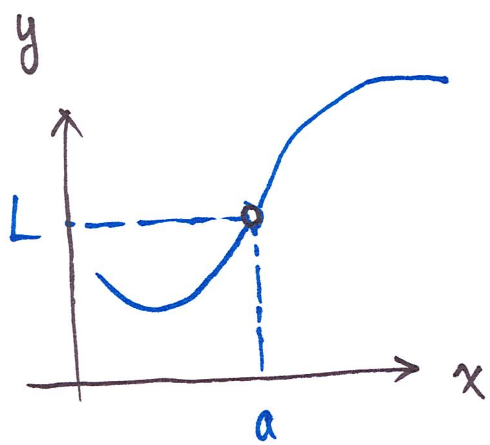
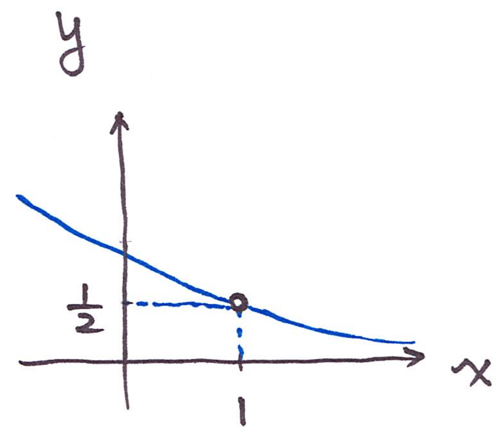
Chapter 2 Limits and Derivatives (5 lectures)

§2.2 Limit of a Function

$$\lim_{x \to a} f(x) = L \left\{ \begin{array}{l} \iff \text{values of } f(x) \text{ can be made arbitrarily close to } L \\ \qquad \text{by taking } x \text{ sufficiently close to } a, \text{ but } x \neq a \\ \iff |f(x) - L| \text{ can be made arbitrarily small} \\ \qquad \text{by taking } 0 < |x-a| \text{ sufficiently small} \\ \iff \end{array} \right.$$



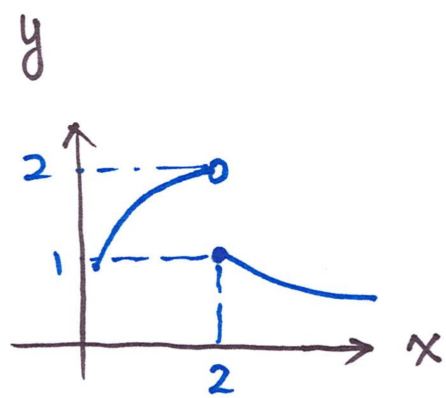


Ex. 1 $\displaystyle \lim_{x \to 1} \frac{x-1}{x^2-1}$

One-sided Limits

$$\lim_{x \to a^-} f(x) = L$$

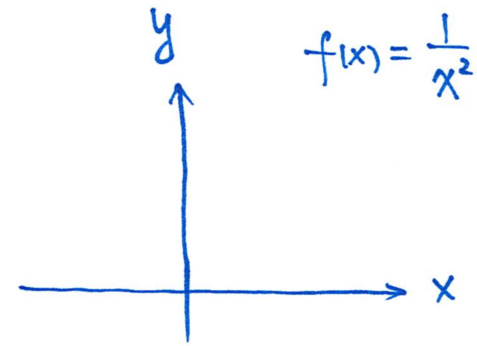
$$\lim_{x \to a^+} f(x) = L$$



$$\lim_{x \to a} f(x) = L \iff$$

Infinite Limits

$$\lim_{x \to a} f(x) = +\infty$$



$$\lim_{x \to a} f(x) = -\infty$$

Ex. $\displaystyle \lim_{x \to 5^+} \frac{x+1}{x-5} =$ $\qquad$ $\displaystyle \lim_{x \to \frac{\pi}{2}} \tan x =$

$\displaystyle \lim_{x \to 5^-} \frac{x+1}{x-5} =$ $\qquad$ $\displaystyle \lim_{x \to -\frac{\pi}{2}} \tan x =$

Vertical Asymptote $\quad x = a$ is a vertical asymptote of the curve $y = f(x)$

$$\Longleftrightarrow \quad \lim_{x \to a} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^{\pm}} f(x) = \pm\infty$$

§2.3 Calculating Limits Using the Limit Laws

Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

(1) $\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$; (2) $\lim_{x\to a}\left[c\,f(x)\right] = c\lim_{x\to a} f(x)$;

(3) $\lim_{x\to a}\left[f(x)\,g(x)\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$; (4) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

(5) $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$; (6) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ for even n $\lim_{x\to a} f(x) > 0$

(7) $\lim_{x\to a} c = c$; (8) $\lim_{x\to a} x = a$; (9) $\lim_{x\to a} x^n = a^n$

(10) $\lim_{x\to a} x^{\frac{1}{n}} = a^{\frac{1}{n}}$ $n > 0$; for even n, $a > 0$.

