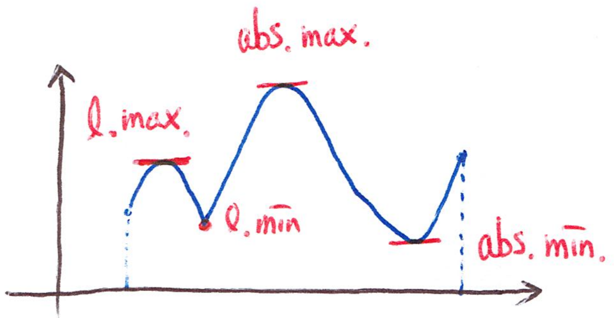
Chapter 4. Application of Differentiation (9 lectures)

§4.1 Maximum and Minimum Values



Def. (absolute max./min) $c \in \mathrm{dom}(f)$

(1) $f(c)$ is the absolute maximum value of f on D

$$\iff f(c) \geq f(x) \quad \forall x \in D$$

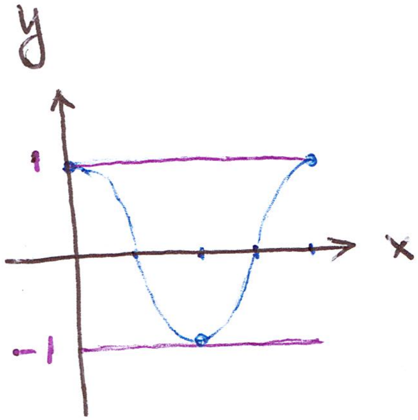
(2) $f(c)$ is the absolute minimum value of f on D

$$\iff f(c) \leq f(x) \quad \forall x \in D$$

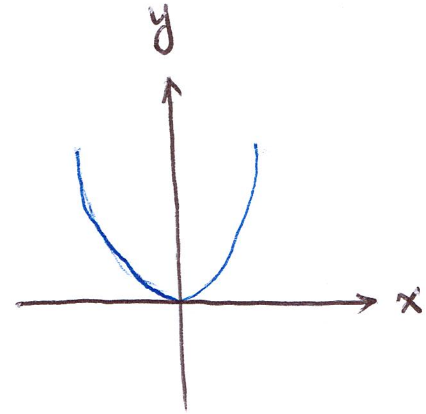
Def. (local max./min.)

(1) $f(c)$ is a local maximum value of $f \iff f(c) \geq f(x)$ when x is near c.

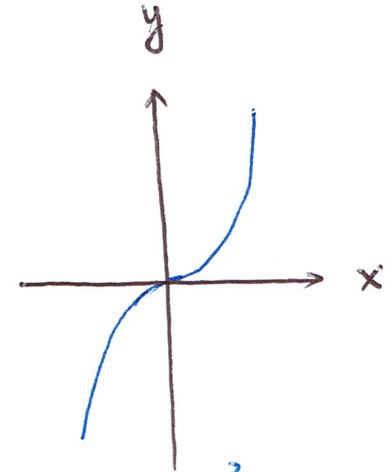
(2) $f(c)$ is a local minimum value of $f \iff f(c) \leq f(x)$ when x is near c.



$y = \cos x$



$y = x^2$



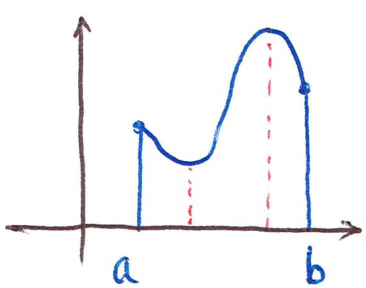
$y = x^3$

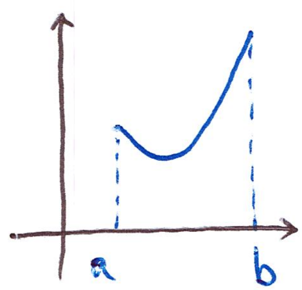
The Extreme Value Thrm

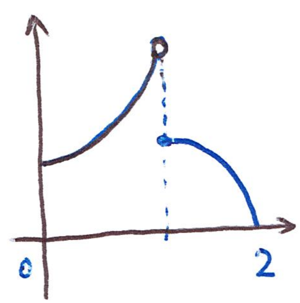
Assume that f is continuous on a closed interval $[a, b]$

