases}$ 3 + 2

Example Problems

5.1. How many solutions are there on the interval $[0, 2\pi]$ for the equation $\sqrt{3}\sin x = \sin(2x)$ $= 2\sin x \cos x \Rightarrow \boxed{0 = \sin x\left(\sqrt{3} - 2\cos x\right)}$



5.2. Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\cos(2x) - \sin(2x) = 0$.
5.3. Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $\cos(2x) - \sin x = 0$.
5.4. Find all the values x on the interval $[0, 2\pi]$ satisfying the equation $3\sin^2(x) - \cos^2(x) = 0$.

6. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems

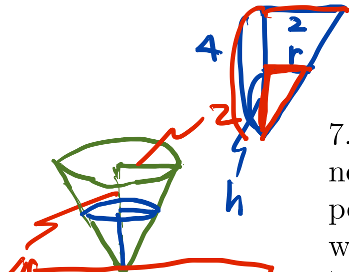
6.1. Find the derivative of the following function.
(i) $y = x^x$ (ii) $y = (\sqrt{x})^{\sin x}$
(iii) $y = x^{\ln x}$ (iv) $y = (\ln x)^x$
(v) $y = (\cot x)^{\sin x}$ (vi) $y = (\sqrt{x})^{\tan x}$

7. **TWO "Related Rates" problems will be given in the Final Exam.**

Of particular importance are:

- Light house problem
- Inverted circular conical tank problem
- Snowball problem
- Angle of elevation problem
- Area (of a rectangle or a triangle) problem

$\frac{r}{2} = \frac{h}{4} \Rightarrow \boxed{r = \frac{1}{2}h}$



$\boxed{V_{(t)} = \frac{1}{3}\pi r^2 h}$

$= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$

$= \frac{\pi}{12}h^3(t)$

$V' = \frac{\pi}{4}h^2 \cdot h'$

$h' = \frac{4V'}{\pi h^2} = \frac{4\cdot 2}{\pi \cdot 3^2}$

7.1. A lighthouse is located on a small island 4 km away from the nearest point P on a straight shoreline and its light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

7.2. A water tank has the shape of an inverted circular cone with base radius 2 m ~~and height~~ 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

$V(t)$

7.3. A snowball melts so that its surface decreases at the rate of 3 cm^2/min. How fast is the volume decreasing when the radius is 9 cm ?

$V'|_{r=9} = \frac{1}{2}\cdot 9 \cdot 3$

7.4. (speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

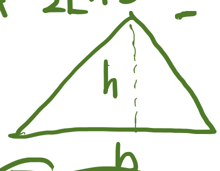
7.5. (angle) A 15-foot ladder is leaning against a vertical wall and its bottom is being pushed toward the wall at the rate of 2 ft/sec. At what rate is the acute angle θ between the ladder and the ground changing when the acute angle the ladder makes with the ground is $\pi/4$?

$A' = \frac{1}{2}[h'b + hb'] = \frac{1}{2}\left[h'b + \frac{2A}{b}b'\right]$

7.7. The base of a triangle is increasing at a rate of 2 cm/min while the area of the triangle is increasing at a rate 5 cm^2/min. At what rate is the altitude of the triangle changing when the base is 10 cm and the area is 50 cm^2 ?

$h' = ?$ $b = 10$, $A = 50$

7.8. Suppose you are recording a rocket launch from a location 3 miles away from the launch site. When the rocket is 1 mile high, you must rotate your camera angle at a rate of $\frac{1}{36}$ radians per second. How fast is the rocket moving (in miles per hour) at that moment ? The unit conversion reads $\frac{1}{36}$ rad/sec $= 100$ rad/hour.



