D EXERCISES

1–6 Convert from degrees to radians.

1. $210°$ **2.** $300°$ **3.** $9°$

4. $-315°$ **5.** $900°$ **6.** $36°$

7–12 Convert from radians to degrees.

7. 4π **8.** $-\frac{7\pi}{2}$ **9.** $\frac{5\pi}{12}$

10. $\frac{8\pi}{3}$ **11.** $-\frac{3\pi}{8}$ **12.** 5

13. Find the length of a circular arc subtended by an angle of $\pi/12$ rad if the radius of the circle is 36 cm.

14. If a circle has radius 10 cm, find the length of the arc subtended by a central angle of $72°$.

15. A circle has radius 1.5 m. What angle is subtended at the center of the circle by an arc 1 m long?

16. Find the radius of a circular sector with angle $3\pi/4$ and arc length 6 cm.

17–22 Draw, in standard position, the angle whose measure is given.

17. $315°$ **18.** $-150°$ **19.** $-\frac{3\pi}{4}$ rad

20. $\frac{7\pi}{3}$ rad **21.** 2 rad **22.** -3 rad

23–28 Find the exact trigonometric ratios for the angle whose radian measure is given.

23. $\frac{3\pi}{4}$ **24.** $\frac{4\pi}{3}$ **25.** $\frac{9\pi}{2}$

26. -5π **27.** $\frac{5\pi}{6}$ **28.** $\frac{11\pi}{4}$

29–34 Find the remaining trigonometric ratios.

29. $\sin\theta = \frac{3}{5}, \quad 0 < \theta < \frac{\pi}{2}$

30. $\tan\alpha = 2, \quad 0 < \alpha < \frac{\pi}{2}$

31. $\sec\phi = -1.5, \quad \frac{\pi}{2} < \phi < \pi$

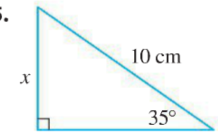
32. $\cos x = -\frac{1}{3}, \quad \pi < x < \frac{3\pi}{2}$

33. $\cot\beta = 3, \quad \pi < \beta < 2\pi$

34. $\csc\theta = -\frac{4}{3}, \quad \frac{3\pi}{2} < \theta < 2\pi$

35–38 Find, correct to five decimal places, the length of the side labeled x.

35.



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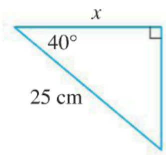

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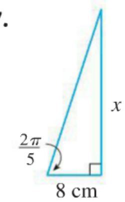

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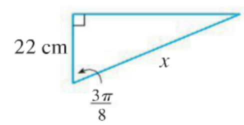


39–41 Prove each equation.

39. (a) Equation 10a (b) Equation 10b

40. (a) Equation 14a (b) Equation 14b

41. (a) Equation 18a (b) Equation 18b
(c) Equation 18c

42–58 Prove the identity.

42. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

43. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$ **44.** $\sin(\pi - x) = \sin x$

45. $\sin\theta\,\cot\theta = \cos\theta$ **46.** $(\sin x + \cos x)^2 = 1 + \sin 2x$

47. $\sec y - \cos y = \tan y\,\sin y$

48. $\tan^2\alpha - \sin^2\alpha = \tan^2\alpha\,\sin^2\alpha$

49. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$

50. $2\csc 2t = \sec t\,\csc t$

51. $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

52. $\frac{1}{1 - \sin\theta} + \frac{1}{1 + \sin\theta} = 2\sec^2\theta$

53. $\sin x\,\sin 2x + \cos x\,\cos 2x = \cos x$

54. $\sin^2 x - \sin^2 y = \sin(x + y)\,\sin(x - y)$

55. $\frac{\sin\phi}{1 - \cos\phi} = \csc\phi + \cot\phi$

56. $\tan x + \tan y = \frac{\sin(x + y)}{\cos x\,\cos y}$

57. $\sin 3\theta + \sin \theta = 2 \sin 2\theta \cos \theta$

58. $\cos 3\theta = 4 \cos^3\theta - 3 \cos \theta$

59–64 If $\sin x = \frac{1}{3}$ and $\sec y = \frac{5}{4}$, where x and y lie between 0 and $\pi/2$, evaluate the expression.

59. $\sin(x + y)$

60. $\cos(x + y)$

61. $\cos(x - y)$

62. $\sin(x - y)$

63. $\sin 2y$

64. $\cos 2y$

65–72 Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

65. $2 \cos x - 1 = 0$

66. $3 \cot^2 x = 1$

67. $2 \sin^2 x = 1$

68. $|\tan x| = 1$

69. $\sin 2x = \cos x$

70. $2 \cos x + \sin 2x = 0$

71. $\sin x = \tan x$

72. $2 + \cos 2x = 3 \cos x$

73–76 Find all values of x in the interval $[0, 2\pi]$ that satisfy the inequality.

73. $\sin x \leqslant \frac{1}{2}$

74. $2 \cos x + 1 > 0$

75. $-1 < \tan x < 1$

76. $\sin x > \cos x$

77–82 Graph the function by starting with the graphs in Figures 14 and 15 and applying the transformations of Section 1.3 where appropriate.

77. $y = \cos\left(x - \dfrac{\pi}{3}\right)$

78. $y = \tan 2x$

79. $y = \dfrac{1}{3}\tan\left(x - \dfrac{\pi}{2}\right)$

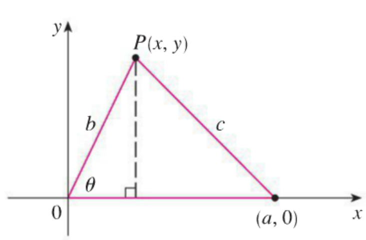
80. $y = 1 + \sec x$

81. $y = |\sin x|$

82. $y = 2 + \sin\left(x + \dfrac{\pi}{4}\right)$

83. Prove the **Law of Cosines**: If a triangle has sides with lengths a, b, and c, and θ is the angle between the sides with lengths a and b, then

$$c^2 = a^2 + b^2 - 2ab \cos\theta$$

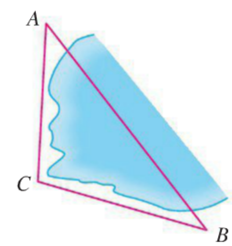


[*Hint:* Introduce a coordinate system so that θ is in standard position, as in the figure. Express x and y in terms of θ and then use the distance formula to compute c.]

