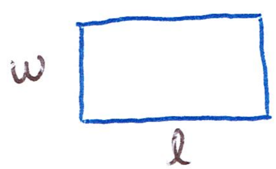
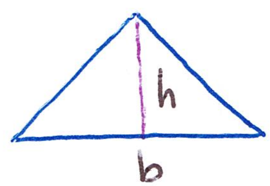
Chapter 5 Integrals (4 lectures)

§5.1 Areas and ~~Distances~~

- area problem

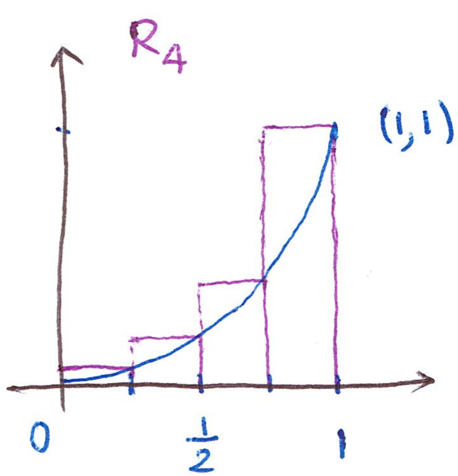


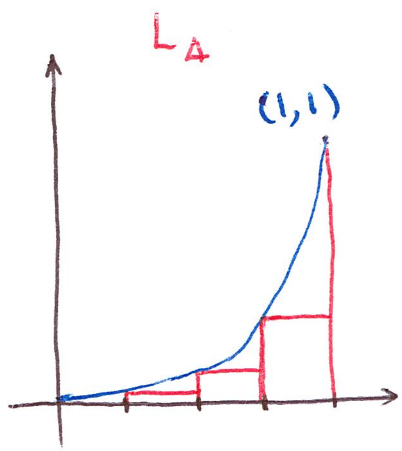
$A = w\ell$



$A = \frac{1}{2} b h$

Ex. 1 Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.





$R_4 =$

$L_4 =$

Ex. 2 Partition of the interval $[0,1]$:

$$\underset{x_0}{0} < \underset{x_1}{\frac{1}{n}} < \underset{x_2}{\frac{2}{n}} < \cdots < \underset{x_n}{\frac{n}{n}} = 1 \qquad \Delta x_i = x_i - x_{i-1} = \frac{1}{n}$$

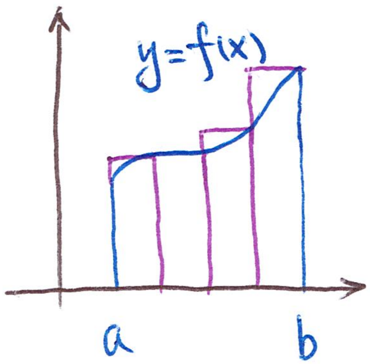
Let

$$R_n = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + \cdots + f(x_n)\,\Delta x_n$$

$$L_n = f(x_0)\,\Delta x_1 + f(x_1)\,\Delta x_2 + \cdots + f(x_{n-1})\,\Delta x_n$$

For $f(x) = x^2$, prove that $\lim\limits_{n\to\infty} R_n = \frac{1}{3}$ and that $\lim\limits_{n\to\infty} L_n = \frac{1}{3}$.

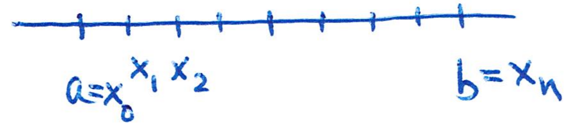
$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6}\, n(n+1)(2n+1)$$



Partition of [a, b]

$$a = x_0 < x_1 < \cdots < x_n = b$$

$$x_i = a + i\,\Delta x, \quad \Delta x = \frac{b-a}{n}$$



Def. The area A of the region S that lies under the graph of the continuous function f

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right]$$

$$= \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\right]$$

$$= \lim_{n\to\infty}\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right]$$

Ex. 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x=0$ and $x=2$.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(b) Estimate the area by taking the sample points to be midpoints and using 4 subintervals and then 10 subintervals.

