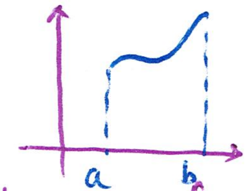
§3.5 Implicit Differentiation

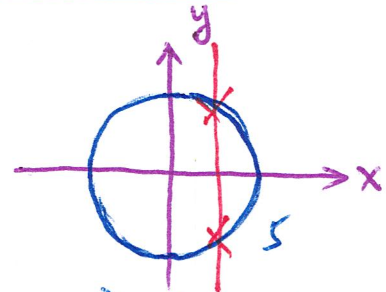


graph $C = \{(x, f(x)) \mid x \in D\}$

- explicit $y = f(x)$
- implicit $\boxed{f(x, y) = \text{constant}}$

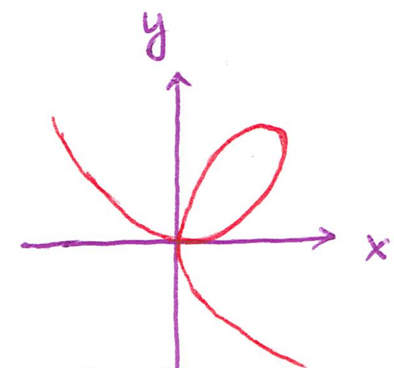
difficult to solve it for y

level curves



$x^2 + y^2 = 25$

$y = \pm\sqrt{25 - x^2}$



$x^3 + y^3 = 6xy$

The folium of Descartes

implicit differentiation

Ex. 1 (a) $x^2 + y^2(x) = 25$, find $\frac{dy}{dx}$

(b) Find an equation of the tagent line to the circle $x^2 + y^2 = 25$ at $(3, 4)$.

Solution 1 (implicit differentiation)

$$\frac{d}{dx}(25) = \frac{d}{dx}\left(x^2 + y^2(x)\right)$$

$$\boxed{0 = 2x + 2\,y(x)\,y'(x)}$$

$$y'(x) = -\frac{x}{y(x)}$$

Solution 2 (solving the equation for y)

Ex. 2 (a) Find y' if $x^3 + y^3 = 6xy$

(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at $(3, 3)$.

(c) At what point in the first quadrant is the tangent line horizontal?

Ex. 3 Find y' if $\sin(x+y(x)) = y^2(x)\cos x$.

$$\frac{d}{dx}\left(\sin(x+y)\right) = \frac{d}{dx}\left(y^2\cos x\right) = 2y\cdot y'\cos x + y^2\cdot(-\sin x)$$

$$\shortparallel$$

$$\cos(x+y)\cdot(1+y')$$

$$\boxed{2y\cos x - \cos(x+y)}\, y' = \cos(x+y) + y^2\sin x$$

$$y' = \frac{\cos(x+y) + y^2\sin x}{2y\cos x - \cos(x+y)}$$

Ex. 4 Find y'' if $x^4+y^4=16$

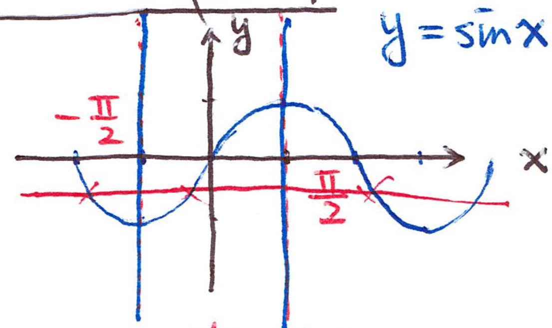
$$\frac{d}{dx}\left(x^4+y^4\right) = \frac{d}{dx}(16) = 0$$

$$\shortparallel$$

$$4x^3+4y^3\cdot y' = 0 \Longrightarrow y' = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$$

- **Inverse Trigonometric Functions (§1.6)**

$y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



one-to-one

inverse $y = \sin^{-1} x$, $-1 \le x \le 1$

$= \arcsin x$

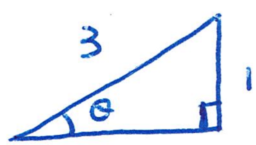
$y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

identities $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$.

Ex. (a) $\sin^{-1}\left(\frac{1}{2}\right) = ?$ and (b) $\tan\left(\sin^{-1}\frac{1}{3}\right) = ?$

(a) let $\sin^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta =$

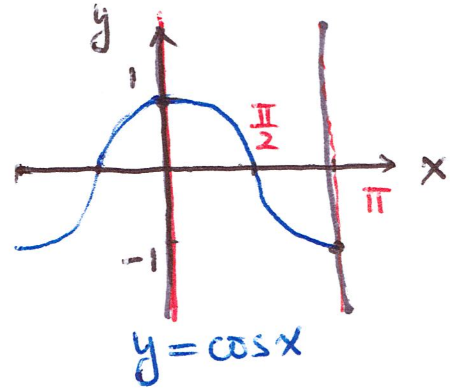
(b) let $\sin^{-1}\left(\frac{1}{3}\right) = \theta \Rightarrow \sin\theta = \frac{1}{3}$