84. In order to find the distance $|AB|$ across a small inlet, a point C was located as in the figure and the following measurements were recorded:

$$\angle C = 103^\circ \qquad |AC| = 820 \text{ m} \qquad |BC| = 910 \text{ m}$$

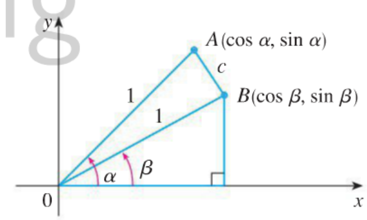
Use the Law of Cosines from Exercise 83 to find the required distance.



85. Use the figure to prove the subtraction formula

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

[*Hint:* Compute c^2 in two ways (using the Law of Cosines from Exercise 83 and also using the distance formula) and compare the two expressions.]



86. Use the formula in Exercise 85 to prove the addition formula for cosine (12b).

87. Use the addition formula for cosine and the identities

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

to prove the subtraction formula (13a) for the sine function.

88. Show that the area of a triangle with sides of lengths a and b and with included angle θ is

$$A = \tfrac{1}{2}ab \sin\theta$$

89. Find the area of triangle ABC, correct to five decimal places, if

$$|AB| = 10 \text{ cm} \qquad |BC| = 3 \text{ cm} \qquad \angle ABC = 107^\circ$$

3. If $f(x) = x^2 - 2x + 3$, evaluate the difference quotient

$$\frac{f(a + h) - f(a)}{h}$$

4. Sketch a rough graph of the yield of a crop as a function of the amount of fertilizer used.

5–8 Find the domain and range of the function. Write your answer in interval notation.

5. $f(x) = 2/(3x - 1)$

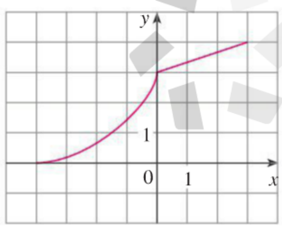
6. $g(x) = \sqrt{16 - x^4}$

7. $h(x) = \ln(x + 6)$

8. $F(t) = 3 + \cos 2t$

9. Suppose that the graph of f is given. Describe how the graphs of the following functions can be obtained from the graph of f.
(a) $y = f(x) + 8$
(b) $y = f(x + 8)$
(c) $y = 1 + 2f(x)$
(d) $y = f(x - 2) - 2$
(e) $y = -f(x)$
(f) $y = f^{-1}(x)$

10. The graph of f is given. Draw the graphs of the following functions.
(a) $y = f(x - 8)$
(b) $y = -f(x)$
(c) $y = 2 - f(x)$
(d) $y = \frac{1}{2}f(x) - 1$
(e) $y = f^{-1}(x)$
(f) $y = f^{-1}(x + 3)$



11–16 Use transformations to sketch the graph of the function.

11. $y = (x - 2)^3$

12. $y = 2\sqrt{x}$

13. $y = x^2 - 2x + 2$

14. $y = \ln(x + 1)$

15. $f(x) = -\cos 2x$

16. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geqslant 0 \end{cases}$

17. Determine whether f is even, odd, or neither even nor odd.
(a) $f(x) = 2x^5 - 3x^2 + 2$
(b) $f(x) = x^3 - x^7$
(c) $f(x) = e^{-x^2}$
(d) $f(x) = 1 + \sin x$

18. Find an expression for the function whose graph consists of the line segment from the point $(-2, 2)$ to the point $(-1, 0)$ together with the top half of the circle with center the origin and radius 1.

19. If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, (d) $g \circ g$, and their domains.

20. Express the function $F(x) = 1/\sqrt{x + \sqrt{x}}$ as a composition of three functions.

21. Life expectancy improved dramatically in the 20th century. The table gives the life expectancy at birth (in years) of males born in the United States. Use a scatter plot to choose an appropriate type of model. Use your model to predict the life span of a male born in the year 2010.

Birth year	Life expectancy	Birth year	Life expectancy
1900	48.3	1960	66.6
1910	51.1	1970	67.1
1920	55.2	1980	70.0
1930	57.4	1990	71.8
1940	62.5	2000	73.0
1950	65.6		

22. A small-appliance manufacturer finds that it costs \$9000 to produce 1000 toaster ovens a week and \$12,000 to produce 1500 toaster ovens a week.
(a) Express the cost as a function of the number of toaster ovens produced, assuming that it is linear. Then sketch the graph.
(b) What is the slope of the graph and what does it represent?
(c) What is the y-intercept of the graph and what does it represent?

23. If $f(x) = 2x + \ln x$, find $f^{-1}(2)$.

24. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

25. Find the exact value of each expression.
(a) $e^{2\ln 3}$
(b) $\log_{10} 25 + \log_{10} 4$
(c) $\tan\left(\arcsin \frac{1}{2}\right)$
(d) $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

26. Solve each equation for x.
(a) $e^x = 5$
(b) $\ln x = 2$
(c) $e^{e^x} = 2$
(d) $\tan^{-1}x = 1$

27. The half-life of palladium-100, ^{100}Pd, is four days. (So half of any given quantity of ^{100}Pd will disintegrate in four days.) The initial mass of a sample is one gram.
(a) Find the mass that remains after 16 days.
(b) Find the mass $m(t)$ that remains after t days.
(c) Find the inverse of this function and explain its meaning.
(d) When will the mass be reduced to 0.01g?