$A = \frac{1}{2}hb$

$h = \frac{2A}{b}$

$h' = 2\frac{A'b - Ab'}{b^2}$

$A' = 5$, $b' = 2$

$V = \frac{4}{3}\pi r \frac{S}{4\pi}$

$= \frac{1}{3}rS$

$\Rightarrow r^2 = \frac{S}{4\pi}$

$S(t) = 4\pi r^2$

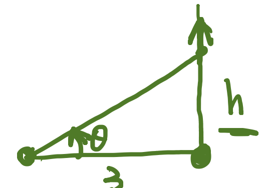
$V(t) = \frac{4}{3}\pi r^3$

$\boxed{S' = 8\pi r \cdot r'}$

$? = V' = 4\pi r^2 \cdot r'$

$= 4\pi r^2 \cdot \frac{S'}{8\pi r}$

$= \frac{1}{2}rS'$



$\theta' = 100$, $h = 1$

$h' = ?$

$\boxed{\tan\theta = \frac{h}{3}}'$

$\sec^2\theta \cdot \theta' = \frac{1}{3}h'$

$h' = \frac{3\theta'}{\cos^2\theta}$

$\cos^2\theta = \left(\frac{3}{\sqrt{10}}\right)^2 = \frac{9}{10}$

8. You are supposed to understand the idea of the linear approximation of a function $f(x)$ at $x = a$

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

and apply it to approximate the value of a function at a given point.

Example Problems

8.1. Find the formula for the linear approximation to $f(x) = \sqrt{x}$ at $a = 9$ and use it to approximate $\sqrt{9.3}$.

8.2. Find the formula for the linear approximation to $f(x) = \sqrt[3]{x}$ at $a = 27$ and use it to approximate $\sqrt[3]{26.8}$.

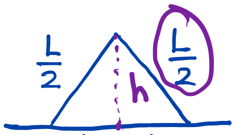
8.3. Write the formula for the linear approximation to $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.05}$.

9. TWO Optimization Problems will be given in the Final Exam.

Of particular importance are:

- Maximize the area of an isosceles triangle
- Minimize the cost of making a box
- Screen problem
- Hinge problem

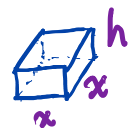
Example Problems



$A = bh = b\sqrt{\left(\frac{L}{2}\right)^2 - b^2}$

$\max_{b} A(b)$

9.1. What is the largest area of an isosceles triangle when we fix the sum of two equal sides to be a constant L ?



9.2. An open box (that is, a box with no top) is to have a square base, and a volume of 16 cubic meters. The material for the base costs \$1 per square meter, and the material for the sides costs \$2 per square meter. Find the cost of the cheapest such box.

$\min C = \$1 \times x^2 + \$2 \times 4 \times xh$

$= x^2 + 8xh$

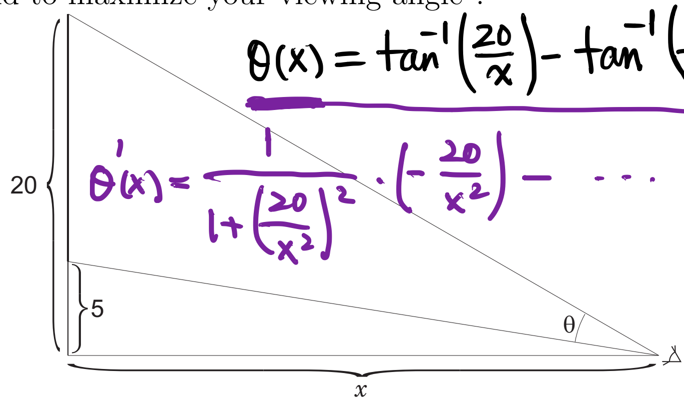
$16 = x^2 h \Rightarrow h = \frac{16}{x^2}$

$\Rightarrow C(x) = x^2 + \frac{8 \cdot 16}{x}$

$\min_{x} C(x)$

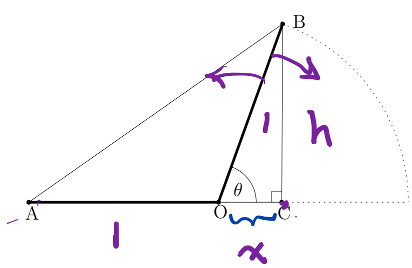
9.3. An auditorium with a flat floor has a large screen on one wall. The lower edge of the screen is 5 feet above eye level and the upper edge of the screen is 20 feet above eye level. How far from the screen should you stand to maximize your viewing angle ?

$\theta(x) = \tan^{-1}\left(\frac{20}{x}\right) - \tan^{-1}\left(\frac{5}{x}\right)$



$\theta'(x) = \frac{1}{1+\left(\frac{20}{x^2}\right)^2} \cdot \left(-\frac{20}{x^2}\right) - \cdots$

9.4. Two wooden bars of equal length AO = BO = 1 ft are connected by a hinge at point O so that one can rotate the bar BO around as shown in the picture below. Find the maximum area of the triangle $\triangle ABC$ when $0 < \theta < \pi$.



