Chapter 3 Differentiation Rules (10 lectures)

§3.1 Derivatives of Polynomials and Exponential Functions

- $\frac{d}{dx}(c) = 0$, c - constant

- $\frac{d}{dx}(x^n) = n x^{n-1}$, (1) $n > 0$ integer; (2) n is a real number

$\underline{n=1}$ $\frac{d}{dx}(x) =$

$\underline{n=2}$ $\frac{d}{dx}(x^2) =$

$\underline{n=3}$ $\frac{d}{dx}(x^3) =$

$\underline{n=-1}$ $\quad \frac{d}{dx}\left(x^{-1}\right) =$

$\underline{n=\frac{1}{2}}$ $\quad \frac{d}{dx}\left(x^{\frac{1}{2}}\right) =$

<u>Examples</u>

$f(x) = \frac{1}{x^2}$, $\quad f'(x) =$

$y = \sqrt[3]{x^2}$, $\quad \frac{dy}{dx} =$

Ex. 3 Find equations of the tangent and normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$. Illustrate by graphing the curve and these lines.

- $$\frac{d}{dx}\left(c\, f(x)\right) = c\,\frac{d}{dx} f(x)$$

- $$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

<u>example</u>

$$\frac{d}{dx}\left(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5\right)$$

$$=$$

Ex. 6 Find the points on the curve $y = x^4 - 6x^2 + 4$, where the tangent line is horizontal.

Ex. 7 The equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$, s – centimeters, t – seconds

(a) Find the acceleration as a function of time. (b) What is the acceleration after 2 seconds?

- exponential functions $f(x) = a^x$, base $a>0$

$$f'(x) = f'(0)\, a^x$$

$$\parallel$$

$$\lim_{h\to 0} \frac{a^{x+h} - a^x}{h} =$$

<u>$a=2$</u> $f'(0) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.69 \Longrightarrow \frac{d}{dx} 2^x \approx$

<u>$a=3$</u> $f'(0) = \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10 \Longrightarrow \frac{d}{dx} 3^x \approx$

- Number e

e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$$\frac{d}{dx} e^x = e^x$$

Ex. 8 $f(x) = e^x - x$, compute f' and f''.

Ex. 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

§3.2 The Product and Quotient Rules

product rule

$$\frac{d}{dx}\Big(f(x)\,g(x)\Big) = f(x)\,\frac{d}{dx}g(x) + g(x)\,\frac{d}{dx}f(x)$$

$$\| $$

$$\lim_{h\to 0} \frac{f(x+h)\,g(x+h) \qquad - f(x)\,g(x)}{h}$$

__Examples__

(1) $f(x) = xe^{x}$, compute $f'(x)$ and $f^{(n)}(x)$

(2) $f(t) = \sqrt{t}\,(a + bt)$, compute $f'(t)$

(3) $f(x) = \sqrt{x}\, g(x)$, $g(4) = 2$, $g'(4) = 3$. Compute $f'(4)$

quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$$

$$\| $$

$$\lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{g(x)\,f(x+h) \qquad\qquad - f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$$

Ex. 4 $y = \dfrac{x^2+x-2}{x^3+6}$, compute y'.

Ex. 5 $f(x) = \dfrac{3x^2+2\sqrt{x}}{x}$, compute $f'(x)$

§3.3 Derivatives of Trigonometric Functions

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

$$\| $$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} =$$

Ex. 1 $y = x^2 \sin x$, compute y'.

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

- $(\sin x)' = \cos x$, • $(\cos x)' = -\sin x$, • $(\tan x)' = \sec^2 x$,
- $(\cot x)' = -\csc x$, • $(\sec x)' = \sec x \tan x$, • $(\csc x)' = -\csc x \cot x$.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' =$$

<u>Ex. 2</u> $f(x) = \dfrac{\sec x}{1 + \tan x}$, $f'(x) = ?$ For what values of x does the graph of f have a horizontal tangent?