28. The population of a certain species in a limited environment with initial population 100 and carrying capacity 1000 is

$$P(t) = \frac{100{,}000}{100 + 900e^{-t}}$$

where t is measured in years.
(a) Graph this function and estimate how long it takes for the population to reach 900.
(b) Find the inverse of this function and explain its meaning.
(c) Use the inverse function to find the time required for the population to reach 900. Compare with the result of part (a).

23–24 How would you "remove the discontinuity" of f? In other words, how would you define $f(2)$ in order to make f continuous at 2?

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$

25–32 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$

26. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1}$

27. $Q(x) = \dfrac{\sqrt[3]{x} - 2}{x^3 - 2}$

28. $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$

29. $A(t) = \arcsin(1 + 2t)$

30. $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$

31. $M(x) = \sqrt{1 + \dfrac{1}{x}}$

32. $N(r) = \tan^{-1}(1 + e^{-r^2})$

33–34 Locate the discontinuities of the function and illustrate by graphing.

33. $y = \dfrac{1}{1 + e^{1/x}}$

34. $y = \ln(\tan^2 x)$

35–38 Use continuity to evaluate the limit.

35. $\lim_{x \to 2} x\sqrt{20 - x^2}$

36. $\lim_{x \to \pi} \sin(x + \sin x)$

37. $\lim_{x \to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$

38. $\lim_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$

39–40 Show that f is continuous on $(-\infty, \infty)$.

39. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \ln x & \text{if } x > 1 \end{cases}$

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

41–43 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

41. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

42. $f(x) = \begin{cases} 2^x & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

44. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

45. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

46. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

47. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

48. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.

(a) Find $(f \circ g)(x)$.

(b) Is $f \circ g$ continuous everywhere? Explain.

49. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}, \quad a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}, \quad a = 2$

(c) $f(x) = [\![\sin x]\!], \quad a = \pi$

50. Suppose that a function f is continuous on [0, 1] except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

51. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

52. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

11. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \to \infty} \frac{2x^2 - 7}{5x^2 + x - 3}$

14. $\lim_{x \to \infty} \sqrt{\frac{9x^3 + 8x - 4}{3 - 5x + x^3}}$

15–42 Find the limit or show that it does not exist.

15. $\lim_{x \to \infty} \frac{3x - 2}{2x + 1}$

16. $\lim_{x \to \infty} \frac{1 - x^2}{x^3 - x + 1}$

17. $\lim_{x \to -\infty} \frac{x - 2}{x^2 + 1}$

18. $\lim_{x \to -\infty} \frac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

19. $\lim_{t \to \infty} \frac{\sqrt{t} + t^2}{2t - t^2}$

20. $\lim_{t \to \infty} \frac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

21. $\lim_{x \to \infty} \frac{(2x^2 + 1)^2}{(x - 1)^2(x^2 + x)}$

22. $\lim_{x \to \infty} \frac{x^2}{\sqrt{x^4 + 1}}$

23. $\lim_{x \to \infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

24. $\lim_{x \to -\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$

25. $\lim_{x \to \infty} \frac{\sqrt{x + 3x^2}}{4x - 1}$

26. $\lim_{x \to \infty} \frac{x + 3x^2}{4x - 1}$

27. $\lim_{x \to \infty} \left(\sqrt{9x^2 + x} - 3x\right)$

28. $\lim_{x \to -\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

29. $\lim_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

30. $\lim_{x \to \infty} \sqrt{x^2 + 1}$

31. $\lim_{x \to \infty} \frac{x^4 - 3x^2 + x}{x^3 - x + 2}$

32. $\lim_{x \to \infty} (e^{-x} + 2\cos 3x)$

33. $\lim_{x \to -\infty} (x^2 + 2x^7)$

34. $\lim_{x \to -\infty} \frac{1 + x^6}{x^4 + 1}$

35. $\lim_{x \to \infty} \arctan(e^x)$

36. $\lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

37. $\lim_{x \to \infty} \frac{1 - e^x}{1 + 2e^x}$

38. $\lim_{x \to \infty} \frac{\sin^2 x}{x^2 + 1}$

39. $\lim_{x \to \infty} (e^{-2x} \cos x)$

40. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

41. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

42. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

43. (a) For $f(x) = \frac{x}{\ln x}$ find each of the following limits.

(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 1^-} f(x)$ (iii) $\lim_{x \to 1^+} f(x)$

(b) Use a table of values to estimate $\lim_{x \to \infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

44. For $f(x) = \frac{2}{x} - \frac{1}{\ln x}$ find each of the following limits.

(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to 0^+} f(x)$

(c) $\lim_{x \to 1^-} f(x)$ (d) $\lim_{x \to 1^+} f(x)$

(e) Use the information from parts (a)–(d) to make a rough sketch of the graph of f.

45. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

46. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

47–52 Find the horizontal and vertical asymptotes of each curve. If you have a graphing device, check your work by graphing the curve and estimating the asymptotes.

47. $y = \frac{5 + 4x}{x + 3}$

48. $y = \frac{2x^2 + 1}{3x^2 + 2x - 1}$

49. $y = \frac{2x^2 + x - 1}{x^2 + x - 2}$

50. $y = \frac{1 + x^4}{x^2 - x^4}$

3–20 Find the limit.