$h = \sqrt{1 - x^2}$

$A = \frac{1}{2}(1+x)h$

$= \frac{1}{2}(1+x)\sqrt{1-x^2}$

$\max_{x} A(x)$

10. You are supposed to understand the notions of continuity and differentiability, and their difference.

Example Problems

10.1. Determine where the followig function is continuous / differentiable.

(i) $f(x) = \begin{cases} \dfrac{x-2}{|x-2|} & \text{if} \quad x \neq 2 \\ 0 & \text{if} \quad x = 2. \end{cases}$

(ii) $g(x) = x|x-2|$ (iii) $h(x) = x(x-2)|x-2| = \begin{cases} -x(x-2)^2, & x<2, \\ x(x-2)^2, & x>2. \end{cases}$

$h'(2^-) = -\left[(x-2)^2 + x\cdot 2(x-2)\right]_{x=2} = 0$

$h'(2^+) = \left[(x-2)^2 + x\cdot 2(x-2)\right]_{x=2} = 0$

h(x) is cont. & diff.

10.2. Consider the function described below:

$$g(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

10.3?

Choose the right statement about the continuity and differentiability of the function $y = g(x)$ at 0.

A. The function g is continuous at 0 and differentiable at 0.
B. The function g is continuous at 0 but not differentiable at 0.
C. The function g is not continuous at 0 but differentiable at 0.
D. The function g is not continuous at 0 and not differentiable at 0.
E. The above description is not sufficient to judge the continuity or the differentiability of the function g.

10.4. Find the values of a and b so that the following function is continuous over $(-\infty, \infty)$:

$$f(x) = \begin{cases} \dfrac{3x^2 + 12x - b}{x^2 + 2x - 3} & \text{if} \quad x < -3 \\ x^2 - a & \text{if} \quad -3 \leq x. \end{cases}$$

$= 3(x^2+4x+3) = 3(x+1)(x+3)$

$x^2+2x-3 = (x-1)(x+3)$

$\boxed{x = 1, \boxed{-3}}$

-3 1

(1) $\left[3x^2+12x-b\right]_{x=-3} = 0$

$27 - 36 - b = 0 \Rightarrow b = -9$

$\lim_{x\to-3^-} f(x) = \lim_{x\to-3^-} \dfrac{3(x+1)}{x-1} = \dfrac{3\cdot(-2)}{-4} = \dfrac{2}{3}$

(2) $f(-3) = (-3)^2 - a = 9 - a = \lim_{x\to-3^-} f(x) = \dfrac{2}{3}$

$a = 9 - \dfrac{2}{3} = \dfrac{25}{3}$

11. Given a function which is one-to-one, with a given domain and range, you are supposed to be able to find the formula for its inverse function, describing its domain and range. You should also understand the relation between the graph of the original function and the one for its inverse.

Example Problems

11.1. Find a formula for the inverse of the following function, and describe the domain and range of the inverse function.

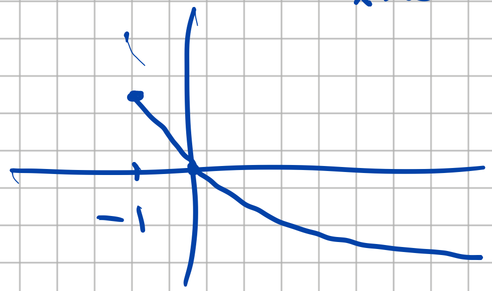
(i) $f(x) = \dfrac{6x-1}{2x+1}$ (ii) $f(x) = \dfrac{e^x}{e^x+1}$ (iii) $f(x) = \dfrac{e^x}{2e^x+1}$

(iv) $\underbrace{f(x) = 1 - \sqrt{x+1}}_{=y}$ for $\boxed{x \geq -1}$ (v) $f(x) = \dfrac{1 - \ln x}{1 + \ln x}$ for $x > e^{-1}$