Ex. 4 $f(x)=\cos x$, compute $f^{(27)}(x)$.

Ex. 5 $\lim\limits_{x\to 0} \dfrac{\sin 7x}{4x} =$

Ex. 6 $\lim\limits_{x\to 0} x\cot x =$

§3.4 The Chain Rule

- composition of functions

$$(f\circ g)(x) = f(g(x)) \quad \text{or} \quad y = f(u) \text{ and } u = g(x)$$

- the chain rule

$$(f\circ g)'(x) = f'(g(x))\, g'(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\| $$

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{[f'(g(x)) + \varepsilon][g(x+h) - g(x)]}{h}$$

f is differentiable at $g(x)$
and $\varepsilon \to 0$ as $h \to 0$

Examples Compute derivative

(1) $f(x) = \sqrt{x^2+1}$

(2) $y = \sin x^2$

(3) $y = \sin^2 x$

(4) $y = \left(g(x)\right)^n$

(5) $y = \left(x^3-1\right)^{100}$

(6) $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$

(7) $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

(8) $y = (2x+1)^5 (x^3 - x + 1)^4$

(9) $y = e^{\sin x}$

• $\boxed{\dfrac{d}{dx}\left(a^x\right) = a^x \ln a}$

$\quad \|$

$\dfrac{d}{dx}\left(\qquad \right) =$

• $\begin{cases} y = f(u) \\ u = g(x) \\ x = h(t) \end{cases}$ $\qquad \dfrac{dy}{dt} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} \cdot \dfrac{dx}{dt}$

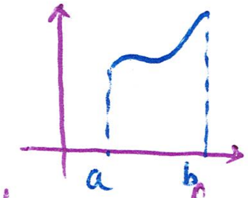
<u>Ex. 8</u> $f(x) = \sin\left(\cos(\tan x)\right)$, compute $f'(x)$.

<u>Ex. 9</u> $y = e^{\sec(3\theta)}$, compute y'.

§3.5 Implicit Differentiation

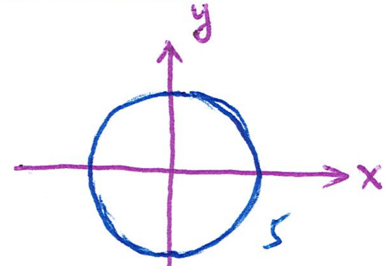
- explicit $y = f(x)$
- implicit $f(x, y) = \text{constant}$

difficult to solve it for y

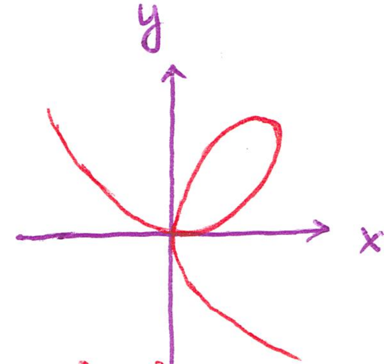


graph $C = \{(x, f(x)) \mid x \in D\}$

level curves



$x^2 + y^2 = 25$



$x^3 + y^3 = 6xy$

The folium of Descartes

implicit differentiation

Ex. 1 (a) $x^2 + y^2 = 25$, find $\frac{dy}{dx}$

(b) Find an equation of the tagent line to the circle $x^2 + y^2 = 25$ at $(3, 4)$.

Solution 1 (implicit differentiation)

Solution 2 (solving the equation for y)

__Ex. 2__ (a) Find y' if $x^3+y^3=6xy$

(b) Find the tangent to the folium of Descartes $x^3+y^3=6xy$ at $(3,3)$.

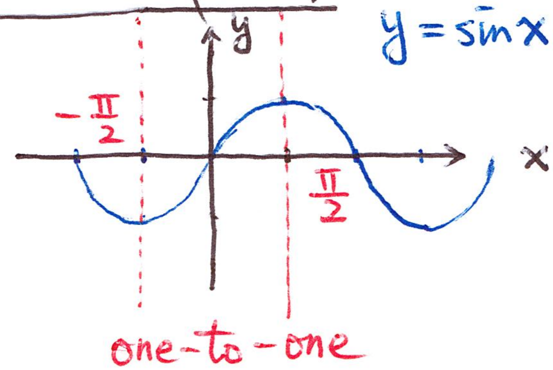
(c) At what point in the first quadrant is the tangent line horizontal?

