§3.4 The Chain Rule

- composition of functions

$$(f\circ g)(x) = f(g(x)) \quad \text{or} \quad y = f(u) \text{ and } u = g(x)$$

outer inner. dep. var. intermed. indep var.

- the chain rule

$$(f\circ g)'(x) = f'(\underline{g(x)})\, g'(x) \quad \text{or} \quad \frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \cdot \frac{du}{dx}$$

$$\| $$

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{[f'(g(x)) + \varepsilon]\,[g(x+h) - g(x)]}{h}$$

f is differentiable at $g(x)$

and $\varepsilon \to 0$ as $h \to 0$

Examples Compute derivatives

$f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x)$

(1) $y = f(x) = \sqrt{x^2+1} = \sqrt{u}$, $u = x^2+1 = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\Big|_{u=g(x)} \cdot \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}\Big|_{u=g(x)} \cdot (2x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

(2) $y = \sin(x^2) = \sin u$, $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u\Big|_{u=x^2} \cdot 2x = 2x\cos(x^2)$$

$$y' = \cos(x^2) \cdot 2x = 2x\cos x^2$$

(3) $y = (\sin x)^2$ $\qquad y' = 2(\sin x) \cdot \cos x = 2\sin x\cos x = \sin(2x)$

$y = u^2$, $u = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u\Big|_{u=\sin x} \cdot \cos x = 2\sin x\cos x$$

(4) $y = \left(g(x)\right)^n = u^n, \quad u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n u^{n-1}\Big|_{u=g(x)} \cdot g'(x) = n\left(g(x)\right)^{n-1} \cdot g'(x)$$

$$y' = n\left(g(x)\right)^{n-1} \cdot g'(x)$$

(5) $y = \left(x^3 - 1\right)^{100}, \quad y' = 100 \cdot \left(x^3 - 1\right)^{99} \cdot 3x^2 = 300x^2\left(x^3 - 1\right)^{99}$

(6) $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}} = (x^2+x+1)^{-\frac{1}{3}}$

$f'(x) = (-\frac{1}{3})(x^2+x+1)^{-\frac{1}{3}-1} \cdot (2x+1)$

(7) $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

$g'(t) = 9\left(\dfrac{t-2}{2t+1}\right)^8 \dfrac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2} = 9\left(\dfrac{t-2}{2t+1}\right)^8 \dfrac{5}{(2t+1)^2}$

$= \dfrac{45(t-2)^8}{(2t+1)^{10}}$

(8) $y = \underline{(2x+1)^5}\,\underline{(x^3-x+1)^4}$

$y' = \left[(2x+1)^5\right]'(x^3-x+1)^4 + (2x+1)^5 \cdot \left[(x^3-x+1)^4\right]'$

$= 5(2x+1)^4 \cdot 2\,(x^3-x+1)^4 + (2x+1)^5 \cdot 4\,(x^3-x+1)^3 \cdot (3x^2-1)$

(9) $y = e^{\sin x}$, $y' = e^{\sin x} \cdot \cos x$

$y = e^u$, $u = \sin x$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \Big|_{u=\sin x} \cos x = e^{\sin x} \cos x$

- $\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$ $\boxed{(e^x)' = e^x}$

$\overset{\shortparallel}{\frac{d}{dx}\left(e^{\ln a^x}\right)} = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} (\ln a) = a^x \ln a$

- $\begin{cases} y = f(u) \\ u = g(x) \\ x = h(t) \end{cases}$ $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$

$y = f(g(h(t)))$

Ex. 8 $f(x) = \sin(\cos(\tan x))$, compute $f'(x)$.

$f'(x) = \cos(\cos(\tan x)) \cdot [-\sin(\tan x)] \cdot \sec^2 x$

Ex. 9 $y = e^{\sec(3\theta)}$, compute y'.

$$y' = e^{\sec(3\theta)} \cdot \sec(3\theta)\tan(3\theta) \cdot 3$$