$\sqrt{x+1} = 1 - y \Rightarrow x + 1 = (1-y)^2 \Rightarrow x = (1-y)^2 - 1$

$\text{dom}(f) = [-1, +\infty)$ $f(-1) = 1$ $\lim f(x) = -\infty$ $\text{range}(f) = (-\infty, 1]$ $y = (1-x)^2 - 1 = f^{-1}(x)$

$x \to \infty$



$\mathrm{dom}(f^{-1}) = \mathrm{range}(f) = (-\infty, 1]$

$\mathrm{range}(f^{-1}) = \mathrm{dom}(f) = [-1, +\infty)$

12. You are supposed to understand the method of Implicit Differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems

12.1. Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the equation $f(x)+x^2(f(x))^3 = 10$ and the condition $f(1) = 2$. Find $f'(1)$.

12.2. Find the equation of the tangent to the curve given by the equation $\cos(xy) = \ln(x)$ at the point $(x, y) = (1, \pi/2)$

12.3. Find the slope of the tangent to the curve given by the equation $y^2\sin(2x) = x^2\cos(2y)$ at a point $(\pi/2, \pi/4)$.

12.4. Find $\dfrac{dy}{dx}$ given $e^{x/y} = 7x - y$.

13. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions. You are also supposed to know the relation between the derivative of the original function and the derivative of its inverse (when it exists).

Example problems

13.1. Compute the derivative of the following function:

(i) $y = \sin(\sin(\sin x))$ (ii) $y = \left(\dfrac{t-2}{2t+1}\right)^9$

(iii) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$ (iv) $y = e^{\sec 3\theta}$

$\boxed{y = f^{-1}(x)} \Rightarrow f(y) = x$

$f'(y)\cdot y' = 1$

$y' = \dfrac{1}{f'(y)} = \boxed{\dfrac{1}{f'(f^{-1}(x))} = y'}$

13.2. Suppose that $F(x) = f^{-1}\left(\{g(x)\}^2\right)$ and that the functions f (which is one-to-one, and hence has its inverse) and g satisfy the following conditions:

$$\begin{cases} f(2) = 9, & f(9) = 5, \\ f'(1) = 4, & f'(2) = 3, & f'(3) = -2 \\ g(1) = 3, & g'(1) = 2 \end{cases}$$

Find $F'(1)$.

$F'(1) = \dfrac{1}{f'(f^{-1}(g^2(1)))} \cdot 2g(1)\cdot g'(1)$

$= \dfrac{2\cdot 3\cdot 2}{f'(f^{-1}(3^2))}$

$= \dfrac{12}{f'(2)} = \dfrac{12}{3}$

13.3. The function is given by the formula $f(x) = 2^{\sqrt{x^2+\ln x}}$. Compute $f'(1)$.

$f^{-1}(9) = 2$

$f'(x) = 2^{\sqrt{x^2+\ln x}} \cdot \ln 2 \cdot \left(\sqrt{x^2+\ln x}\right)'$

$\qquad\qquad\qquad\qquad \| $

$\dfrac{1}{2}(x^2+\ln x)^{-\frac{1}{2}} \cdot \left(2x + \dfrac{1}{x}\right)$

$$f'(1) = 2 \cdot \ln 2 \cdot \frac{1}{2}(1)^{-\frac{1}{2}} \cdot (2+1)$$

14. Given the position function of a particle, you are supposed to be able to compute its velocity, acceleration, understanding its physical meaning. You should be able to determine when a particle is speeding up or down, whether it is accelerating or decelerating. You are also supposed to be able to compute the "total" distance travelled during the given period.

NOTE: The last part of computing the total distance requires integration, which was discussed after Exam 3. However, this subject is classified as an item in "Materials up to Exam 3 (including Eam 3)" by convention.

Example Problems

14.1. The position of a particle is given by the function $s = f(t) = t^3 - 6t^2 + 9t$. Find the total distance traveled during the first 6 seconds.
~~14.2.~~ The velocity function, which is the 1st derivative of a position function $f(t)$ is given by the following formula $v = f'(t) = 6(t-1)(t-3)$. Can you determine the total distance traveled during the fisrt 5 seconds ?
14.3. A rock is thrown upward so that its height (in ft) after t seconds is given by $h(t) = 48t - 16t^2$. What is the velocity of the rock when its height is 32 ft on its way up ?