Ex. 3 Find y' if $\sin(x+y) = y^2 \cos x$.

Ex. 4 Find y'' if $x^4 + y^4 = 16$

- Inverse Trigonometric Functions (§1.6)

$y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



inverse $y = \sin^{-1} x$, $-1 \le x \le 1$

$= \arcsin x$

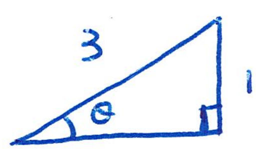
$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

identities $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$.

Ex. (a) $\sin^{-1}\left(\frac{1}{2}\right) = ?$ and (b) $\tan\left(\sin^{-1}\frac{1}{3}\right) = ?$

(a) let $\sin^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta =$

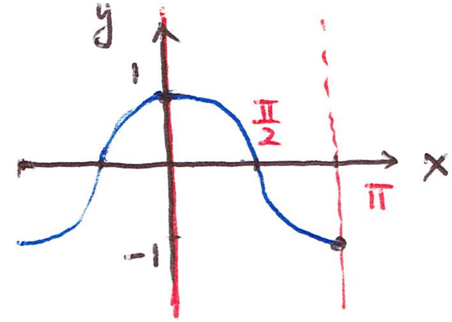
(b) let $\sin^{-1}\left(\frac{1}{3}\right) = \theta \Rightarrow \sin\theta = \frac{1}{3}$



- $y = \cos^{-1} x = \arccos x, \quad x \in [-1, 1]$

  $y \in [0, \pi]$

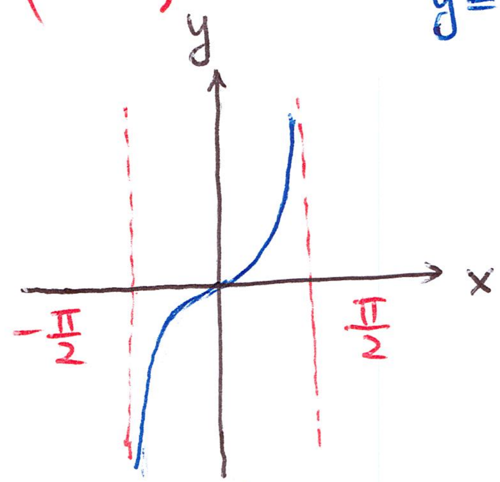
  <u>identities</u> $\cos^{-1}(\cos x) = x$ and $\cos(\cos^{-1} x) = x$.



$y = \cos x$

- $y = \tan^{-1} x, \quad x \in (-\infty, +\infty)$

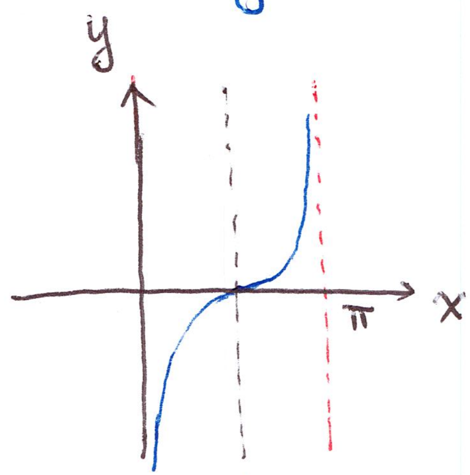
  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$y = \tan x$

- $y = \cot^{-1} x, \quad x \in (-\infty, +\infty)$

  $y \in (0, \pi)$



$y = \cot x$

- <u>derivatives of inverse trigonometric functions</u>

$y = \sin^{-1} x$ $\qquad \frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