3. $\lim_{x\to 1} e^{x^3-x}$

4. $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3}$

5. $\lim_{x\to -3} \dfrac{x^2-9}{x^2+2x-3}$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3}$

7. $\lim_{h\to 0} \dfrac{(h-1)^3+1}{h}$

8. $\lim_{t\to 2} \dfrac{t^2-4}{t^3-8}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4}$

10. $\lim_{v\to 4^+} \dfrac{4-v}{|4-v|}$

11. $\lim_{u\to 1} \dfrac{u^4-1}{u^3+5u^2-6u}$

12. $\lim_{x\to 3} \dfrac{\sqrt{x+6}-x}{x^3-3x^2}$

13. $\lim_{x\to\infty} \dfrac{\sqrt{x^2-9}}{2x-6}$

14. $\lim_{x\to -\infty} \dfrac{\sqrt{x^2-9}}{2x-6}$

15. $\lim_{x\to\pi^-} \ln(\sin x)$

16. $\lim_{x\to -\infty} \dfrac{1-2x^2-x^4}{5+x-3x^4}$

17. $\lim_{x\to\infty} \left(\sqrt{x^2+4x+1}-x\right)$

18. $\lim_{x\to\infty} e^{x-x^2}$

19. $\lim_{x\to 0^+} \tan^{-1}(1/x)$

20. $\lim_{x\to 1} \left(\dfrac{1}{x-1}+\dfrac{1}{x^2-3x+2}\right)$

21–22 Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

21. $y = \dfrac{\cos^2 x}{x^2}$

22. $y = \sqrt{x^2+x+1} - \sqrt{x^2-x}$

23. If $2x-1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x\to 1} f(x)$.

24. Prove that $\lim_{x\to 0} x^2\cos(1/x^2) = 0$.

25–28 Prove the statement using the precise definition of a limit.

25. $\lim_{x\to 2} (14-5x) = 4$

26. $\lim_{x\to 0} \sqrt[3]{x} = 0$

27. $\lim_{x\to 2} (x^2-3x) = -2$

28. $\lim_{x\to 4^+} \dfrac{2}{\sqrt{x-4}} = \infty$

29. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3-x & \text{if } 0 \leqslant x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$ (iii) $\lim_{x\to 0} f(x)$

(iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$ (vi) $\lim_{x\to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

30. Let

$$g(x) = \begin{cases} 2x-x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2-x & \text{if } 2 < x \leqslant 3 \\ x-4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geqslant 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(b) Sketch the graph of g.

31–32 Show that the function is continuous on its domain. State the domain.

31. $h(x) = xe^{\sin x}$

32. $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$

33–34 Use the Intermediate Value Theorem to show that there is a root of the equation in the given interval.

33. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

34. $\cos\sqrt{x} = e^x - 2$, $(0, 1)$

35. (a) Find the slope of the tangent line to the curve $y = 9 - 2x^2$ at the point $(2, 1)$.

(b) Find an equation of this tangent line.

36. Find equations of the tangent lines to the curve

$$y = \frac{2}{1-3x}$$

at the points with x-coordinates 0 and -1.

37. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.

(a) Find the average velocity over each time period.

(i) $[1, 3]$ (ii) $[1, 2]$ (iii) $[1, 1.5]$ (iv) $[1, 1.1]$

(b) Find the instantaneous velocity when $t = 1$.

38. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 800$, where P is measured in pounds per square inch and V is measured in cubic inches.

(a) Find the average rate of change of P as V increases from 200 in^3 to 250 in^3.

(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

39. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.

(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point $(2, 4)$.

(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

40. Find a function f and a number a such that

$$\lim_{h\to 0} \frac{(2+h)^6 - 64}{h} = f'(a)$$

41. The total cost of repaying a student loan at an interest rate of $r\%$ per year is $C = f(r)$.

(a) What is the meaning of the derivative $f'(r)$? What are its units?

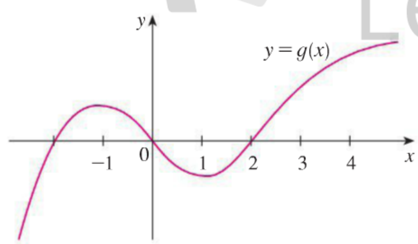
14. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

15. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, and $t = 3$.

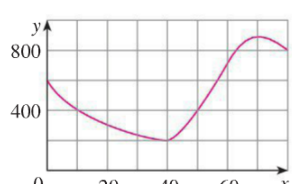
16. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
(i) $[4, 8]$ (ii) $[6, 8]$
(iii) $[8, 10]$ (iv) $[8, 12]$
(b) Find the instantaneous velocity when $t = 8$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$



18. The graph of a function f is shown.
(a) Find the average rate of change of f on the interval $[20, 60]$.
(b) Identify an interval on which the average rate of change of f is 0.
(c) Which interval gives a larger average rate of change, $[40, 60]$ or $[40, 70]$?
(d) Compute $\dfrac{f(40) - f(10)}{40 - 10}$; what does this value represent geometrically?



19. For the function f graphed in Exercise 18:
(a) Estimate the value of $f'(50)$.
(b) Is $f'(10) > f'(30)$?
(c) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.

20. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

21. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

22. If the tangent line to $y = f(x)$ at (4, 3) passes through the point (0, 2), find $f(4)$ and $f'(4)$.

23. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

24. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

25. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

26. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ±1, and f is odd.

27. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

28. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point $(1, -1)$.

29. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point (2, 2).
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

30. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points (2, 8) and (3, 9).
(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

31–36 Find $f'(a)$.