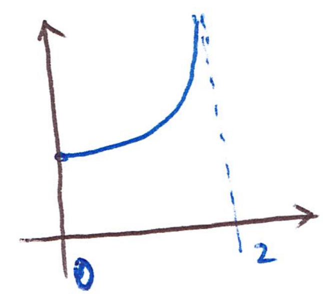
$\Longrightarrow$ f attains an absolute max. value $f(c)$ and an absolute minimum value $f(d)$

at some numbers c and d in $[a, b]$





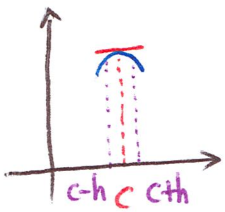




Fermat's Theorem

Assume that f has a local maximum or minimum at c and $f'(c)$ exists

$$\Longrightarrow \quad f'(c) = 0$$



$$\frac{f(c+h)-f(c)}{h} \begin{cases} \leq 0 & h>0 \\ \geq 0 & h<0 \end{cases}$$

Ex. 5 $\quad f(x) = x^3$

Ex. 6 $\quad f(x) = |x|$

Def. (critical number)

$c \in dom(f)$ is a critical number $\Longleftrightarrow$ either $f'(c)=0$ or $f'(c)$ does not exist.

Ex. 7 Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x)$.

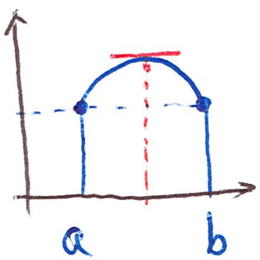
Ex. 8 Find the absolute extreme values of $f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$.

<u>Ex. 9</u> Find the absolute extreme value of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

§4.2 The Mean Value Theorem

Rolle's Theorem

Assume that $\begin{cases} (1)\ f \text{ is continuous on } [a,b] \\ (2)\ f \text{ is differentiable on } (a,b) \\ (3)\ f(a)=f(b) \end{cases} \Longrightarrow \exists\, c \in (a,b)$ such that $f'(c)=0$



Proof

Case (a) $f(x)=k$ constant $\Longrightarrow$

Case (b) For some $x \in (a,b)$, $f(x) > f(a)$.

f is continuous on $[a,b] \Longrightarrow \exists\, c \in [a,b]$ s.t. $f(c)$ is the abs. max.

$\Longrightarrow f(c) \geq f(x) > f(a) = f(b)$

$\Longrightarrow c \neq a$ or b $\Longrightarrow c \in (a,b)$

Case (c) For some $x \in (a,b)$, $f(x) < f(a)$

<u>Ex. 1</u> A ball is thrown directly upward.

$s = f(t)$ — position of a moving object.

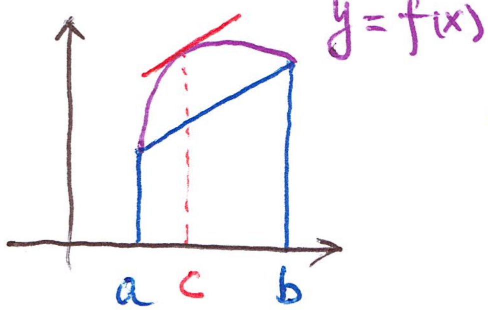
$f(a) = f(b) \Longrightarrow \exists\, c \in (a,b)$ s.t. $f'(c) = 0$ — velocity.

<u>Ex. 2</u> Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

The Mean Value Theorem

Assume that $\begin{cases} (1)\ f \text{ is continuous on } [a,b] \\ (2)\ f \text{ is differentiable on } (a,b) \end{cases} \Longrightarrow \exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Proof (making use of Rolle's Thrm)

the equation of the secant line : $y =$

an auxiliary function: $h(x) =$

Ex. 3 Use $f(x) = x^3 - x$ on $[0,2]$ to illustrate the Mean Value Thrm.

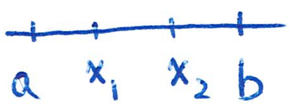
Ex. 4 Application of MVT to a moving object.

Ex. 5 Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x.

How large can $f(2)$ possibly be?