$\Downarrow$

$\sin y = x \Longrightarrow$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Ex. 5 Differentiate

(a) $y = \dfrac{1}{\sin^{-1} x}$, (b) $f(x) = x \arctan \sqrt{x}$

§3.6 Derivatives of Logarithmic Functions

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}, \qquad \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}.$$

$$y = \log_a x \implies x = a^y \implies$$

Examples Differentiate

(1) $y = \ln\left(x^3+1\right)$

(2) $y = \ln\left(\sin x\right)$

(3) $f(x) = \sqrt{\ln x}$

(4) $f(x) = \log_{10}(2+\sin x)$

(5) $y = \ln \dfrac{x+1}{\sqrt{x-2}}$

(6) $f(x) = \ln|x|$ for $x \neq 0$

- Logrithmic Differentiation $\quad y = f(x)$

(1) $\ln y = \ln f(x) = \cdots$, (2) implicit differentiation, (3) solve the equation for y'

Ex. 7 $\quad y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

Ex. 8 $\quad y = x^{\sqrt{x}}$

Solution 1 $\quad \ln y = \ln x^{\sqrt{x}}$

Solution 2 $\quad y = x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}}$

$= e^{\sqrt{x}\ln x}$

§3.7 Rates of Change in the Natural and Social Sciences

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- Physics $\quad s = f(t)$ — the position function of a moving particle in a straight line

velocity $\quad v(t) = \dfrac{ds}{dt}$, $\quad$ acceleration $\quad a(t) = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2}$

Ex. 1 The position of a particle is given by $\quad s = f(t) = t^3 - 6t^2 + 9t \; \left\langle \begin{array}{l} s - \text{meters} \\ t - \text{seconds} \end{array} \right.$

(a) Find the velocity at time t.

(b) What is the velocity after 2 s? After 4 s?

(c) When is the particle at rest?

(d) When is the particle moving forward (in the positive direction)?

(e) Draw a diagram to represent the motion of the particle.

(f) Find the total distance traveled by the particle during the first 5 s.

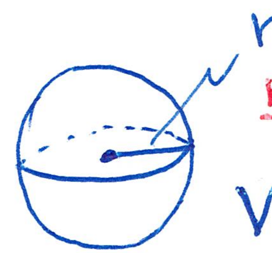
(g) Find the acceleration at time t and after 4 s.

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 5$.

(i) When is the particle speeding up? When is it slowing down?

§3.9 Related Rates (2 lectures)

Ex. 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?



relation

$$V = \frac{4}{3}\pi r^3$$

Given the rate of increase of the volume is 100 cm^3/s

V — volume

Unknown the rate of increase of the radius when the diameter is 50 cm

r — radius

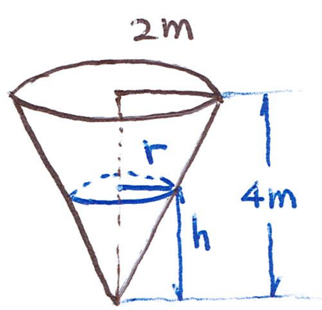
Ex. 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Given