31. $f(x) = 3x^2 - 4x + 1$

32. $f(t) = 2t^3 + t$

33. $f(t) = \dfrac{2t + 1}{t + 3}$

34. $f(x) = x^{-2}$

35. $f(x) = \sqrt{1 - 2x}$

36. $f(x) = \dfrac{4}{\sqrt{1 - x}}$

37–42 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

37. $\lim_{h\to0} \dfrac{\sqrt{9 + h} - 3}{h}$

38. $\lim_{h\to0} \dfrac{e^{-2+h} - e^{-2}}{h}$

39. $\lim_{x \to 2} \dfrac{x^6 - 64}{x - 2}$

40. $\lim_{x \to 1/4} \dfrac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

41. $\lim_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$

42. $\lim_{\theta \to \pi/6} \dfrac{\sin\theta - \frac{1}{2}}{\theta - \pi/6}$

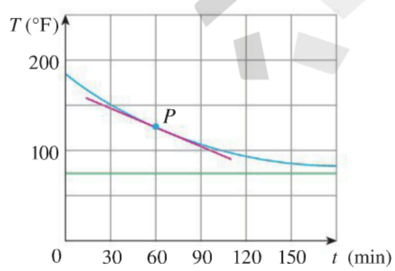
43–44 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

43. $f(t) = 80t - 6t^2$

44. $f(t) = 10 + \dfrac{45}{t + 1}$

45. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

46. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



47. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (mg/mL)	0.33	0.24	0.18	0.12	0.07

(a) Find the average rate of change of C with respect to t over each time interval:
 (i) [1.0, 2.0] (ii) [1.5, 2.0]
 (iii) [2.0, 2.5] (iv) [2.0, 3.0]
 In each case, include the units.
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

48. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	2004	2006	2008	2010	2012
N	8569	12,440	16,680	16,858	18,066

(a) Find the average rate of growth
 (i) from 2006 to 2008
 (ii) from 2008 to 2010
 In each case, include the units. What can you conclude?
(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.

49. The table shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.
(a) Compute and interpret the average rate of change from 1990 to 2005. What are the units?
(b) Estimate the instantaneous rate of change in 2000 by taking the average of two average rates of change. What are its units?

Years since 1985	Thousands of barrels of oil per day
0	60,083
5	66,533
10	70,099
15	76,784
20	84,077
25	87,302

Source: US Energy Information Administration

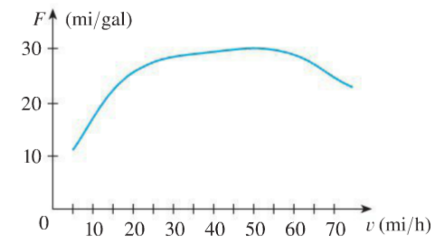
50. The table shows values of the viral load $V(t)$ in HIV patient 303, measured in RNA copies/mL, t days after ABT-538 treatment was begun.

t	4	8	11	15	22
$V(t)$	53	18	9.4	5.2	3.6

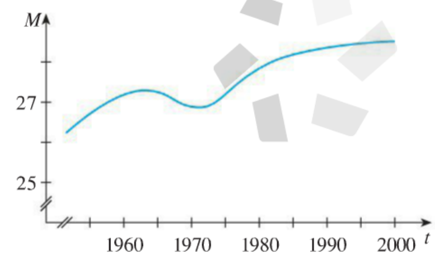
(a) Find the average rate of change of V with respect to t over each time interval:
 (i) [4, 11] (ii) [8, 11]
 (iii) [11, 15] (iv) [11, 22]
 What are the units?
(b) Estimate and interpret the value of the derivative $V'(11)$.

Source: Adapted from D. Ho et al., "Rapid Turnover of Plasma Virions and CD4 Lymphocytes in HIV-1 Infection," *Nature* 373 (1995): 123–26.

14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in miles per gallon and speed v is measured in miles per hour.
(a) What is the meaning of the derivative $F'(v)$?
(b) Sketch the graph of $F'(v)$.
(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average age of first marriage of Japanese men varied in the last half of the 20th century. Sketch the graph of the derivative function $M'(t)$. During which years was the derivative negative?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$ **17.** $f(x) = e^x$ **18.** $f(x) = \ln x$

19. Let $f(x) = x^2$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by using a graphing device to zoom in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
(c) Use the values from parts (a) and (b) to graph f'.
(d) Guess a formula for $f'(x)$.
(e) Use the definition of derivative to prove that your guess in part (d) is correct.

21–31 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = 3x - 8$

22. $f(x) = mx + b$

23. $f(t) = 2.5t^2 + 6t$

24. $f(x) = 4 + 8x - 5x^2$

25. $f(x) = x^2 - 2x^3$

26. $g(t) = \dfrac{1}{\sqrt{t}}$

27. $g(x) = \sqrt{9 - x}$

28. $f(x) = \dfrac{x^2 - 1}{2x - 3}$

29. $G(t) = \dfrac{1 - 2t}{3 + t}$

30. $f(x) = x^{3/2}$

31. $f(x) = x^4$

32. (a) Sketch the graph of $f(x) = \sqrt{6 - x}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.
(b) Use the graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?
(d) Use a graphing device to graph f' and compare with your sketch in part (b).

33. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

34. (a) If $f(x) = x + 1/x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. The unemployment rate $U(t)$ varies with time. The table gives the percentage of unemployed in the US labor force from 2003 to 2012.
(a) What is the meaning of $U'(t)$? What are its units?
(b) Construct a table of estimated values for $U'(t)$.

t	$U(t)$	t	$U(t)$
2003	6.0	2008	5.8
2004	5.5	2009	9.3
2005	5.1	2010	9.6
2006	4.6	2011	8.9
2007	4.6	2012	8.1

Source: US Bureau of Labor Statistics

3.1 EXERCISES

1. (a) How is the number e defined?
(b) Use a calculator to estimate the values of the limits

$$\lim_{h \to 0} \frac{2.7^h - 1}{h} \quad \text{and} \quad \lim_{h \to 0} \frac{2.8^h - 1}{h}$$

correct to two decimal places. What can you conclude about the value of e?

2. (a) Sketch, by hand, the graph of the function $f(x) = e^x$, paying particular attention to how the graph crosses the y-axis. What fact allows you to do this?
(b) What types of functions are $f(x) = e^x$ and $g(x) = x^e$? Compare the differentiation formulas for f and g.
(c) Which of the two functions in part (b) grows more rapidly when x is large?

3–32 Differentiate the function.