§5.2 The Definite Integral

Definition of a Definite Integral f is a function defined on $[a,b]$,

let $a = x_0 < x_1 < \cdots < x_n = b$ with $x_i = a + i\Delta x$, $\Delta x = \frac{b-a}{n}$ be a partition of $[a,b]$,

let $x_i^* \in [x_{i-1}, x_i]$ be sample points,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that the limit exists independent of choices of x_i^*.

a — lower limit

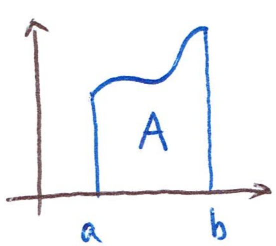
b — upper limit

f — integrand

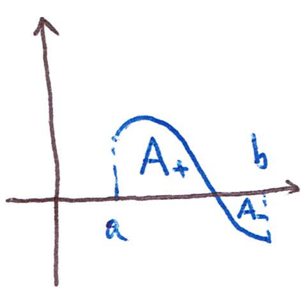
$\int$ — integral sign

f is integrable on $[a,b]$ $\Longleftrightarrow$ the limit exists.

$\sum_{i=1}^{n} f(x_i^*)\Delta x$ — Riemann sum



$A = \int_a^b f(x)dx$



$\int_a^b f(x)dx = A_+ - A_-$

Theorem f is continuous on $[a,b]$ or f has only a finite number of jump discontinuities

$\Longrightarrow f$ is integrable on $[a,b]$.

Theorem f is integrable on $[a,b]$

$$\Longrightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x.$$

Ex. 1 Express $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(x_i^3 + x_i \sin x_i\right)\Delta x$ as an integral on the interval $[0,\pi]$.

- evaluating integrals

identities

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1), \quad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

<u>Ex. 2</u> (a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Ex. 3 (a) Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.

(b) Use $\sum_{i=1}^{n} e^{1+\frac{2i}{n}} = \left[e^{\frac{3n+2}{n}} - e^{\frac{n+2}{n}}\right] / (e^{\frac{2}{n}}-1)$ to evaluate the expression.

Ex. 4 Evaluate (a) $\int_0^1 \sqrt{1-x^2}\,dx$ and (b) $\int_0^3 (x-1)\,dx$ by interpreting each in terms of areas.

§5.3 The Fundamental Theorem of Calculus

a connection between differential calculus and integral calculus

The Fundamental Therorem of Calculus, Part I

Assume that f is continuous on $[a,b]$ $\Longrightarrow$ $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$

and differentiable on (a,b).

Moreover, $g'(x) = f(x) = \frac{d}{dx}\int_a^x f(t)\,dt$

Proof

- $$\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

- On $[x, x+h]$ for $h>0$, $\exists\, u, v \in [x, x+h]$ such that

$$f(u) \leq \frac{g(x+h)-g(x)}{h} \leq f(v)$$

<u>Ex. 2</u> Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Ex. 3 Fresnel function

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$$

Ex. 4 Find $\frac{d}{dx} \int_1^{x^4} \sec t \, dt$.

The Fundamental Theorem of Calculus, Part II

Assume that f is continuous on $[a,b]$
and that $F'(x) = f(x)$ $\Longrightarrow$ $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

Proof Let $g(x) = \displaystyle\int_a^x f(t)\,dt$

- $F(x) = g(x) + C$ on $[a, b]$

- $F(b) - F(a) =$

Ex. 5 Evaluate the integral $\displaystyle\int_1^3 e^x\,dx$.

Ex. 6 Find the area under the parabola $y = x^2$ from 0 to 1.

Ex. 7 Evaluate $\int_3^6 \frac{1}{x}\,dx$.

Ex. 8 Find the area under the cosine curve from 0 to b, where $0 \le b \le \frac{\pi}{2}$.

Ex. 9 What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_{-1}^{3} = -\left[\frac{1}{3} + 1\right] = -\frac{4}{3}$$

- <u>differentiation and integration as inverse processes</u>

<u>Fundamental Thrm of Calculus</u> Assume that f is continuous on $[a, b]$

$\Longrightarrow$ (1) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

(2) $\int_a^b F'(x)\,dx = F(b) - F(a)$

§5.4 Indefinite Integrals and the Net Change Theorem

- indefinite integral

$$\int f(x)\,dx = F(x) \iff F'(x) = f(x)$$

table of indefinite integrals

$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

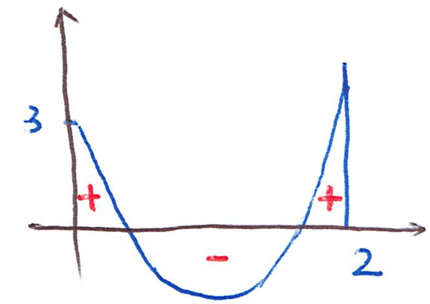
$$\int \frac{1}{x^2}\,dx = \begin{cases} -\frac{1}{x} + C_1 & \text{if } x < 0 \\ -\frac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

Ex. 1 $\int (10x^4 - 2\sec^2 x)\,dx$

Ex. 2 $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

Ex. 3 $\int_0^3 (x^3 - 6x)\,dx$

Ex. 4 $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.