Examples

$$\lim_{x \to 5} \left(2x^2 - 3x + 4\right) =$$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} =$$

Direct Substitution Property Assume that $f(x)$ is a polynomial or rational function

$$\lim_{x \to a} f(x) = f(a)$$

- $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ if $f(x) = g(x)$ when $x \neq a$

<u>examples</u>

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} =$$

$$\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} =$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} =$$

$$\lim_{u \to 2} \frac{\sqrt{4u+1} - 3}{u-2}$$

• $\lim\limits_{x\to a} f(x) = L \iff \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$

<u>Ex. 7</u> Show that $\lim\limits_{x\to 0} |x| = 0$

<u>Ex. 8</u> Show that $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

<u>Ex. 9</u>

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x>4 \\ 8-2x & \text{if } x<4 \end{cases}$$

$\lim\limits_{x\to 4} f(x)$

- $f(x) \le g(x)$ for all $x \ne a$ near a $\Longrightarrow$ $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

- (The Squeeze Thrm) $f(x) \le g(x) \le h(x)$ $\forall$ $x \ne a$ but near a

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x) \Longrightarrow \lim_{x\to a} g(x) = L$$

example 11 show that $\lim\limits_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} = 0.$

- $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Ex. 7 Show that $\lim_{x \to 0} |x| = 0$

Ex. 8 Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Ex. 9

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

$$\lim_{x \to 4} f(x)$$

- $f(x) \le g(x)$ for all $x \neq a$ near a $\Longrightarrow$ $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$

- (The Squeeze Thrm) $f(x) \le g(x) \le h(x)$ $\forall$ $x \neq a$ but near a

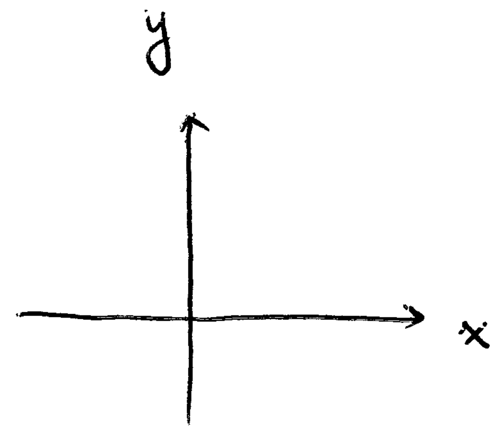
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x) \Longrightarrow \lim_{x\to a} g(x) = L$$

<u>example 11</u> show that $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin\left(\frac{\pi}{x}\right)} = 0$.

§2.5 Continuity

Def. $f(x)$ is continuous at $x = a$

$\iff$



Examples 2 where are each of the following functions discontinuous? why?

(a) $$f(x) = \frac{x^2 - x - 2}{x - 2}$$

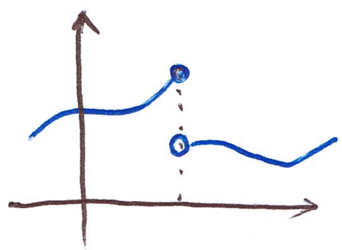
(b) $$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(c) $$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

(d) $f(x) = [\![x]\!] = n$ if $n \leq x < n+1$

<u>Def.</u> $f(x)$ is continuous from the right at $x=a \Longleftrightarrow$

$f(x)$ is continuous from the left at $x=a \Longleftrightarrow$



<u>Ex. 3</u> $f(x) = [\![x]\!]$ is continuous from right but left at integers $x=n$.

Def. $f(x)$ is continuous on an interval $(a, b]$

$\Longleftrightarrow$ (1) $f(x)$ is continuous at every $x \in (a, b)$

(2) $f(x)$ is continuous at $x = b$ from the left.

Ex. 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $[-1, 1]$.