$v(t)$: 0 $+$ 1 $-$ 3 $+$ 5

$f(t) = \int v(t)\,dt + C$
$= 6\int (t^2 - 4t + 3)\,dt$
$= 6\left[\frac{1}{3}t^3 - 2t^2 + 3t\right] + C$
$S_1 = |f(1) - f(0)|$
$= |6[\frac{1}{3} - 2 + 3]|$
$S_2 = |f(3) - f(1)|$
$S_3 = |f(5) - f(3)|$

15. You are supposed to be able to use the 1st Derivative Test, as well as the 2nd Derivative Test, to find the local maximum and local minimum of a function.

Example Problems

15.1. The first derivative of a function f is given by

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5).$$

Find the values of x for which the function f takes
(a) local maximum, and
(b) local minimum.

15.2. Consider the function $f(x) = x^8(x-4)^7$.
(a) Find the critical numbers of the function f.
(b) What does the Second Derivative Test tell you about the behavior of f at these critical numbers ?
(c) What does the First Derivative Test tell you that the Second Derivative test does not ?
15.3. How many inflection points does the graph of the function $y = f(x) = x^5 - 5x^4 + 25x$ have ?
15.4. We have a function whose first derivative is given by the formula $f'(x) = (x-1)^3(x+1)^3$. Find the x-coordinates of the local extrema and the inflection points of the function.

16. You are supposed to be able to sketch the graph of a function by computing the 1st derivative (increasing or decreasing) and 2nd derivative (concave up or down). You are also supposed to be able to determine the horizontal/vertical/slant asymptotes of the graph.

Example Problems

16.1. Draw the graph of the following function:

(a) $y = f(x) = \dfrac{x}{x^2 - 16}$ (b) $y = f(x) = \dfrac{x}{x^2 + 16}$

(c) $y = f(x) = \dfrac{1}{x^2 - 16}$ (d) $y = f(x) = \dfrac{x^2}{x^2 - 16}$

(e) $y = f(x) = \ln|x^2 - 10x + 24|$ (vi) $y = f(x) = x^3 e^{-x^2/6}$

16.2. Find the equations of the horizontal, vertical, and slant asymptotes of the function

$$f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}.$$

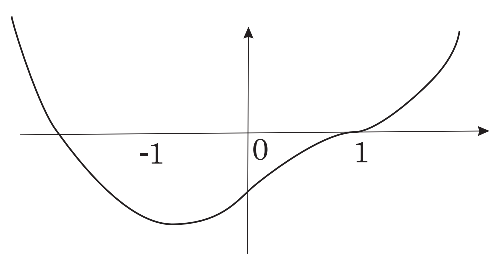
16.3. Which of the following graphs can be the graph of a function $y = f(x)$ whose 2nd derivative is given by the formula

$$f''(x) = x(x-1)^2(x+1) \ ?$$

What happens if we change the formula for the 2nd derivative to

$$f''(x) = -x(x+1)^2(x-1) \ ?$$

A.



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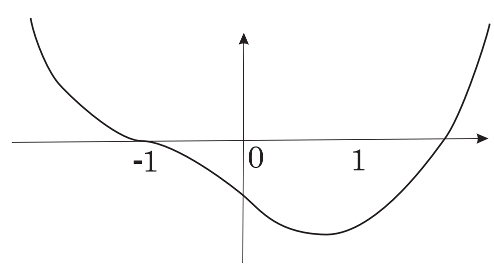

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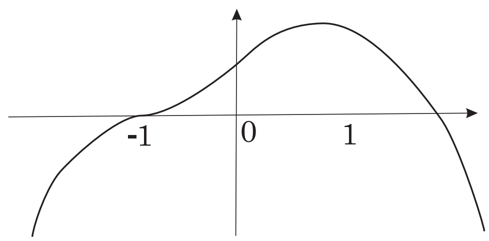

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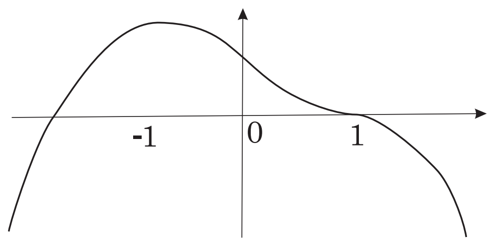