Ex. 5 $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dt$

- applications

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

(see examples

Ex. 6 A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ m/s

(a) Find the displacement of the particle during the time period $1 \le t \le 4$

(b) Find the distance traveled during this time period.

§5.5 The Substitution Rule

$$\int 2x\sqrt{1+x^2}\,dx$$

The Substitution Rule

$u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I $\Longrightarrow$ $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

Ex. 1 $\int x^3\cos(x^4+2)\,dx$

Ex. 2 $\int \sqrt{2x+1}\, dx$

Ex. 3 $\int \frac{x}{\sqrt{1-4x^2}}\, dx$

Ex. 4 $\int e^{5x} dx$

Ex. 5 $\int \sqrt{1+x^2}\, x^5 \, dx$

Ex. 6 $\int \tan x \, dx$

- definite integrals

$$\int_a^b f(g(x))\,g'(x)\,dx \overset{u=g(x)}{=\!=} \int_{g(a)}^{g(b)} f(u)\,du$$

Proof Let $F' = f$

$\Rightarrow \left(F(g(x))\right)' =$

$\Rightarrow$ LHS $=$

RHS $=$

Ex. 7 $\int_0^4 \sqrt{2x+1}\,dx$

Ex. 8 $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

Ex. 9 $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

- symmetry

Integrals of Symmetric Functions f is continuous on $[-a, a]$

(a) f is even $\Longrightarrow$ $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) f is odd $\Longrightarrow$ $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

Ex. 10 $\displaystyle\int_{-2}^{2} (x^6+1)\,dx$

Ex. 11 $\displaystyle\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4}\,dx$

§3.8 Exponential Growth and Decay

Let $y(t)$ be the value of a quantity y at time t

$$\frac{dy}{dt} = ky$$

law of natural growth if $k>0$

law of natural decay if $k<0$

the solution

$$y(t) = y(0)\, e^{kt}$$

- population growth $P(t)$ denotes the size of a population at time t.

$$\frac{dP}{dt} = kP \qquad \text{or} \qquad \frac{1}{P}\frac{dP}{dP} = k$$

relative growth rate

- ## Radioactive Decay

$m(t)$ is the mass remaining from an initial mass m_0 of the substance after time t.

$$\frac{dm}{dt} = km \quad \text{with } k<0 \quad \Longrightarrow \quad m(t) = m_0 e^{kt}$$

Ex. 2 The half-life of radium-266 is 1590 years.

(a) A sample of radium-266 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.

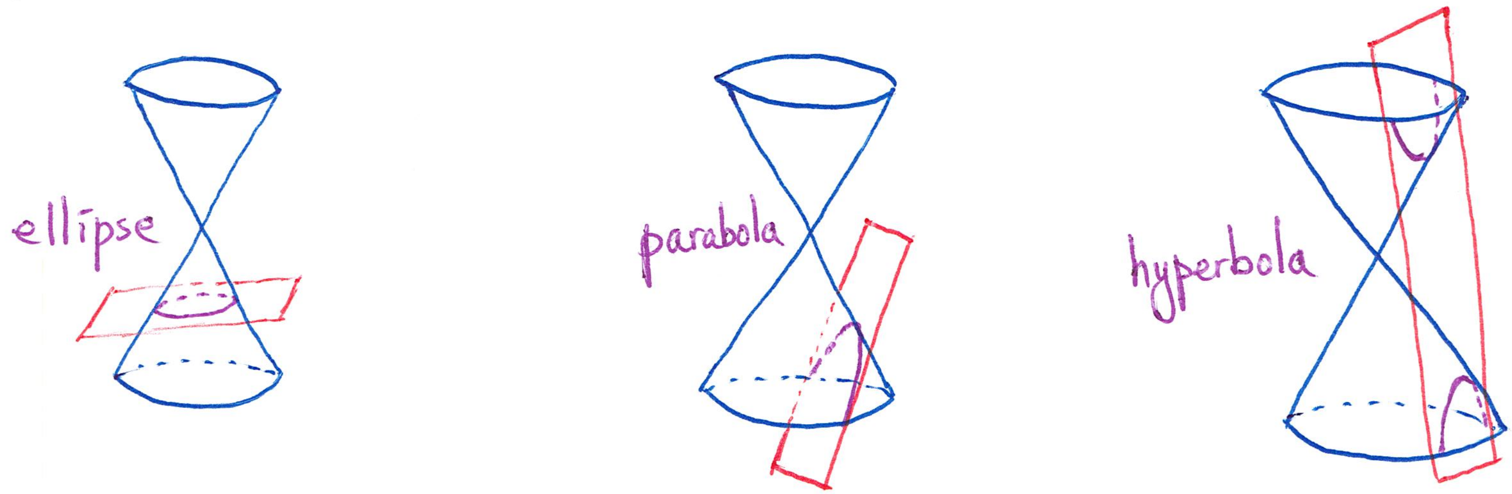
(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