<u>Theorem</u> Assume that $f(x)$ and $g(x)$ are continuous at $x=a$

$\Longrightarrow$ $f(x) \overset{+}{\underset{\times}{-}} g(x)$, are continuous at $x = a$ (for $\frac{f}{g}$, if $g(a) \neq 0$)

$c f(x)$ is continuous at $x = a$

<u>Proof of $f+g$ is cont.</u>

<u>Theorem</u> The following types of functions are continuous at every number in their domain

(1) polynomials, (2) rational functions, (3) root functions, (4) trigonometric functions

(5) inverse trigonometric functions, (6) exponential and logarithmic functions

<u>Theorem</u>

$$\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right) = f\left(g(a)\right)$$

Ex. (#24, P125) How to define $f(2)$ in order to make f is continuous at 2?

$$f(x) = \frac{x^3 - 8}{x^2 - 4}$$

Ex. (#35, P125) Use continuity to evaluate the limit

$$\lim_{x \to 2} x\sqrt{20 - x^2}$$

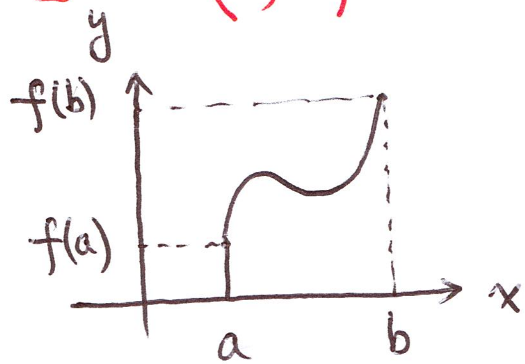
Ex. (#45, P125) For what value of the constant c is f continuous on $(-\infty, +\infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Intermediate Value Thrm

Assume that f is continuous on $[a,b]$, N is between $f(a)$ and $f(b)$, and $f(a) \neq f(b)$

$\Longrightarrow$ $\exists\, c \in (a,b)$ such that $f(c) = N$.



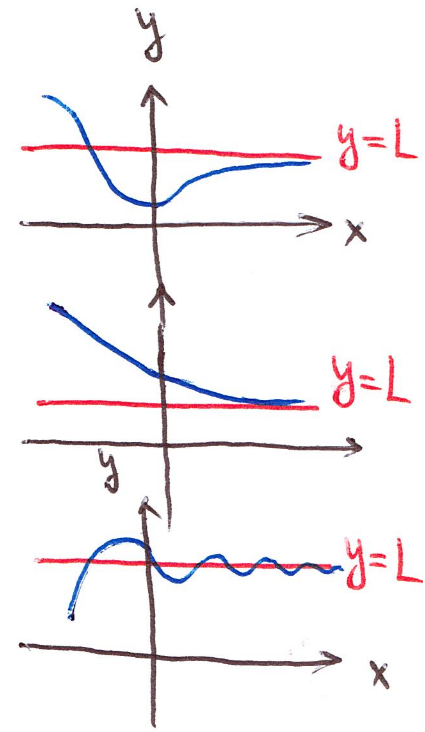
Ex. 10 Show that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a root between 1 and 2.

Ex. (#54, P126) $\ln x = x - \sqrt{x}$, $(2, 3)$

§2.6 Limits at Infinity; Horizontal Asymptotes

$$\lim_{x \to +\infty} f(x) = L$$

$$\lim_{x \to -\infty} f(x) = L$$



$y=L$ — horizontal asymptote $\iff \lim\limits_{x \to \pm\infty} f(x) = L$

Examples

$$\lim_{x \to +\infty} \frac{x^2-1}{x^2+1} =$$

$$\lim_{x \to \infty} \frac{1}{x} =$$

$$\lim_{x \to -\infty} \frac{x^2-1}{x^2+1} =$$

$$\lim_{x \to -\infty} \frac{1}{x} =$$

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2},$$

$$\lim_{x \to +\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \quad \text{for } r > 0$$

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0 \quad \text{for } r > 0 \text{ and } x^r \text{ is defined.}$$

Examples

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Ex. 4 Find the horizantol and vertical asymptotes of the graph of $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Ex. 5 $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

Ex. 7 $\lim\limits_{x\to 0^-} e^{\frac{1}{x}}$

Ex. 8 $\lim\limits_{x\to\infty} \sin x$

- ## Infinite Limits at Infinity

$$\lim_{x \to +\infty} f(x) = \pm\infty \quad , \quad \lim_{x \to -\infty} f(x) = \pm\infty$$