Theorem $f'(x) = 0 \ \forall x \in (a,b) \Longrightarrow f$ is constant on (a,b)

Proof $\forall x_1, x_2 \in (a,b)$ and $x_1 < x_2$



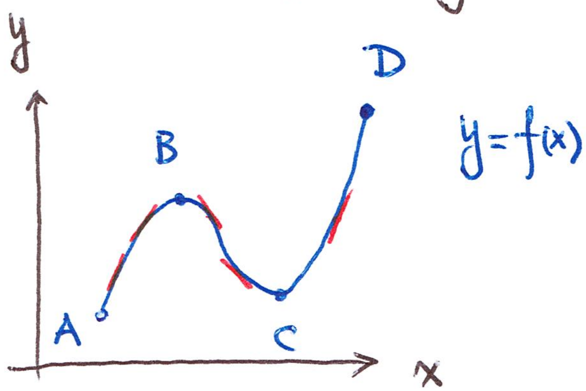
$f(x_2) - f(x_1) =$

Corollary $f'(x) = g'(x) \ \forall x \in (a,b) \Longrightarrow f(x) = g(x) + c$, where c is a constant.

Ex. 6 Prove the identity $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ by using calculus.

§4.3 How Derivatives Affect the Shape of a Graph

- what does $f'(x)$ say about f ?



Increasing / Decreasing Test

(a) $f'(x) > 0$ on an interval $I \Longrightarrow$

(b) $f'(x) < 0$ on an interval $I \Longrightarrow$

Proof of (a) $\forall\ x_1, x_2 \in I$ and $x_1 < x_2$

$$f(x_2) - f(x_1) =$$

Ex. 1 Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

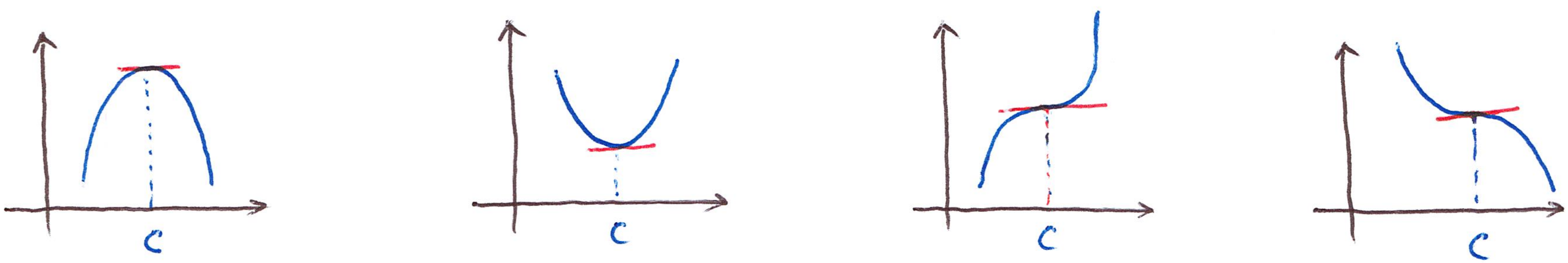
The First Derivative Test

Assume that c is a critical number of a continuous function f.

(a) f' changes from positive to negative at c $\Longrightarrow$

(b) f' changes from negative to positive at c $\Longrightarrow$

(c) f' does not change sign at c $\Longrightarrow$



Ex. 2 Find the local minimum and maximum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

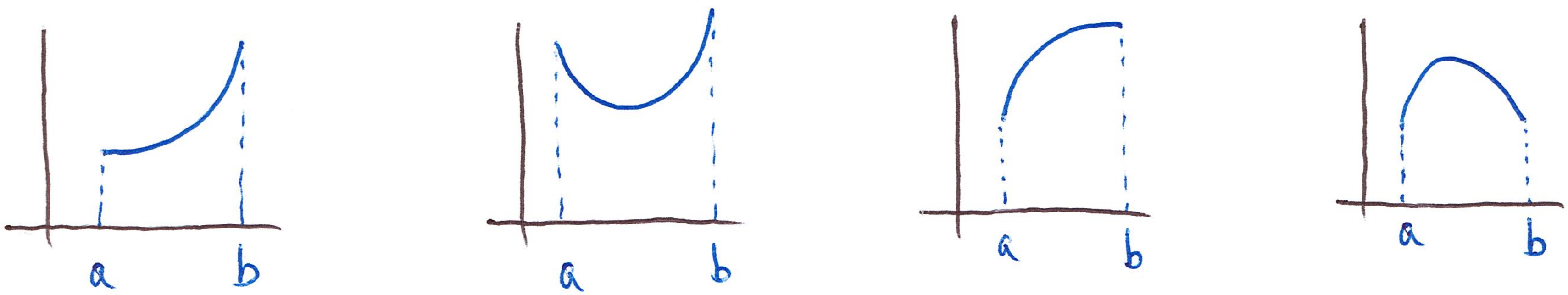
Ex. 3 Find the local maximum and minimum values of $g(x) = x + 2\sin x$ on $[0, 2\pi]$.