Unknown

relation

Ex. 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



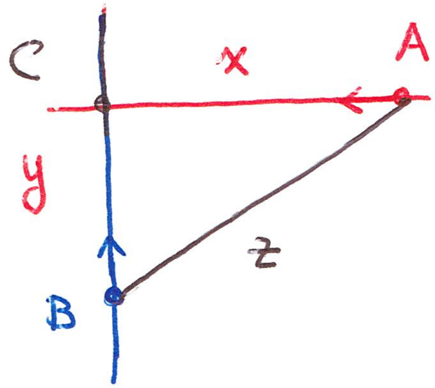
Given

Unknown

relation

$$V = \frac{1}{3}\pi r^2 h$$

Ex. 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

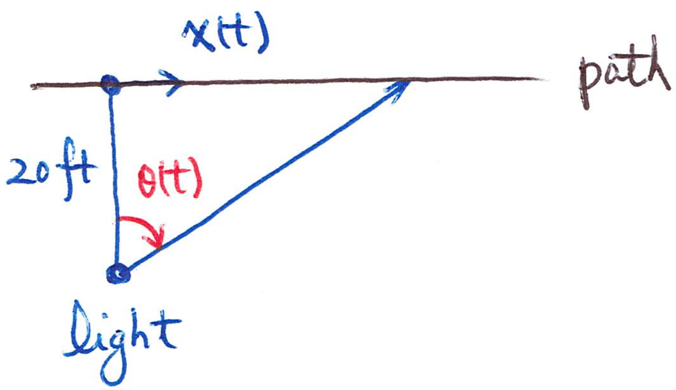


Given

Unknown

relation

Example 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Given

Unknown

relation

#14 If a snow ball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm

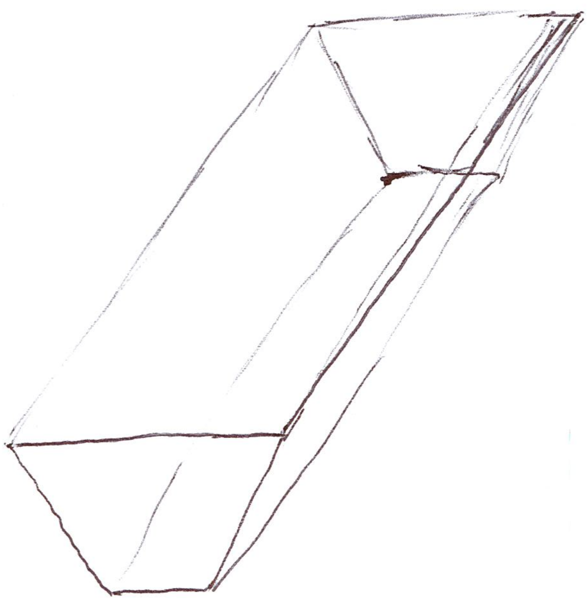
#15 A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

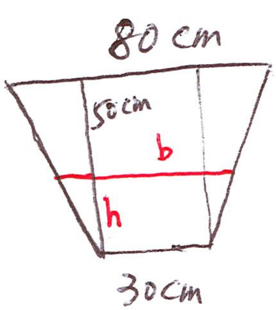
#17 Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

#18 A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

#25 Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min at the same ~~tu~~ time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the water height is 2 m, find the rate at which water is being pumped into the tank.

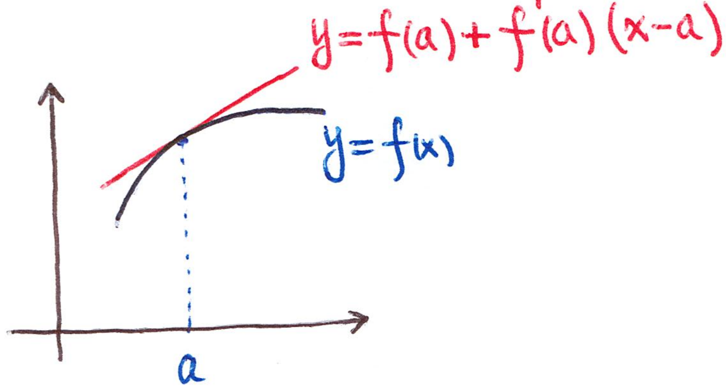
#27 A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has height 50 cm. If the trough is being filled with water ~~with~~ at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?





$$A = \frac{30+b}{2} h$$

§3.10 Linear Approximations and Differentials



when x near a

$$f(x) \approx L(x) \equiv f(a)+f'(a)(x-a)$$

Linearization of f at a.