- $y = \cos^{-1} x = \arccos x, \quad x \in [-1, 1]$

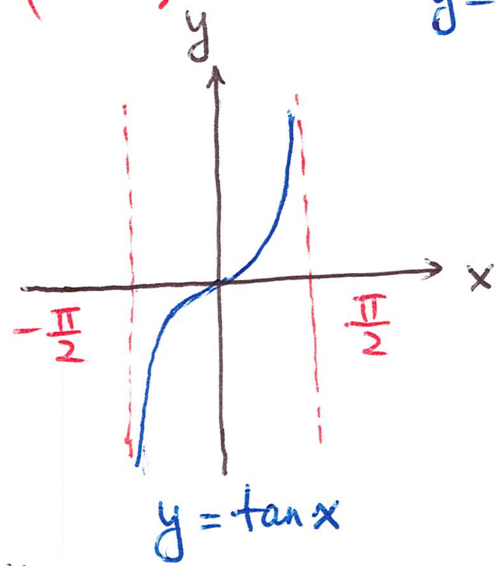
  $y \in [0, \pi]$

  identities $\cos^{-1}(\cos x) = x$ and $\cos(\cos^{-1} x) = x$.



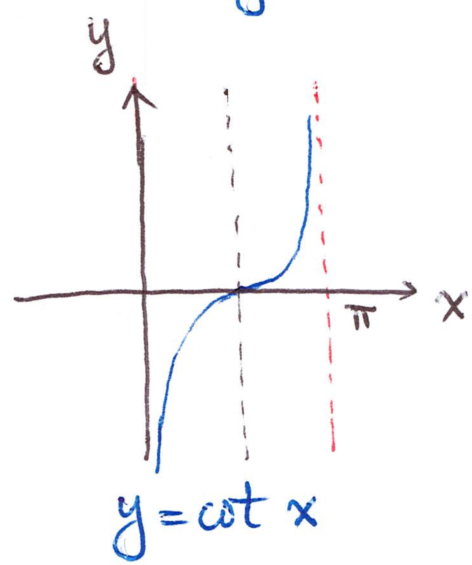
- $y = \tan^{-1} x, \quad x \in (-\infty, +\infty)$

  $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



- $y = \cot^{-1} x, \quad x \in (-\infty, +\infty)$

  $y \in (0, \pi)$



- derivatives of inverse trigonometric functions

$$\boxed{\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}}$$

$y = \sin^{-1}x$

$\Downarrow$

$\boxed{\sin y(x) = x} \Longrightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}(x) = 1$

$\frac{d}{dx}(\sin y) = \cos y \cdot y'$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin^2 y + \cos^2 y = 1$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$= \sqrt{1-x^2}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}}$$

$y = \tan^{-1}x \Longrightarrow \boxed{\tan y = x} \Longrightarrow$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y) = \left(\sec^2 y\right) \cdot y'$$

$\frac{d}{dx}(x) = 1$

$$y' = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Ex. 5 Differentiate

(a) $y = \dfrac{1}{\sin^{-1} x}$, $\sin x$ (b) $f(x) = x \arctan \sqrt{x}$

(a) $$y' = \frac{d}{dx}\left(\sin^{-1} x\right)^{-1} = (-1)\left(\sin^{-1} x\right)^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{1}{\sqrt{1-x^2}\left(\sin^{-1} x\right)^2}$$

(b) $$f'(x) = \arctan\sqrt{x} + x\,\frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \arctan\sqrt{x} + \frac{\cancel{x}\ \ x^{\frac{1}{2}}}{2\left(1+|x|\right)\cancel{x^{\frac{1}{2}}}}$$

§3.6 Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}, \qquad \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}.$$

$$y = \log_a x \implies x = a^{y(x)} \implies 1 = a^y \cdot \ln a \cdot y' \implies y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

Examples Differentiate

(1) $y = \ln(x^3+1)$, $\quad y' = \frac{1}{x^3+1} \cdot \left(3x^2 + 0\right) = \frac{3x^2}{x^3+1}$

(2) $y = \ln(\sin x)$, $\quad y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$

(3) $f(x) = \sqrt{\ln x}$; $f'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$

(4) $f(x) = \log_{10}(2+\sin x)$, $f'(x) = \frac{1}{(2+\sin x)\ln 10} \cdot \cos x$

(5) $y = \ln \frac{x+1}{\sqrt{x-2}} = \ln(x+1) - \frac{1}{2}\ln(x-2)$, $y' = \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2}$

$\cancel{f(x) = |x|}$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

(6) $f(x) = \ln|x|$ for $x \neq 0$

$$= \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

For $x > 0$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$$

For $x < 0$

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x}$$