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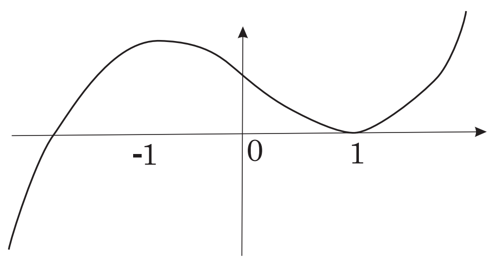
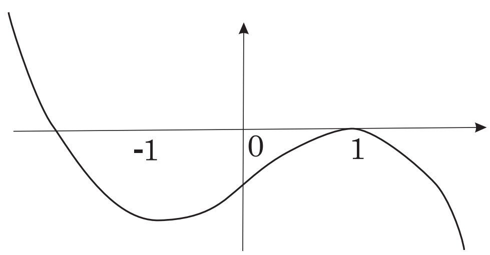
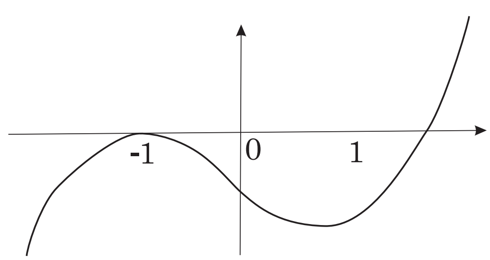


F.

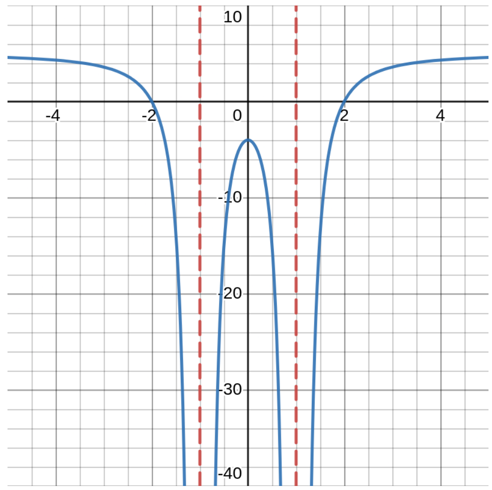


G.

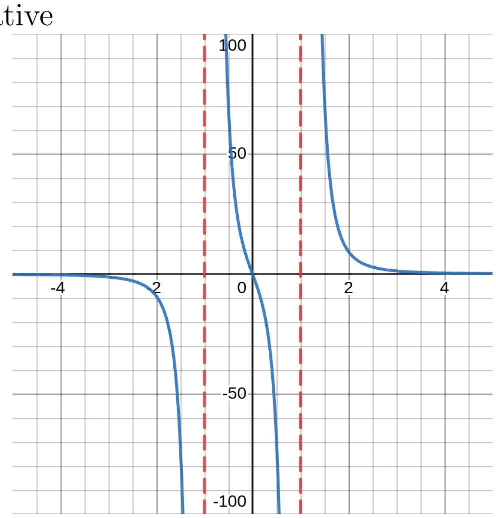


16.5. The graphs of the 1st derivative $y = f'(x)$ and the 2nd derivative $y = f''(x)$ of a rational function $y = f(x) = \dfrac{p(x)}{q(x)}$ are given as below:

the graph of the 1st derivative



the graph of the 2nd derivative



Draw the (rough) graph of the function $y = f(x)$.

17. You are supposed to know how to find the absolute maximum and absolute minimum of a function f defined on the closed interval $[a, b]$, by comparing the values on the end points $f(a), f(b)$ and the values on the critical value(s) $f(c)('s)$. You should know what the definition of a critical value is.

Example Problems:

1.1. Find the absolute maximum and absolute minimum values of the function f on the given interval.
(a) $f(x) = x^{2/3}$ on $[-1, 8]$
(b) $f(x) = \dfrac{\cos x}{2 - \sin x}$ on $[-\pi/2, \pi/2]$
(c) $f(x) = 2x^3 + 3x^2 - 12x$ on $[-1, 2]$
(d) $f(t) = 2\cos t + \sin 2t$ on $[0, 2\pi]$

18. You are supposed to know the statement of the Mean Value Theorem as well as its meaning, and also to know under what conditions you can apply the Mean Value Theorem. You are also supposed to be able to know how to apply this corollary of the Mean Value Theorem to compute some value which is seemingly difficult to determine otherwise: If $f'(x) = 0$ for all values of $x \in (a, b)$, then a continuous function f on the closed interval $[a, b]$ is actually a constant.