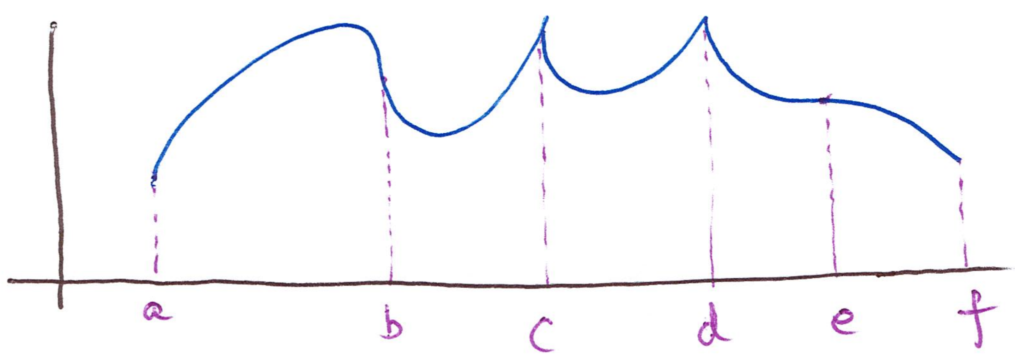
- what does $f''(x)$ say about f?

Def. (concave upward/downward)

(1) f is concave upward on an interval $I \iff$ the graph of f lies above all of its tangents

(2) f is concave downward on an interval $I \iff$ the graph of f lies below all of its tangents.





Concavity Test

(a) $f''(x) > 0 \ \forall x \in I \implies$ the graph of f is concave on I

(b) $f''(x) < 0 \ \forall x \in I \implies$ the graph of f is concave on I

Def. (inflection point)

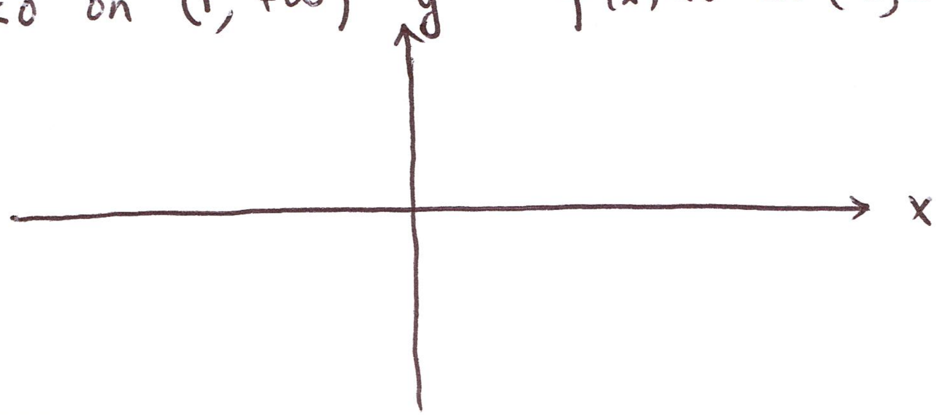
A point P on a curve $y = f(x)$ is an inflection point $\iff$ (1) f is continuous at P
(2) the curve changes its concavity.

Ex. 5 Sketch a possible graph of a function f that satisfies the following conditions:

(i) $f'(x) > 0$ on $(-\infty, 1)$
$f'(x) < 0$ on $(1, +\infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$
$f''(x) < 0$ on $(-2, 2)$

(iii) $\lim\limits_{x \to -\infty} f(x) = -2$
$\lim\limits_{x \to \infty} f(x) = 0$



The Second Derivative Test

f'' is continuous near c.

(a) $f'(c) = 0$ and $f''(c) > 0 \Longrightarrow f$ has a local minimum at c.