Ex. 9 $\displaystyle\lim_{x \to \infty} x^3 =$

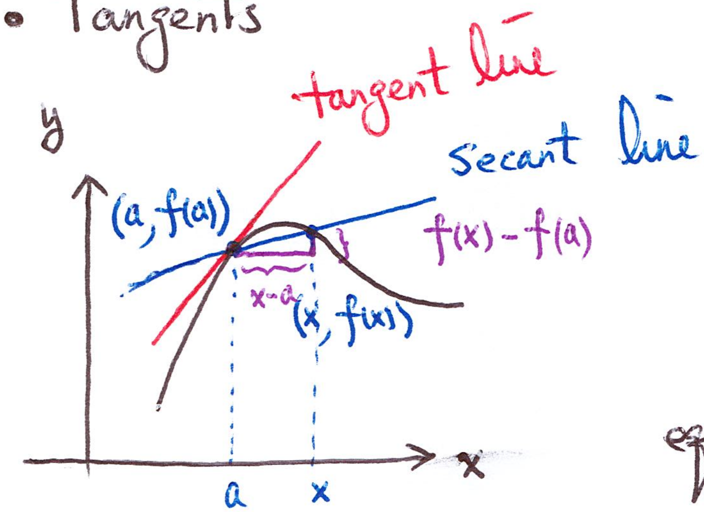
$\displaystyle\lim_{x \to -\infty} x^3 =$

Ex. 10 $\displaystyle\lim_{x \to \infty} \left(x^2 - x\right) =$

Ex. 11 $\displaystyle\lim_{x \to \infty} \frac{x^2 + x}{3 - x} =$

§2.7 Derivatives and Rate of Change

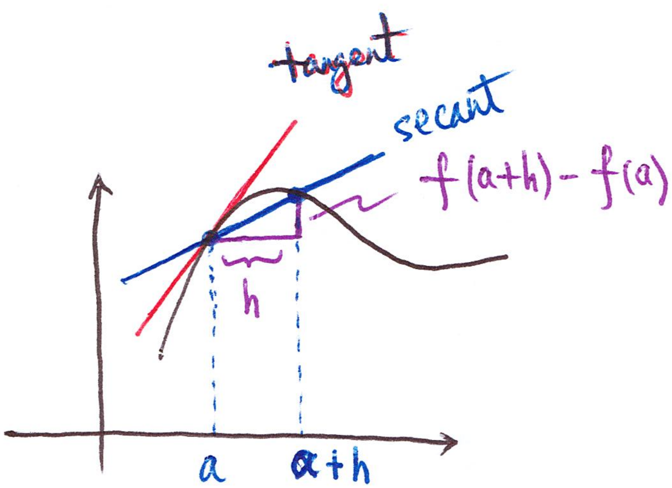
- Tangents



the slope of the secant line =

the slope of the tangent line =

equation of tangent line passing through (a, f(a))



Ex. 1 Find an equation of the tangent line to $y = x^2$ at pt $(1,1)$.

Ex. 2 Find an equation of the tangent line to the hyperbola $y = \frac{1}{x}$ at $(3,1)$.

- Velocities

$s = f(t)$ the position function of a moving object along a straight line

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

instantaneous velocity =
at $t = a$

Ex. 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Galileo's Law

$s(t) = 4.9t^2$ m

• Derivatives • Rates of Change

the derivative of a function f at $x=a$

$$f'(a) = \lim_{x \to a} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex. 4 & 5 $y = f(x) = x^2 - 8x + 9$

(a) $f'(a) = ?$ (b) the eq. of the tangent line at $(3, -6)$?

§2.8 The Derivative as a Function

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex. 1 See Fig. 2 on P153.

Examples compute $f'(x)$ for (a) $f(x) = x^3 - x$, (b) $f(x) = \sqrt{x}$, (c) $f(x) = \frac{1-x}{2+x}$

- Other Notations $\qquad y = f(x)$

derivative $\quad f'(x) = y' = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx} f = Df(x) = D_x f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Def. A function f is differential at a $\iff$ $f'(a)$ exists.

f is differential on an open interval (a,b) $\iff$ $f'(x)$ exists for all $x \in (a,b)$

Ex. 5 Where is $f(x) = |x|$ differentiable ?

Theorem f is differentiable at $a \Longrightarrow f$ is continuous at a.

Proof

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a)$$

- **Not-Differentiable Functions**

- Higher Derivatives

the 2^{nd} derivatives

the 3^{rd} derivatives

Ex. 6&7 $f(x) = x^3 - x$, $f''(x) = ?$, $f'''(x) = ?$, $f^{(4)}(x) = ?$