3. $f(x) = 2^{40}$

4. $f(x) = e^5$

5. $f(x) = 5.2x + 2.3$

6. $g(x) = \frac{7}{4}x^2 - 3x + 12$

7. $f(t) = 2t^3 - 3t^2 - 4t$

8. $f(t) = 1.4t^5 - 2.5t^2 + 6.7$

9. $g(x) = x^2(1 - 2x)$

10. $H(u) = (3u - 1)(u + 2)$

11. $g(t) = 2t^{-3/4}$

12. $B(y) = cy^{-6}$

13. $F(r) = \dfrac{5}{r^3}$

14. $y = x^{5/3} - x^{2/3}$

15. $R(a) = (3a + 1)^2$

16. $h(t) = \sqrt[4]{t} - 4e^t$

17. $S(p) = \sqrt{p} - p$

18. $y = \sqrt[3]{x}(2 + x)$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}}$

20. $S(R) = 4\pi R^2$

21. $h(u) = Au^3 + Bu^2 + Cu$

22. $y = \dfrac{\sqrt{x} + x}{x^2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t}$

25. $j(x) = x^{2.4} + e^{2.4}$

26. $k(r) = e^r + r^e$

27. $G(q) = (1 + q^{-1})^2$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v}$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3}$

31. $z = \dfrac{A}{y^{10}} + Be^y$

32. $y = e^{x+1} + 1$

33–36 Find an equation of the tangent line to the curve at the given point.

33. $y = 2x^3 - x^2 + 2$, $(1, 3)$

34. $y = 2e^x + x$, $(0, 2)$

35. $y = x + \dfrac{2}{x}$, $(2, 3)$

36. $y = \sqrt[4]{x} - x$, $(1, 0)$

37–38 Find equations of the tangent line and normal line to the curve at the given point.

37. $y = x^4 + 2e^x$, $(0, 2)$

38. $y^2 = x^3$, $(1, 1)$

39–40 Find an equation of the tangent line to the curve at the given point. Illustrate by graphing the curve and the tangent line on the same screen.

39. $y = 3x^2 - x^3$, $(1, 2)$

40. $y = x - \sqrt{x}$, $(1, 0)$

41–42 Find $f'(x)$. Compare the graphs of f and f' and use them to explain why your answer is reasonable.

41. $f(x) = x^4 - 2x^3 + x^2$

42. $f(x) = x^5 - 2x^3 + x - 1$

43. (a) Graph the function

$$f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30$$

in the viewing rectangle $[-3, 5]$ by $[-10, 50]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of f'. (See Example 2.8.1.)
(c) Calculate $f'(x)$ and use this expression, with a graphing device, to graph f'. Compare with your sketch in part (b).

44. (a) Graph the function $g(x) = e^x - 3x^2$ in the viewing rectangle $[-1, 4]$ by $[-8, 8]$.
(b) Using the graph in part (a) to estimate slopes, make a rough sketch, by hand, of the graph of g'. (See Example 2.8.1.)
(c) Calculate $g'(x)$ and use this expression, with a graphing device, to graph g'. Compare with your sketch in part (b).

45–46 Find the first and second derivatives of the function.

45. $f(x) = 0.001x^5 - 0.02x^3$

46. $G(r) = \sqrt{r} + \sqrt[3]{r}$

47–48 Find the first and second derivatives of the function. Check to see that your answers are reasonable by comparing the graphs of f, f', and f''.

47. $f(x) = 2x - 5x^{3/4}$

48. $f(x) = e^x - x^3$

49. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find
(a) the velocity and acceleration as functions of t,
(b) the acceleration after 2 s, and
(c) the acceleration when the velocity is 0.

50. The equation of motion of a particle is $s = t^4 - 2t^3 + t^2 - t$, where s is in meters and t is in seconds.
(a) Find the velocity and acceleration as functions of t.
(b) Find the acceleration after 1 s.
(c) Graph the position, velocity, and acceleration functions on the same screen.

51. Biologists have proposed a cubic polynomial to model the length L of Alaskan rockfish at age A:

$$L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21$$

where L is measured in inches and A in years. Calculate

$$\left.\frac{dL}{dA}\right|_{A=12}$$

and interpret your answer.

52. The number of tree species S in a given area A in the Pasoh Forest Reserve in Malaysia has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

where A is measured in square meters. Find $S'(100)$ and interpret your answer.

Source: Adapted from K. Kochummen et al., "Floristic Composition of Pasoh Forest Reserve, A Lowland Rain Forest in Peninsular Malaysia," *Journal of Tropical Forest Science* 3 (1991):1–13.

53. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P of the gas is inversely proportional to the volume V of the gas.
(a) Suppose that the pressure of a sample of air that occupies 0.106 m^3 at 25°C is 50 kPa. Write V as a function of P.
(b) Calculate dV/dP when $P = 50$ kPa. What is the meaning of the derivative? What are its units?

54. Car tires need to be inflated properly because overinflation or underinflation can cause premature tread wear. The data in the table show tire life L (in thousands of miles) for a certain type of tire at various pressures P (in lb/in^2).

P	26	28	31	35	38	42	45
L	50	66	78	81	74	70	59

(a) Use a calculator to model tire life with a quadratic function of the pressure.
(b) Use the model to estimate dL/dP when $P = 30$ and when $P = 40$. What is the meaning of the derivative? What are the units? What is the significance of the signs of the derivatives?

55. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

56. For what value of x does the graph of $f(x) = e^x - 2x$ have a horizontal tangent?

57. Show that the curve $y = 2e^x + 3x + 5x^3$ has no tangent line with slope 2.

58. Find an equation of the tangent line to the curve $y = x^4 + 1$ that is parallel to the line $32x - y = 15$.

59. Find equations of both lines that are tangent to the curve $y = x^3 - 3x^2 + 3x - 3$ and are parallel to the line $3x - y = 15$.