(b) $f'(c) = 0$ and $f''(c) < 0 \Longrightarrow f$ has a local maximum at c.

Ex. 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Ex 7 Sketch the graph of the function $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

Ex. 8 Use the 1st and 2nd derivatives of $f(x) = e^{\frac{1}{x}}$, together with asymptotes, to sketch its graph.

§4.4 Indeterminate Forms and l'Hospital's Rule

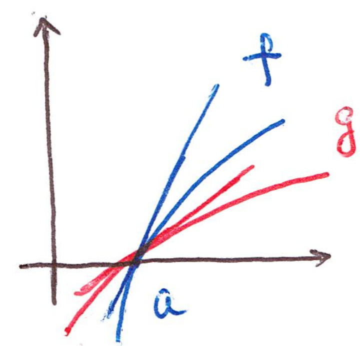
- **indeterminate forms**

$$\lim_{x \to a} \frac{f(x)}{g(x)} : \qquad \frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

examples

$$\lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \qquad\qquad \lim_{x \to 1} \frac{\ln x}{x - 1} =$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \qquad\qquad \lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} =$$



L'Hospital's Rule

Assume that f and g are differentiable and $g'(x) \neq 0$ on an open interval $I \ni a$ and that $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

$$\Longrightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Proof of L'Hospital's Rule in the special case that $f(a)=g(a)=0$, f' and g' are cont., $g'(a)\neq 0$

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} =$$

Examples

(1) $\displaystyle\lim_{x\to 1}\frac{\ln x}{x-1} =$

(2) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2} =$

(3) $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^{\frac{1}{3}}} =$

(4) $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3} =$

(5) $\lim\limits_{x \to \pi^-} \dfrac{\sin x}{1 - \cos x} =$

- indeterminate products $\quad 0 \cdot \infty$

$$\lim_{x \to a} f(x)\,g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}} = \lim_{x \to a} \frac{f'(x)}{\left(\frac{1}{g(x)}\right)'}$$

Ex. 6 $\displaystyle \lim_{x \to 0^+} x \ln x =$

- indeterminate differences $\quad \infty - \infty$

Ex. 7

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} (\sec x - \tan x)$$

- <u>indeterminate powers</u>

$$\lim_{x\to a}\left[f(x)\right]^{g(x)} : \quad 0^0, \quad \infty^0, \quad \text{or } 1^{\infty}$$

$$= \lim_{x\to a} e^{\ln f(x)^{g(x)}} = \lim_{x\to a} e^{g(x)\ln f(x)}$$

<u>Ex. 8</u> $\displaystyle\lim_{x\to 0^-}(1+\sin 4x)^{\cot x}$

<u>Ex. 9</u> $\displaystyle\lim_{x\to 0^+} x^x$

§4.5 Summary of Curve Sketching

- **Guidelines for Sketching a Curve**

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increasing/Decreasing

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

Ex. 1 Sketch the curve $y = \dfrac{2x^2}{x^2-1}$.

Ex. 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x+1}}$.

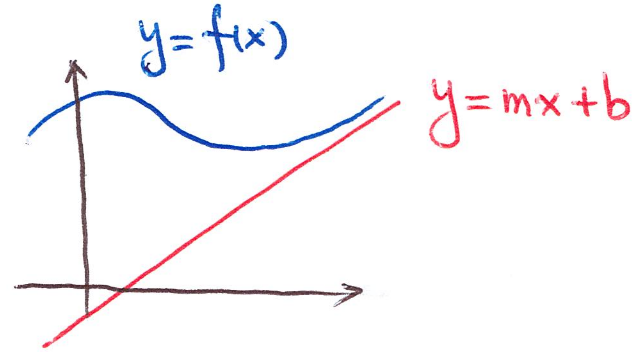
Ex. 3 Sketch the graph of $f(x) = xe^{x}$.

Ex. 4 Sketch the graph of $f(x) = \frac{\cos x}{2 + \sin x}$.

Ex. 5 Sketch the graph of $y = \ln(4 - x^2)$.