Linear approximation of f at a

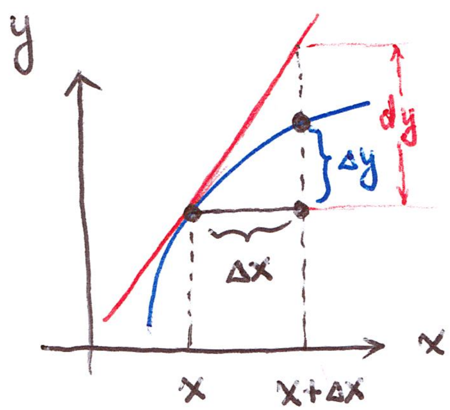
Ex. 1 Find the linearization of the function $f(x)=\sqrt{x+3}$ at $a=1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations over- or under-estimates?

- **differentials** $\quad y = f(x)$, $f(x)$ is differentiable

differential dx is an independent variable

differential

$$dy = f'(x)\,dx \quad \text{is a dependent variable}$$



let $dx = \Delta x$

$$\Longrightarrow \quad \Delta y = f(x+\Delta x) - f(x) \approx f'(x)\Delta x = f'(x)dx = dy$$

Ex. 3 $\quad y = f(x) = x^3 + x^2 - 2x + 1$. Compare the values of Δy and dy when x changes

(a) from 2 to 2.05

(b) from 2 to 2.01

Examples

#14 Find the differential of $y = e^{\tan(\pi t)}$

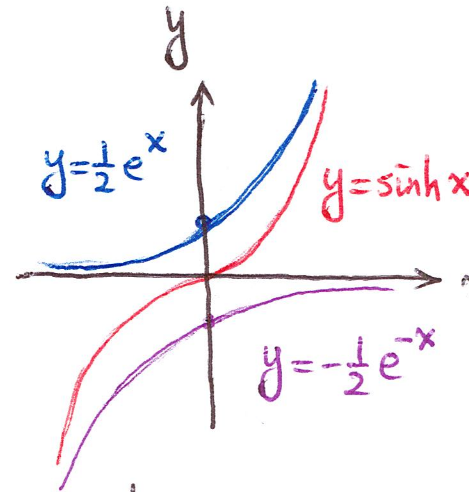
#15 Find the differential dy and evaluate dy for the given values of x and dx:

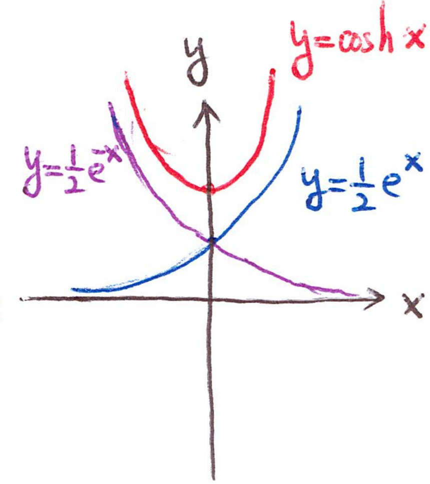
$y = e^{\frac{x}{10}}$, $x = 0$, $dx = 0.1$

§3.11 Hyperbolic Functions

- hyperbolic sine $\quad \sinh x = \dfrac{e^{x} - e^{-x}}{2}$

- hyperbolic cosine $\quad \cosh x = \dfrac{e^{x} + e^{-x}}{2}$





- $\tanh = \dfrac{\sinh x}{\cosh x}$, $\quad \coth x = \dfrac{\cosh x}{\sinh x}$, $\quad \operatorname{sech} x = \dfrac{1}{\cosh x}$, $\quad \operatorname{csch} x = \dfrac{1}{\sinh x}$

- hyperbolic identities

$\sinh(-x) = -\sinh x$ (odd function)

$\cosh(-x) = \cosh x$ (even function)

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sin(-x) =$

$\cos(-x) =$

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

Ex. 1 Prove (a) $\cosh^2 x - \sinh^2 x = 1$ and (b) $1 - \tanh^2 x = \operatorname{sech}^2 x$.

Proof

- derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) =$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Ex. 2 $\frac{d}{dx}(\cosh \sqrt{x}) =$