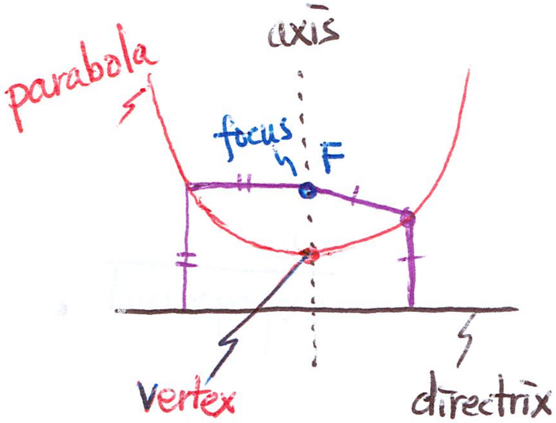
Ex. 1 Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

§10.5 Conic Sections

Conic sections or Conics are the intersections of cone and plane.

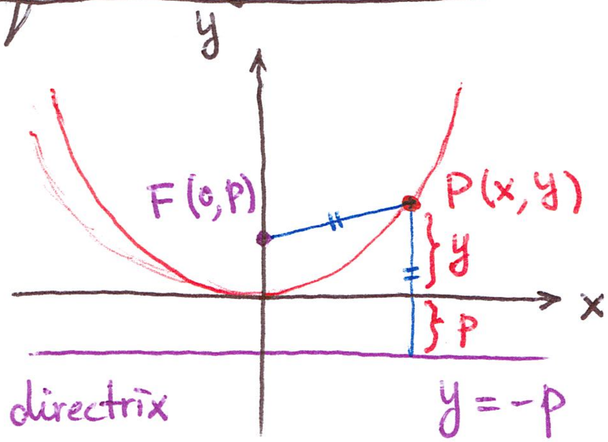


- parabolas



A parabola is the set of points in a plane that are equidistance from a fixed pt F (focus) and a fixed line (directrix)

- equation of parabola vertex at the origion



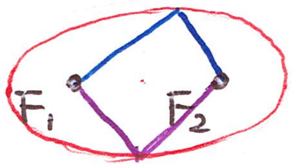
$$|PF| = |y+p|$$

the equation of the parabola with focus $(0,p)$ and directrix $y=-p$

$$x^2 = 4py \quad \text{or} \quad y = ax^2 \quad \text{with } a = \frac{1}{4p}.$$

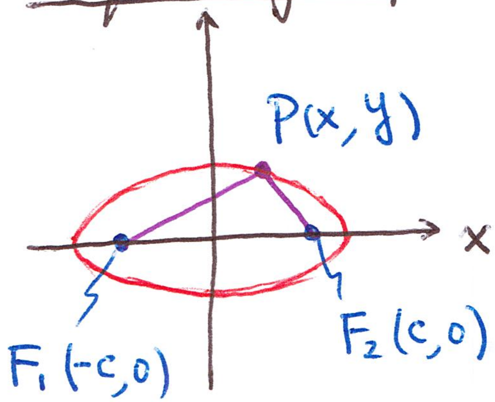
Ex. 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

- ellipses



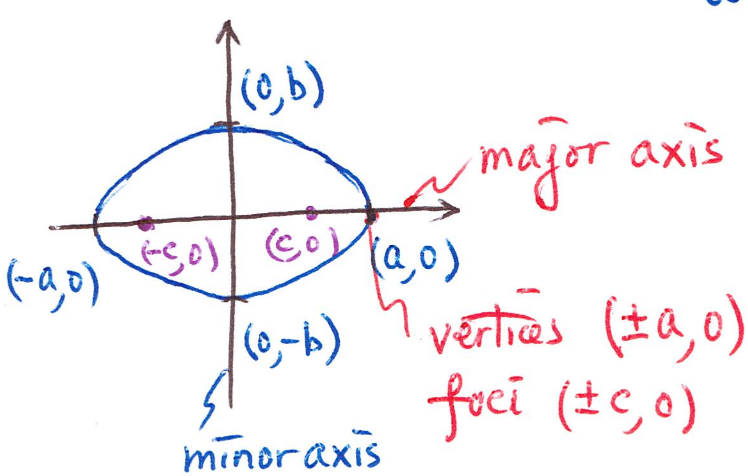
An ellipses is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant.