- <u>slant asymptotes</u> $\quad y = mx + b$

$$\lim_{x \to \infty} \left[f(x) - (mx + b) \right] = 0$$



<u>Ex. 6</u> Sketch the graph of $f(x) = \dfrac{x^3}{x^2+1}$

§4.7 Optimization Problems

- steps in solving optimization problems

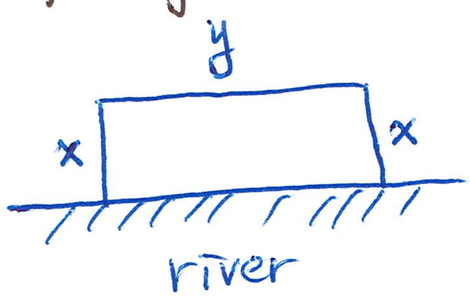
(1) understand the problem: unknowns, given quantities, given conditions

(2) draw a diagram: Identify the given and required quantities on the diagram

(3) Introduce notations:

...

Ex. 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



Ex. 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

$\boxed{1\,L = 1000\ cm^3}$

1^{st} derivative test for absolute extreme values

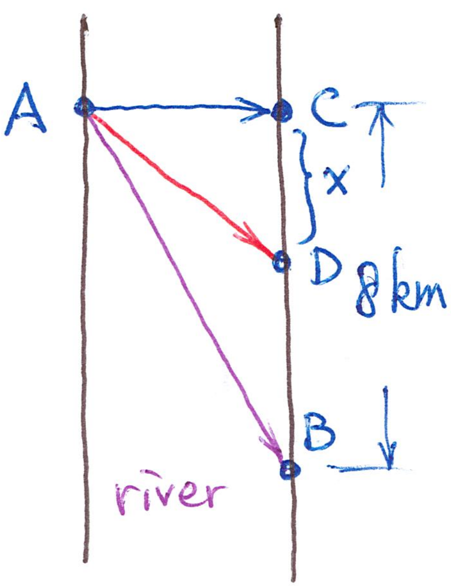
Assume that c is a critical number of a continuous function f defined on I.

(a) $f'(x) > 0 \ \forall \ x < c$ and $f'(x) < 0 \ \forall x > c \implies f(c)$ is the absolute maximum value of f.

(b) $f'(x) < 0 \ \forall \ x < c$ and $f'(x) > 0 \ \forall x > c \implies f(c)$ is the absolute minimum value of f.

Ex. 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Ex. 4 A man lauches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (the speed of the water is negligible.)



<u>Ex. 5</u> Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

#14 (P337) A box with a square base and open top must have a volume of $32{,}000 \text{ cm}^3$. Find the dimensions of the box that minimize the amount of material used.

#26 (P338) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

#3[illegible] (P338) A right circular cylinder is inscribed in a sphere of ~~rad~~ radius r. Find the largest possible ~~[illegible]~~ volume of such a cylinder.

#32 (P338) A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest volume of such a cylinder.

#54 (P.339) Find an equation of the line through the point $(3, 5)$ that cuts off the least area from the first quadrant.

§4.9 Antiderivatives

Def: A function F is an antiderivative of f on an interval I

$$\iff \quad F'(x) = f(x) \quad \forall x \in I.$$

Thrm $F'(x) = f(x) \implies (F(x) + C)' = f(x)$ where C is a constant.

Ex.1 Find the most general antiderivative of

(a) $f(x) = \sin x$, (b) $f(x) = \frac{1}{x}$, (c) $f(x) = x^n$, $n \neq -1$.

Function	Particular Antider.
$c\,f(x)$	$c\,F(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n \ (n \neq -1)$	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$
$\frac{1}{1+x^2}$	$\tan^{-1}x$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$

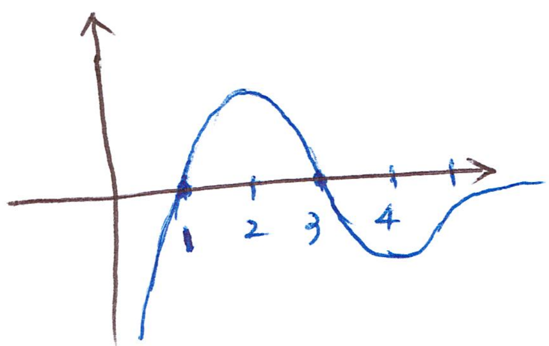
Ex. 2 Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

Ex. 3 Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

Ex. 4 Find f if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

Ex. 5 The graph of f is given. Make a rough sketch of an antiderivative, given $F(0) = 2$.



- rectilinear motion

Ex. 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Ex. 7 A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?