Example Problems

18.1. Consider the function $f(x) = x^4 - 2x^2 + 5x + 3$ over the interval $[-2, 2]$. How many values of $c \in (-2, 2)$ satisfy the statement of the Mean Value Theorem: $f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)}$?

18.2. Consider the function $f(x) = x(x^4 - 5)$ over the interval $[a, b] = [0, 2]$. What value(s) of $c \in (a, b)$ satisfies (satisfy) the statement of the Mean Value Theorem: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

18.3. Consider the function $y = f(x) = x^{2/3}$ over the interval $[-1, 1]$. Does it satisfy the conditions for the Mean Value Theorem to hold ? Do we have any value $c \in (-1, 1)$ such that $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)}$?

18.4. Determine the exact value of $\tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}(7)$.

19. You are supposed to be able to compute the (right/left hand side) limit, understanding its proper meaning, and using the Squeeze Theorem. You are also supposed to be able to determine the exact value of the limit who has an indeterminate form (e.g. $0/0, \pm\infty/\pm\infty, \infty-\infty$) using some proper technique.

Example Problems

19.1. Compute the following limits:

(i) $\lim_{x\to 3} \dfrac{x^2-9}{x^2-7x+12}$ (ii) $\lim_{x\to 5^-} \dfrac{x^2-4x-5}{|x-5|}$

(iii) $\lim_{x\to(\pi/2)^+} e^{\tan x}$ (iv) $\lim_{x\to 0} \left(\dfrac{\sqrt{5x^2+9}-3}{x^2}\right)$

(v) $\lim_{x\to 0} \left(\dfrac{5}{x^2-x}+\dfrac{5}{x}\right)$ (vi) $\lim_{x\to\infty} \left(\sqrt{x^2+3x+1}-x\right)$

(vii) $\lim_{x\to\infty} \dfrac{\sin x}{x}$ (viii) $\lim_{x\to 8} \dfrac{x-8}{\sqrt[3]{x}-2}$

(ix) $\lim_{x\to\infty} \dfrac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$ (ix*) $\lim_{x\to-\infty} \dfrac{x^3+3x+2}{2x^3+\sqrt{9x^6+4x+5}}$

20. You are supposed to know how to compute the limits using L'Hospital's Rule, under the provision that the limits are formally of the form $\dfrac{0}{0}, \dfrac{\pm\infty}{\pm\infty}$.

Example Problems

20.1. Compute the following limits:

(a) $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$ (b) $\lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$

(c) $\lim_{x\to 0} \dfrac{\sin x}{1-x^2}$ (d) $\lim_{x\to 0} \dfrac{7^x-6^x}{3^x-2^x}$

(e) $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$

21. You are suppose to know how to compute the limits of the indeterminate form $\pm\infty \times 0, \infty-\infty$.

Example Problems

21.1. Compute the following limits:

(a) $\lim_{x\to 0^+} \sin(x)\cdot\ln(2x)$ (b) $\lim_{x\to\infty} \left[x\cdot\ln\left(\dfrac{2\tan^{-1}(x)}{\pi}\right)\right]$

(c) $\lim_{x\to\left(\frac{\pi}{2}\right)^-} [(2x-\pi)\cdot\tan x]$ (d) $\lim_{x\to\infty} \left(\sqrt{x^2+2x+3}-x\right)$

(e) $\lim_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right)$

22. You are supposed to be able to compute the limits $\lim\limits_{x\to a}[f(x)]^{g(x)}$ formally of the form $0^0, \infty^0, 1^\infty$.

Example Problems

22.1. Compute the following limits:

(a) $\lim_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{7x}$ (b) $\lim_{x\to\infty}\left(\dfrac{2x+1}{2x-1}\right)^{3x+1}$

(c) $\lim_{x\to\infty}(2x+e^{5x})^{1/x}$ (d) $\lim_{x\to 0^+}\tan(5x)^{\sin x}$

(e) $\lim_{x\to 0}(1+3x)^{1/x}$