- an equation of ellipses



$$|PF_1| + |PF_2| = \text{constant} = 2a$$

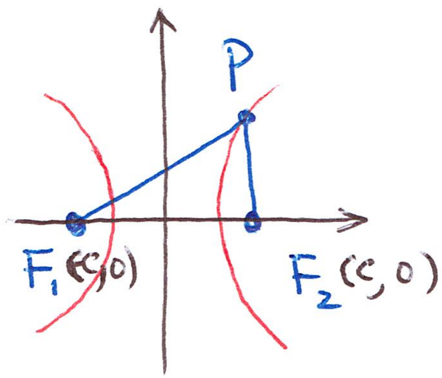
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{with } b^2 = a^2 - c^2.$$



Ex. 2 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

Ex. 3 Find an equation of the ellipses with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

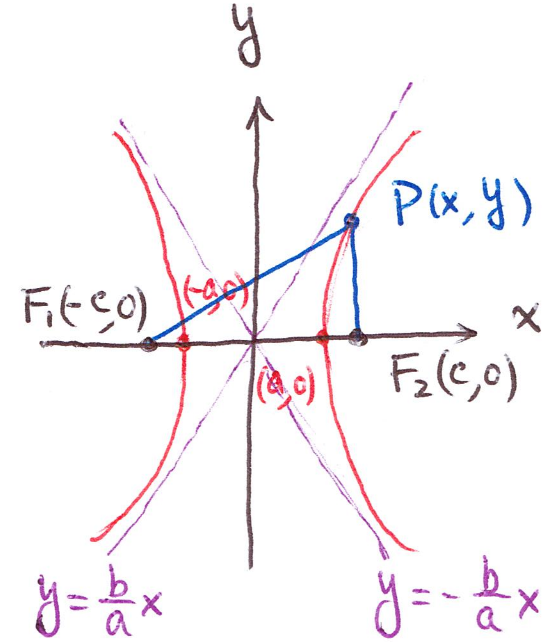
- hyperbolas



A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 is a constant.

the equation of hyperbola with the foci $(\pm c, 0)$

$$|PF_1| - |PF_2| = \pm 2a$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with } c^2 = a^2 + b^2$$

asymptotes $\quad y = \pm \frac{b}{a} x$

Ex. 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

Ex. 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptotes $y = 2x$.

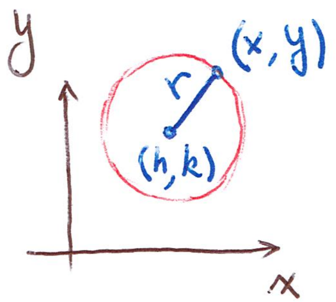
- shifted conics replacing x and y by $x-h$ and $y-k$

Ex. 6 Find an equation of the ellipses with foci $(2,-2)$, $(4,-2)$ and vertices $(1,-2)$, $(5,-2)$

Ex. 7 Sketch the conic $9x^2-4y^2-72x+8y+176=0$ and find its foci.

Appendix C Graphs of Second-Degree Equations

- Circles $(x-h)^2 + (y-k)^2 = r^2$



Ex. 1 Find an equation of the circle with radius 3 and center $(2, -5)$.

Ex. 2 Sketch the graph of the equation $x^2 + y^2 + 2x - 6y + 7 = 0$ by first showing that it represents a circle and then finding its center and radius.

• <u>parabolas</u> $y = ax^2 + bx + c$

<u>Ex. 3</u> Draw the graph of $y = x^2$.

<u>Ex. 4</u> Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$.

- ellipse $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Ex. 5 Sketch the graph of $9x^2 + 16y^2 = 144$

Hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Ex. 6 Sketch the curve $9x^2 - 4y^2 = 36$.

Shifted Conics

circle $(x-h)^2+(y-k)^2=r^2$

ellipse $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

parabola $y-k=a(x-h)^2$

hyperbola $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$

Ex. 7 Sketch $y=2x^2-4x+1$

Ex. 8 Sketch $x=1-y^2$