60. At what point on the curve $y = 1 + 2e^x - 3x$ is the tangent line parallel to the line $3x - y = 5$? Illustrate by graphing the curve and both lines.

61. Find an equation of the normal line to the curve $y = \sqrt{x}$ that is parallel to the line $2x + y = 1$.

62. Where does the normal line to the parabola $y = x^2 - 1$ at the point $(-1, 0)$ intersect the parabola a second time? Illustrate with a sketch.

63. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines intersect the parabola.

64. (a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
(b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why.

65. Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$.)

66. Find the nth derivative of each function by calculating the first few derivatives and observing the pattern that occurs.
(a) $f(x) = x^n$ (b) $f(x) = 1/x$

67. Find a second-degree polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

68. The equation $y'' + y' - 2y = x^2$ is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find constants A, B, and C such that the function $y = Ax^2 + Bx + C$ satisfies this equation. (Differential equations will be studied in detail in Chapter 9.)

69. Find a cubic function $y = ax^3 + bx^2 + cx + d$ whose graph has horizontal tangents at the points $(-2, 6)$ and $(2, 0)$.

70. Find a parabola with equation $y = ax^2 + bx + c$ that has slope 4 at $x = 1$, slope -8 at $x = -1$, and passes through the point $(2, 15)$.

3.2 EXERCISES

1. Find the derivative of $f(x) = (1 + 2x^2)(x - x^2)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree?

2. Find the derivative of the function

$$F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2}$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

3–26 Differentiate.

3. $f(x) = (3x^2 - 5x)e^x$

4. $g(x) = (x + 2\sqrt{x})e^x$

5. $y = \dfrac{x}{e^x}$

6. $y = \dfrac{e^x}{1 - e^x}$

7. $g(x) = \dfrac{1 + 2x}{3 - 4x}$

8. $G(x) = \dfrac{x^2 - 2}{2x + 1}$

9. $H(u) = (u - \sqrt{u})(u + \sqrt{u})$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2})$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$

12. $f(z) = (1 - e^z)(z + e^z)$

13. $y = \dfrac{x^2 + 1}{x^3 - 1}$

14. $y = \dfrac{\sqrt{x}}{2 + x}$

15. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3}$

16. $y = \dfrac{1}{t^3 + 2t^2 - 1}$

17. $y = e^p(p + p\sqrt{p})$

18. $h(r) = \dfrac{ae^r}{b + e^r}$

19. $y = \dfrac{s - \sqrt{s}}{s^2}$

20. $y = (z^2 + e^z)\sqrt{z}$

21. $f(t) = \dfrac{\sqrt[3]{t}}{t - 3}$

22. $V(t) = \dfrac{4 + t}{te^t}$

23. $f(x) = \dfrac{x^2e^x}{x^2 + e^x}$

24. $F(t) = \dfrac{At}{Bt^2 + Ct^3}$

25. $f(x) = \dfrac{x}{x + \dfrac{c}{x}}$

26. $f(x) = \dfrac{ax + b}{cx + d}$

27–30 Find $f'(x)$ and $f''(x)$.

27. $f(x) = (x^3 + 1)e^x$

28. $f(x) = \sqrt{x}\,e^x$

29. $f(x) = \dfrac{x^2}{1 + e^x}$

30. $f(x) = \dfrac{x}{x^2 - 1}$

31–32 Find an equation of the tangent line to the given curve at the specified point.

31. $y = \dfrac{x^2 - 1}{x^2 + x + 1}$, $(1, 0)$

32. $y = \dfrac{1 + x}{1 + e^x}$, $(0, \frac{1}{2})$

33–34 Find equations of the tangent line and normal line to the given curve at the specified point.

33. $y = 2xe^x$, $(0, 0)$

34. $y = \dfrac{2x}{x^2 + 1}$, $(1, 1)$

35. (a) The curve $y = 1/(1 + x^2)$ is called a **witch of Maria Agnesi**. Find an equation of the tangent line to this curve at the point $(-1, \frac{1}{2})$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

36. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point $(3, 0.3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

37. (a) If $f(x) = (x^3 - x)e^x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

38. (a) If $f(x) = e^x/(2x^2 + x + 1)$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

39. (a) If $f(x) = (x^2 - 1)/(x^2 + 1)$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

40. (a) If $f(x) = (x^2 - 1)e^x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

41. If $f(x) = x^2/(1 + x)$, find $f''(1)$.

42. If $g(x) = x/e^x$, find $g^{(n)}(x)$.

43. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.
(a) $(fg)'(5)$ (b) $(f/g)'(5)$ (c) $(g/f)'(5)$

44. Suppose that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$. Find $h'(4)$.
(a) $h(x) = 3f(x) + 8g(x)$ (b) $h(x) = f(x)g(x)$
(c) $h(x) = \dfrac{f(x)}{g(x)}$ (d) $h(x) = \dfrac{g(x)}{f(x) + g(x)}$

3.3 EXERCISES

1–16 Differentiate.

1. $f(x) = x^2 \sin x$

2. $f(x) = x \cos x + 2 \tan x$

3. $f(x) = e^x \cos x$

4. $y = 2 \sec x - \csc x$

5. $y = \sec\theta \, \tan\theta$

6. $g(\theta) = e^\theta(\tan\theta - \theta)$

7. $y = c \cos t + t^2 \sin t$

8. $f(t) = \dfrac{\cot t}{e^t}$

9. $y = \dfrac{x}{2 - \tan x}$

10. $y = \sin\theta \, \cos\theta$

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta}$

12. $y = \dfrac{\cos x}{1 - \sin x}$

13. $y = \dfrac{t \sin t}{1 + t}$

14. $y = \dfrac{\sin t}{1 + \tan t}$

15. $f(\theta) = \theta \cos\theta \sin\theta$

16. $f(t) = te^t \cot t$

17. Prove that $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$.

18. Prove that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

19. Prove that $\dfrac{d}{dx}(\cot x) = -\csc^2 x$.

20. Prove, using the definition of derivative, that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

21–24 Find an equation of the tangent line to the curve at the given point.

21. $y = \sin x + \cos x, \quad (0, 1)$

22. $y = e^x \cos x, \quad (0, 1)$

23. $y = \cos x - \sin x, \quad (\pi, -1)$

24. $y = x + \tan x, \quad (\pi, \pi)$

25. (a) Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\pi/2, \pi)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

26. (a) Find an equation of the tangent line to the curve $y = 3x + 6 \cos x$ at the point $(\pi/3, \pi + 3)$.
(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

27. (a) If $f(x) = \sec x - x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by graphing both f and f' for $|x| < \pi/2$.

28. (a) If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.
(b) Check to see that your answers to part (a) are reasonable by graphing f, f', and f''.

29. If $H(\theta) = \theta \sin\theta$, find $H'(\theta)$ and $H''(\theta)$.

30. If $f(t) = \sec t$, find $f''(\pi/4)$.

31. (a) Use the Quotient Rule to differentiate the function

$$f(x) = \frac{\tan x - 1}{\sec x}$$

(b) Simplify the expression for $f(x)$ by writing it in terms of $\sin x$ and $\cos x$, and then find $f'(x)$.
(c) Show that your answers to parts (a) and (b) are equivalent.

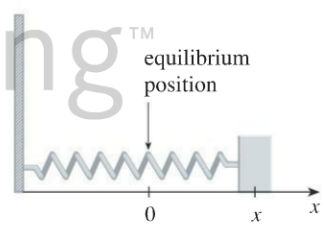
32. Suppose $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x) \sin x$ and $h(x) = (\cos x)/f(x)$. Find
(a) $g'(\pi/3)$ (b) $h'(\pi/3)$

33–34 For what values of x does the graph of f have a horizontal tangent?

33. $f(x) = x + 2 \sin x$

34. $f(x) = e^x \cos x$

35. A mass on a spring vibrates horizontally on a smooth level surface (see the figure). Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x in centimeters.
(a) Find the velocity and acceleration at time t.
(b) Find the position, velocity, and acceleration of the mass at time $t = 2\pi/3$. In what direction is it moving at that time?



36. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2 \cos t + 3 \sin t$, $t \geqslant 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)
(a) Find the velocity and acceleration at time t.
(b) Graph the velocity and acceleration functions.
(c) When does the mass pass through the equilibrium position for the first time?
(d) How far from its equilibrium position does the mass travel?
(e) When is the speed the greatest?

37. A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

38. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object.

If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

where μ is a constant called the *coefficient of friction*.
(a) Find the rate of change of F with respect to θ.
(b) When is this rate of change equal to 0?
(c) If $W = 50$ lb and $\mu = 0.6$, draw the graph of F as a function of θ and use it to locate the value of θ for which $dF/d\theta = 0$. Is the value consistent with your answer to part (b)?

39–50 Find the limit.

39. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x}$

40. $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sin \pi x}$

41. $\displaystyle\lim_{t\to 0} \frac{\tan 6t}{\sin 2t}$

42. $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$

43. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{5x^3 - 4x}$

44. $\displaystyle\lim_{x\to 0} \frac{\sin 3x \sin 5x}{x^2}$

45. $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta + \tan\theta}$

46. $\displaystyle\lim_{x\to 0} \csc x \sin(\sin x)$

47. $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{2\theta^2}$

48. $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x}$

49. $\displaystyle\lim_{x\to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

50. $\displaystyle\lim_{x\to 1} \frac{\sin(x - 1)}{x^2 + x - 2}$

51–52 Find the given derivative by finding the first few derivatives and observing the pattern that occurs.

51. $\dfrac{d^{99}}{dx^{99}}(\sin x)$

52. $\dfrac{d^{35}}{dx^{35}}(x \sin x)$

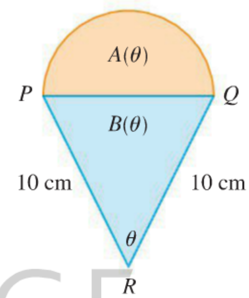
53. Find constants A and B such that the function $y = A\sin x + B\cos x$ satisfies the differential equation $y'' + y' - 2y = \sin x$.

54. (a) Evaluate $\displaystyle\lim_{x\to\infty} x \sin\frac{1}{x}$.
(b) Evaluate $\displaystyle\lim_{x\to 0} x \sin\frac{1}{x}$.
(c) Illustrate parts (a) and (b) by graphing $y = x\sin(1/x)$.

55. Differentiate each trigonometric identity to obtain a new (or familiar) identity.
(a) $\tan x = \dfrac{\sin x}{\cos x}$ (b) $\sec x = \dfrac{1}{\cos x}$
(c) $\sin x + \cos x = \dfrac{1 + \cot x}{\csc x}$

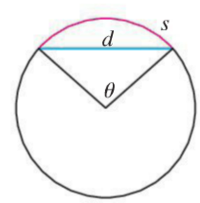
56. A semicircle with diameter PQ sits on an isosceles triangle PQR to form a region shaped like a two-dimensional ice-cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find

$$\lim_{\theta\to 0^+} \frac{A(\theta)}{B(\theta)}$$



57. The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find

$$\lim_{\theta\to 0^+} \frac{s}{d}$$



58. Let $f(x) = \dfrac{x}{\sqrt{1 - \cos 2x}}$.
(a) Graph f. What type of discontinuity does it appear to have at 0?
(b) Calculate the left and right limits of f at 0. Do these values confirm your answer to part (a)?

3.4 The Chain Rule

Suppose you are asked to differentiate the function

$$F(x) = \sqrt{x^2 + 1}$$

The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$.